Vietnam Journal of Mathematics
Volume 23, Number 1, 1995

A Short Communication

ON DIRECT AND INVERSE THEOREMS IN MULTIVARIATE TRIGONOMETRIC APPROXIMATION*

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1. As well known, the classical direct and inverse theorem of best univariate approximation establishes a connection between the smoothness properties of functions, defined via moduli of smoothness, and the speed of the best approximation by trigonometric polynomials of order at most n (i.e. with frequencies from $\{0, 1, \dots, n\}$). This theorem has some direct multivariate generalizations with simple frequency domains in terms of norm equivalence of Besov spaces (see [9]). Generally, in multivariate trigonometric polynomial approximation, because of the complicatedness of multivariate smoothness and of the various possibilities for restricting their frequencies we must first of all understand what frequency domains should be selected for a best method of approximation for a set of functions with common smoothness. Hyperbolic crosseses are such domains for a finite smoothness, characterized by the L_p-boundedness of one or several mixed derivatives and differences (see [4], [5], [10], [11] for detailed descriptions of history and results on hyperbolic cross approximation).

In the direct and inverse setting of such a problem, it is of great interest to characterize the smoothness properties which guarantee a given speed of the approximation by trigonometric polynomials with frequencies from a given sequence of hyperbolic crosses. The authors of the manuscript [2] have deal with this problem for the regular hyperbolic crosses $\{k \in Z^d : \prod_{i=1} \max(1, |k_j|) \leq n\}$. Let A be a finite subset of $\mathbf{R}^d$. In this note we formulate a direct and inverse theorem for the following family of hyperbolic crosses

* This work is supported by Project 1.5.5 of the National Program for Fundamental Researches in Natural Sciences

$$\Gamma_\xi(A) := \left\{ \cup \square_s : \langle a, s \rangle \le \xi,\ s \in \mathbf{N}^d,\ a \in A \right\}, \quad \xi \ge 0,$$

where

$$\square_s := \left\{ k \in \mathbf{Z}^d : \lfloor 2^{s_j - 2} \rfloor \le |k_j| < 2^{s_j - 1} \right\}.$$

2. A function f in $L_p(\mathbf{T}^d)$, $1 \le p \le \infty$, can be considered as a regular distribution on the d-dimensional torus $\mathbf{T}^d$ (see, e.g., [12] for details about distributions on $\mathbf{T}$). For any $a \in \mathbf{R}^d$ the distributional derivative $f^{(a)}$ in the Weil sense is defined by

$$f^{(a)} := \sum_{k \in \mathbf{Z}^d_a} \hat{f}(k) \prod_{j=1}^{d} (ik_j)^{a_j} e^{i\langle k, \cdot \rangle}$$

where $\hat{f}(k)$ is the k-th Fourier coefficient of f, $\mathbf{Z}^d_a = \prod_{j=1}^{d} \mathbf{Z}_{a_j}$, $\mathbf{Z}_t = \mathbf{Z} \setminus \{0\}$ for $t \ne 0$ and $Z_t = \mathbf{Z}$ for $t = 0$ and

$$(iu)^t := |u|^t \exp\left(\frac{i\pi t}{2} \operatorname{sign} u\right).$$

Let A be a finite subset of $\mathbf{R}^d$. We denote by W^A_p the space of all functions on $\mathbf{T}^d$ such that

$$|f|_{W^A_p} := \sum_{a \in A} \|f^{(a)}\|_p$$

is finite where $\|\cdot\|_p$ denotes the norm of $L_p(\mathbf{T}^d)$. We say that the functions in W^A_p have the common mixed smoothness A. Our requirement on the smoothness A is that

$$a^e \in A \quad \text{for} \quad \forall a \in A \quad \text{and} \quad \forall e \subset I_a, \tag{1}$$

where $I_a := \{j : 1 \le j \le d,\ a_j > 0\}$ and $a^e \in \mathbf{R}^d$ is defined by $a^e_j = a_j$ for $j \in e$ and $a^e_j = 0$ for $j \notin e$. This requirement allows us to be free from the restriction to operate only functions in $L^0_p(\mathbf{T}^d)$ which consists of all functions in $L_p(\mathbf{T}^d)$ with zero mean value at each variable.

3. If G is a subset of $\mathbf{Z}^d$, we let $\mathcal{J}(G)$ denote the L_p-closure of the span of the harmonics $e^{i\langle k,\cdot\rangle}$, $k \in G$. For a finite G, $\mathcal{J}(G)$ is the space of all trigonometric polynomials with frequencies in G. We shall use the abbreviated notation: $\mathcal{J}_\xi(A) = \mathcal{J}(\Gamma_\xi(A))$. One can prove that $\Gamma_\xi(A)$ is a finite subset of $\mathbf{Z}^d$ if and only if

$$\min_{1\le j\le d} \max_{a\in A} a_j > 0. \tag{2}$$

For any $\xi > 0$, we let

$$E_\xi(A,f)_p := \inf_{\varphi\in \mathcal{J}_\xi(A)} \|f-\varphi\|_p$$

denote the best L_p-approximation of f by elements from $\mathcal{J}_\xi(A)$. If $1 < p < \infty$ and $\alpha > 0$, we have for any $\xi > 0$

$$E_\xi(A,f) \le C(A,p)2^{-\alpha\xi}|f|_{W_p^{\alpha A}}, \tag{3}$$

$$|\varphi|_{W_p^{\alpha A}} \le C'(A,p)2^{\alpha\xi}\|\varphi\|_p, \quad \varphi\in \mathcal{J}_\xi(A), \tag{4}$$

where $\alpha A := \{x \in \mathbf{R}^d : x = \alpha a,\ a\in A\}$. Moreover, the degree of (3)-(4) is exact. There inequalities were first proved in [1] ($p = 2$) and then in [7] ($p \ne 2$) in the case when $\alpha = 1$ and $A = \{a\}$ with $a \in \mathbf{N}^d$. The inequality (3) was proved in [8] for $\alpha = 1$ and $A = \{a\}$ with $a_j > 0$. Inequalities (with $\alpha = 1$), similar to (3)-(4) was given in [6] for functions from $L_p^0(\mathbf{T}^d)$. The inequalities (3)-(4) can be proved by an analogous method to those in [6] - [8], using the Littlewood-Paley theorem and its generalizations.

The inequality (3) shows that if the function f has the smoothness αA, then the degree of $E_\xi(A,f)_p$, $1 < p < \infty$, is not greater than $2^{-\alpha\xi}$. Moreover, the degree $2^{-\alpha\xi}$ in (3) with $n \approx \text{card}\,\Gamma_\xi(A) \le n$ coincides with the degree of n-widths of the unit ball of $W_p^{\alpha A}$ (for $\alpha = 1$, see [1], [6], [7], [3]). However, as in the univariate case, the space $W_p^{\alpha A}$ is smaller than the set of all functions f such that $E_\xi(A,f)_p \le C2^{-\alpha\xi}$. We are interested in characterization of the smoothness properties of functions f which govern a preassigned degree of $E_\xi(A,f)_p$. If $\alpha > 0$ and $0 < q \le \infty$, we let $\mathcal{E}_{p,q}(A,\alpha)$ denote the space of all functions $f \in L_p(\mathbf{T}^d)$ such that

$$|f|_{\mathcal{E}_{p,q}(A,\alpha)} := \begin{cases} \left(\sum_{n=1}^{\infty} \{2^{\alpha n}E_n(A,f)_p\}^q\right)^{1/q}, & q<\infty \\ \sup_{1\le n\le\infty} 2^{\alpha n}E_n(A,f)_p, & q=\infty \end{cases}$$

is finite. We define the "norm" in $\mathcal{E}_{p,q}(A,\alpha)$ by

$$\|f\|_{\mathcal{E}_{p,q}(A,\alpha)} := \|f\|_p + |f|_{\mathcal{E}_{p,q}(A,\alpha)}.$$

4. We shall give a characterization of the smoothness properties of the space $\mathcal{E}_{p,q}(A,\alpha)$, by introducing a "modulus of smoothness" which is defined via the convolution with a certain distribution. The author of [2] gave such a characterization for the regular hyperbolic cross approximation by suggesting new moduli of smoothness which is defined with the aid of higher-order mixed difference operators and the convolution with the symmetric multivariate B-splines.

For a natural number r, we let the distribution $\lambda_t = \lambda_t(A,r)$, $t \in \mathbf{R}$, on $\mathbf{T}^d$, be defined by its Fourier coefficients $\hat{\lambda}_t(k)$ as follows

$$\hat{\lambda}_t(k) = \left(\frac{e^{it\mu(k)} - 1}{1 + |t\mu(k)|^d}\right)^r$$

with

$$\mu(k) := \prod_{j=1}^{d} \operatorname{sign} k_j \sup_{a\in A} \prod_{j=1}^{d} |k_j|^{a_j}, \quad k \in \mathbf{Z}^d.$$

We define the operator $\Delta_t^r(A)$ for distributions f on $\mathbf{T}^d$ as the convolution of f with λ_t:

$$\Delta_t^r(A)f := f * \lambda_t,$$

and the "modulus of smoothness" $\Omega^r(A,f,h)_p$, $0 \le h \le \pi$, for functions in $L_p(\mathbf{T}^d)$ by

$$\Omega^r(A,f,h)_p = \sup_{|t|\le h} \|\Delta_t^r(A)f\|_p.$$

For $\alpha > 0$ and $0 < q \le \infty$, let $B_{p,q}(A,\alpha,r)$ denote the space of all functions on $\mathbf{T}^d$ such that

$$|f|_{B_{p,q}(A,\alpha,r)} := \begin{cases} \left(\sum_{n=0}^{\infty}\left\{2^{\alpha n}\Omega^r(A,f;2^{-n})_p\right\}^q\right)^{1/q}, & q < \infty \\ \sup_{0\le n<\infty} 2^{\alpha n}\Omega^r(A,f;2^{-n})_p, & q = \infty. \end{cases}$$

is finite. We define the "norm" in $B_{p,q}(A,\alpha,r)$ by

$$\|f\|_{B_{p,q}(A,\alpha,r)} := \|f\|_p + |f|_{B_{p,q}(A,\alpha,r)}.$$

5. The main results of this note read as follows

Theorem 1. *Let A be a finite subset of $\mathbf{R}^d$, and a natural number r, $\alpha > 0$, $1 < p < \infty$, $0 < q \leq \infty$. Assume that there hold the conditions (1)-(2). Then for any $r > \alpha$, we have*

$$\mathcal{E}_{p,q} = B_{p,q}(A, \alpha).$$

Moreover, for functions f in $\mathcal{E}_{p,q}(A, \alpha)$

$$\|f\|_{\mathcal{E}_{p,q}(A,\alpha)} \asymp \|f\|_{B_{p,q}(A,\alpha,r)}.$$

Theorem 2. *Under the hypotheses of Theorem 1, for any natural number n, the following direct and inverse inequalities*

$$\text{(i) } E_n(A,f)_p \leq C\Big(\sum_{m=n+1}^{\infty} \{\Omega^r(A,f;2^{-m})_p\}^{p^*}\Big)^{1/p^*}$$

$$\text{(ii) } \Omega^r(A,f;2^{-n+1}) \leq C' 2^{-rn}\Big(\sum_{m=0}^{n} \{2^{rm} E_m(A,f)_p\}^{p^*}\Big)^{1/p^*}$$

hold with some constants C and C' independent of f where $p^* = \min(p, 2)$.

Theorem 2 extends results proved by A. F. Timan, M. F. Timan and Stechkin (see [3]) for univariate functions. Some weaker inequalities were obtained in [2] for the best regular hyperbolic cross L_p-approximation and the modulus of smoothness defined in this work. In order to prove Theorem 2, we used the Littlewood-Paley theorem the Marcinkiewicz multiplier theorem, the Bernstein type inequality (4) and the following facts. Under the hypotheses of Theorem 1, let $D_\xi = \Gamma_\xi(A) \setminus \Gamma_{\xi-1}(A)$. Then for any $t \in \mathbf{R}$ and $\xi \geq 0$, we have

$$\|\Delta_t^r(A)f\|_p \leq C(A,r,p)|t|^r |f|_{W_p^r A}, \quad f \in L_p(\mathbf{T}^d);$$

$$\|\varphi\|_p \leq C'(A,r,p)\|\Delta_{2^{-\xi}}^r(A)f\|_p, \quad \varphi \in J(D_\xi).$$

Acknowledgment. I would like to thank Professor Vladimir Temlyakov for valuable discussions stimulating the appearance of this note.

REFERENCES

1. K. I. Babenko, *On the approximation of periodic functions of several variables by trigonometric polynomials,* DAN SSSR **132** (1960), 247-250.
2. R. A. Devore, P. P. Petrushev, V. V. Temlyakov, *Multivariate trigonometric polynomial approximation with frequencies from the hyperbolic cross* (manuscript).
3. Dinh Dung, *The number of integral points in some sets and approximation of functions of several variables,* Mat. Zametki **36** (1984), 479-491.
4. Dinh Dung, *Approximation of functions of several variables on the torus by trigonometric polynomials,* Mat. Sb. **131** (1986), 251-271.
5. Dinh Dung, *Approximation of function of several variables by means of harmonic analysis,* Doctor's Dissetation, Moscow State Univ., Moscow 1985.
6. E. M. Galaev, *Approximation of classes of periodic functions with several bounded derivatives by Fourier sums,* Mat. Zametki **23** (1978), 197-211.
7. B. S. Mityagin, *Approximation of functions in the spaces L_p and C on the torus,* Mat. Sb. **58** (100) (1962), 379-413.
8. N. S. Nikol'skaya, *Approximation differentiable functions of several variables by Fourier sums in metric L_p,* Sib. Mat. J. **15** (1974), 395-412
9. S. N. Nikol'skii, *Approximation of functions of several variables and embedding theorems,* Springer, Berlin, 1975.
10. V. N. Temlyakov, *Approximation of functions with bounded mixed derivatives* **178** (1986) (English translation in Proc. Steklov Inst. Math., 1989, No 1)
11. V. N. Temlyakov, *Approximation of periodic functions,*Nova Science Publishers, NY, 1993.
12. V. M. Tikhomirov, *Some problem of approximation theory,* Moscow State Univ. Press, Moscow, 1970.
13. A. F. Timan, *Theory of approximation of functions of a real variable,* Moscow, 1960.

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Received December 20, 1994