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SOME ASPECTS OF THE THEORY OF STOCHASTIC INTEGRALS [1]

DANG HUNG THANG

Abstract. *The aim of the paper is to give a brief survey on some directions of research in the theory of stochastic integration with the emphasis on some topics related to our work. Many important stochastic integrals, which are central in the modern theory of stochastic analysis and crucial for applications are presented. The random operators between Banach spaces, a natural framework of stochastic integrals, are also treated.*

1. INTRODUCTION

The purpose of the paper is to give a brief survey of some aspects of the theory of stochastic integration and to present some of our contributions in this context. We does not attempt to present all aspects as well as the comprehensive history of the development of the theory of stochastic integration but rather just some directions of research which are of our interest and related to our work.

Historically, the first stochastic integral in the probability theory is the integral of a square integrable function with respect to (w.r.t.) the Wiener proces. This integral was introduced by N. Wiener in 1923 [31] and called the Wiener stochastic integral. The Wiener stochastic integral has been generalized in many directions. Genarally speaking, the aim of these generalizations is to define stochastic integrals so that the class of integrators as well as the class of integrands must be as wide as possible and, at the same time, the stochastic integral should enjoy many good properties. Several stochastic integrals (e.g. the Ito integral, the Stratonovich integral) are central in the modern theory of stochastic analysis and crucial for many other applications.

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Sections 2, 3, 4 and 5 treat some directions of research in the theory of stochastic integrals and in the last section we introduce the notion of random operators in Banach spaces as a natural framework of stochastic integrals.

2. STOCHASTIC INTEGRAL OF NON-RANDOM FUNCTIONS WITH RESPECT TO A RANDOM PROCESS

Let $\{W(t),\ 0 \leq t \leq 1\}$ be the Wiener process (or the Brownian motion) on the interval $[0,1]$. In many applications, there arises the need of considering the integral of the form

$$\int_0^1 f(t)\, dW(t), \tag{1.1}$$

where $f(t)$ is a function defined on $[0,1]$. Because all sample paths of the Wiener process have unbounded variation, the Stieltjes integral $\int_0^1 f(t)\, dW(t,\omega)$ can not be performed on every path. However, Wiener was successful in finding a reasonable mathematical definition of the integral (1.1). The construction of the Wiener integral is the following. First, if f is a simple function

$$f(t) = \sum_{i=1}^{n} c_i 1_{[t_i, t_{i+1})},$$

then $\int_0^1 f(t) dW$ is defined by

$$\int_0^1 f(t)\, dW = \sum_{i=1}^{n} c_i \{W(t_{i+1}) - W(t_i)\}.$$

By the independence of the differences we have the identity

$$\mathbf{E}\Big|\int_0^1 f(t)\, dW\Big|^2 = \int_0^1 |f(t)|^2\, dt.$$

This proves that the mapping $f \to \int_0^1 f\,dW$ is an isometry from the linear space S of simple functions into $L_2(\Omega)$. Next, because S is dense in $L_2[0,1]$ this map admids an extension to the whole space $L_2[0,1]$. For each $f \in L_2[0,1]$, $\int_0^1 f(t)\,dW$ is defined as the image of f under this map.

In other words, $\int_0^1 f(t)\,dW$ can be defined as the limit in the mean square of the integral sums of the form

$$\sum_{i=0}^{n} f(t_i)(W(t_{i+1}) - W(t_i))$$

when the gauge of the partition tends to zero.

By a similar construction, H. Crame developed a general theory of stochastic integral w.r.t. a complex-valued process with orthogonal increaments. Recall that by a process with orthogonal increaments we mean a process $X(t), t \in \mathbf{R}$ satisfying $\mathbf{E}|X(t)|^2 < \infty$ and

$$\mathbf{E}[X(t_2) - X(t_1)]\overline{[X(t_4) - X(t_3)]} = 0,$$

whenever $(t_1,t_2),(t_3,t_4)$ are disjoint intervals.

It is shown that $\int_{\mathbf{R}} f(t)\,dX$ exists for complex-valued functions f satisfying

$$\int_{\mathbf{R}} |f(t)|^2\,dG(t) < \infty,$$

where G is a non-decreasing function associated with X by the formula

$$\mathbf{E}|X(t) - X(s)|^2 = G(t) - G(s)$$

and the following isometry holds

$$\mathbf{E}\Big|\int_{\mathbf{R}} f(t)\,dX\Big|^2 = \int_{\mathbf{R}} |f(t)|^2\,dG(t).$$

This integral plays a pivotal role in the theory of stationary processes. Namely, if $X(t)$ is a stationary process then there exists a complex-valued process with orthogonal increament $S(t)$ such that

$$X(t) = \int_{\mathbf{R}} e^{itu}\, dS(u).$$

This formula is called the spectral representation of the stationary process $X(t)$.

As a continuous analogue of weight sum of i.i.d random variables of the form $\sum a_i \xi_i$ the stochastic integral w.r.t. a Levy process was firstly studied by Levy and developed by Urbanik and Woczynski [34]. Recall that a process $X(t)$ is said to be a Levy process if it has independent increaments and the distribution of $X(t) - X(s)$ depends only on $t - s$. If $f(t)$ is a simple function , $f(t) = \sum_{i=0}^{n-1} c_i \mathbf{1}_{[t_i, t_{i+1})}$ then the stochastic integral $\int_0^1 f(t)\, dX(t)$ is defined by

$$\int_0^1 f(t) dX(t) = \sum_{i=0}^{n-1} c_i \big[X(t_{i+1}) - X(t_i)\big].$$

A function f is said to be X-integrable if there exists a sequence (f_n) of simple functions such that $f_n(t)$ converges to $f(t)$ almost everywhere and the sequence $\left\{\int_0^1 f_n\, dX\right\}$ converges in probability. In this case we put

$$\int_0^1 f(t)\, dX(t) = P - \lim_n \int_0^1 f_n(t)\, dX(t).$$

It was shown that the limit, if it exists, is independent of the choice of a particular approximating sequence (f_n). The following theorem gives a full analytic description of X-integrable functions.

Theorem (Urbanik,Woyczynski [34]). *Let $X(t)$ be a symmetric Levy process and the characteristic function $\phi(u)$ of the increament $X(t) - X(s)$ be given by*

$$\phi(u) = \exp\left\{|t-s|\int_0^\infty (\cos uv - 1)\frac{1+v^2}{v^2}\,dG(v)\right\},$$

where the function $G(v)$ is monotone non-decreasing bounded continuous on the left with $G(0) = 0$ (G is called the Levy-Khinchin function corresponding to X). Then f is X-integrable if and only if f belongs to the Orlicz space $L_\Phi[0,1]$, where

$$\Phi(x) = \int_{1/x}^\infty \frac{G(u)}{u^3}\,du.$$

In the case X is a p-stable motion i.e. $\phi(u) = \exp\{-|t-s||u|^p\}$ then $L_\Phi[0,1] = L_p[0,1]$

Vakhania and Kandelski [32] gave a definition of a stochastic integral for operator-valued functions w.r.t. a Levy process taking values in a Hilbert space H and satisfying $\mathbf{E}\|X(1)\|^2 < \infty$. Let R denote the covariance operator associated with X. For any bounded linear operator $A \in L(H,H)$ define

$$\|A\|_* = \left[\mathrm{Tr}(ARA^*)\right]^{1/2} + \left[\mathrm{Tr}(A^*RA)\right]^{1/2}.$$

Then the set $\{A : \|A\|_* = 0\}$ is a linear semi-group in the linear group $L(H,H)$. The function $A \to \|A\|_*$ is a norm in the corresponding factor group. We shall not distinguish between a coset and the individual operators in the coset. Let $\mathcal{A}$ denote the completion of $L(H,H)$ in this norm. Consider the space $L_2\left[[0,1],\mathcal{A}\right]$ of functions $f : [0,1] \to \mathcal{A}$ such that

$$\int_0^1 \|f(t)\|_*^2\,dt < \infty.$$

The standard extension procedure yields a stochastic integral $\int_0^1 f(t)\,dX(t)$ for each $f \in L_2\left[[0,1],\mathcal{A}\right]$. It was shown that $\int_0^1 f(t)dX(t)$ is a H-

valued infinitely divisible random variable with the characteristic function given by

$$F(h) = \exp \int_0^1 K[f^*(t)(h)]\, dt, \quad h \in H,$$

where $K(h) = \ln G(h)$ and $G(h)$ is the characteristic function of $X(1)$.

Rao [18] used this stochastic integral to obtain the characterization theorem for Wiener processes taking values in a Hilbert space H throught indentical distributions of $\int_0^1 f(t)\, dX(t)$ and $\int_0^1 g(t)\, dX(t)$. Namely, he proved that under a slight assumption, $\int_0^1 f(t)\, dX$ and $\int_0^1 g(t)\, dX$ are indentically distributed if and only if X is a H-valued Wiener process and

$$\int_0^1 f(t) R f^*(t)\, dt = \int_0^1 g(t) R g^*(t)\, dt,$$

where R is the covariance operator associated with X. Here the integral is understood to be a Bochner integral under convergence in the space $\mathcal{A}$.

3. RANDOM MEASURES AND STOCHASTIC INTEGRALS WITH RESPECT TO THEM

Let $(S, \mathcal{S})$ be a measurable space. A mapping M from $\mathcal{S}$ into $L_0(\Omega)$ is called a random measure if for every sequence $\{A_n\}$ of disjoint sets in $\mathcal{S}$, the random variables $M(A_n)$ are independent and we have

$$M\Big(\bigcup_n A_n\Big) = \sum_n M(A_n)$$

where the series is assumed to converge in probability. In addition, if $M(A)$ is a symmetric random variable for every $A \in \mathcal{S}$ then we call M a symmetric random measure.

By definition, a random measures on $(S, \mathcal{S})$ can be viewed as a real stochastic process indexed by the parameter set $\mathcal{S}$. Two random measures M and N are called a version of each other if for each $A \in \mathcal{S}$

$$P\{\omega : M(A)(\omega) = N(A)(\omega)\} = 1$$

For each ω fixed, the set function $A \to M(A)(\omega)$ is called a sample path of M. We say that a random measure is regular if its all paths are measures of bounded variation. There are simple examples of random measures having no regular version. For instance, let $W = W(t)$, $0 \leq t \leq 1$ be the Wiener process. Define $M\left\{ \bigcup_{i=1}^{n} (a_i, b_i) \right\} = \sum_{i=1}^{n} [W(a_i) - W(b_i)]$ if (a_i, b_i) are disjoint intervals, we get a random set function which by the Prekopa theorem [12] can be extended to a random measure on σ-algebra $\mathcal{B}$ of Borel set of $[0,1]$. M has no regular version since almost every path of the Wiener process is of unbounded variation. A point process is a good example of a regular random measure. A study of non-regular random measures requireds a different approach from ones taken in the theory of point processes presented in [7].

Now we want to define the stochastic integral of the form

$$\int_S f(t)dM$$

where $f(t)$ is a non-random function defined on S.

If M is regular then it is natural to define the integral as the usual integral of the function $f(t)$ with respect to the measure $dM(t, \omega)$, ω being fixed, if such an integral exists. Actually, this construction is not possible for the general case. The reason is that many random measures are not regular, and then $dM(t, \omega)$ does not define a measure. The following definition of the stochastic integral was proposed first by Urbanik and Woyczynski [34]

Definition. a) Let $f = \sum_{i=1}^{n} x_i 1_{A_i}$ be a real simple function on S, where $A_i \in \mathcal{S}$ are disjoint. Then, for every $A \in \mathcal{S}$, we define

$$\int_A f\, dM = \sum_{i=1}^{n} x_i M(A \cap A_i).$$

b) A measurable function $f : S \to \mathbf{R}$ is said to be M-integrable if there exists a sequence $\{f_n\}$ of simple functions as in (a) such that

i) $f_n \to f$ M-almost every where i.e. $M\{t : f_n(t) \not\to f(t)\} = 0$ a.s.

ii) For every $A \in S$, the sequence $\left\{\int_A f_n\, dM\right\}$ converges in probability as $n \to \infty$.

If f is M-integrable, then we put

$$\int_A f dM = P - \lim_{n\to\infty} \int_A f_n dM$$

We note that $\int_A f\, dM$ is well-defined i.e. it does not depend on the approximating sequence $\{f_n\}$.

The study of Wiener-type integral $\int f\, dM$ of non-random functions with respect to a random measure M under various hypotheses on the random measure M has a long history (e.g. Urbanik, Woczynski [34] Urbanik [33] Schilder [22] Rajput and Rama-Murthy [14]). The most general results obtained by Rajput and Rosinski [15] were concerned with a systemmatic study of the case where M is an arbitrary infinitely divisible random measure i.e. for each $A \in S$ $M(A)$ is a infinitely divisible. The main results of [15] are the following:

a) To give a characterization of the space of M-integrable functions. It was proved that for every $A \in S$, the characteristic function of $M(A)$ can be written in the Levy form

$$\Phi_A(t) = \exp\left\{it\nu_0(A) - \frac{1}{2}t^2\nu_1(A) + \int_{\mathbf{R}} (e^{itx} - 1 - it\tau(x))F(A, dx)\right\},$$

where $\nu_0 : S \to \mathbf{R}$ is a signed-measure, $\nu_1 : S \to [0, \infty)$ is a non-negative measure, $F(dt, dx)$ is an σ-finite measure on $S \times \mathcal{B}$. The measues ν_0, ν_1 and F are called the deterministic characteristics of M. A necessary and sufficient condition for a function to be M-integrable is given in terms of these deterministic characteristics.

b) The identification of the space of M-integrable functions as a certain Musielak-Orlicz space.

c) This stochastic integral is used to obtain the spectral representation for arbitrary discrete parameter infinitely divisible processes as well as for continuous parameter infinitely divisible processes, which are separable in probability. Namely, for any infinitely divisible process X one can choose a random measure M and a family of non-random functions $\{f_t\}$ such that $X_t \simeq \left\{ \int f_t \, dM \right\}$ i.e. X_t and the process $\int f_t \, dM$ have the same finite-dimensional distributions. This spectral representation, when specialized to stable and semistable processes yields, in a unified way, all known spectral representations for these processes.

Rosinski [16, 17] extended the above definition to the case in which f takes values in a Banach space B. He obtained the characterization of M-integrable functions in the case M is generated by a single symmetric infinitely divisible i.e. for each A, the characterisric function (ch.f.) of $M(A)$ is of the form

$$\Phi_(A)(t) = \exp\left\{ -\lambda(A)\left[t^2\sigma^2 + \int_{\mathbf{R}} (1 - \cos tx)\, dm(x)\right]\right\},$$

where m is a Levy measure on $\mathbf{R}$.

Theorem (Rosinski [16]). *Let B be a Banach space. A function $f : S \to B$ is M-integrable if and only if*

i) For each $a \in X'$

$$\int_S K[(f(t), a)]\, d\lambda(t) < \infty ,$$

where $K(t) = \sigma^2 t^2 + \int_{\mathbf{R}} (1 - \cos tx) dm(x)$;

ii) The function

$$\Psi(a) = \exp\left\{ - \int_S K[(f(t), a)]\, d\lambda(t)\right\}$$

is the ch.f. of a Radon measure on B. In this case, $\Psi(a)$ is the ch.f. of $\int_S f\, dM$.

Equivalently, f is M-integrable if and only if

i) The function

$$Q(a) = \int_S |(f(t), a)|^2 d\lambda(t)$$

is a covariance Gaussian function;

ii) The measure μ on B defined by

$$\mu(A) = (\lambda \times m)\{(t, v) : f(t)v \in A \backslash 0\}$$

is a Levy measure on B.

The Rosinski's theorem includes the earlier results on stochastic integrals of Banach space-valued functions w.r.t. p-stable random measures obtained before by Hoffman-Jorgensent [4], Okazaki [11].

Vector random measures arise naturally as a Banach space generalization of random measures. Let X be a Banach space. A mapping $F :\to\to L_0^X(\Omega)$ is called a X-valued random measure if for every sequence $\{A_n\}$ of disjoint sets in S, the X -valued r.v's $F(A_n)$ are independent and we have

$$F(\bigcup_n A_n) = \sum_n F(A_n),$$

where the series is assumed to converges a.s. in the norm topology of X. F is said to be symmetric if $F(A)$ is a symmetric random variable for every $A \in S$. A finite positive measure μ on (S, S) is called a control measure for F if $F(A) = 0$ a.s. whenever $\mu(A) = 0$. Vector random measures can be also regarded as a randomization of (non-random) vector measure studied by many authors. From the latter point of view, one question arises naturally: How to extend the basis theorems of the theory of non-random vector measures to the random context. The following theorems are random analogues of the Pettis theorem and the Vitali-Hahn- Saeks theorem.

Theorem (Thang [25]). *Let F be a X-valued symmetric random measure on $(S, \mathcal{S})$ and μ be a control measure for F. Then F is μ-continuous, i.e.*

$$P - \lim_{\mu(E)\to 0} F(E) = 0.$$

Theorem (Thang [25]). *Let $\{F_n\}$ be a sequence of X-valued symmetric random measures such that*

$$P - \lim_n F_n(E) = F(E)$$

exists for each $E \in \mathcal{S}$. Suppose that there exists a control measure μ for every F_n. Then the mapping $E \to F(E)$ is also a X-valued symmetric random measure with the control measure μ.

Two X-valued random measures F and G are said to be a modification of each other if two X-valued stochastic processes $F = \{F(E)\}$ and $G = \{G(E)\}$, $E \in \mathcal{S}$ have the same finite dimensional distributions. In this case we write $F \simeq G$.

Theorem (Thang [25]). *Let $\{F_n\}$ $(n \geq 0)$ be X-valued symmetric random measures with the same control measure μ such that for each $E \in \mathcal{S}$ the distribution of $F_n(E)$ converges weakly to the distribution of $F_0(E)$. Under a slight assumption, we can find X-valued random measures $\{G_n\}$ such that $G_n \simeq F_n$ for each $n \geq 0$ and*

$$P - \lim_n F_n(E) = F_0(E) \quad \text{for each} \quad E \in \mathcal{S}.$$

It still remains an open problem to examine the validity of a random analogue of the Random-Nykodym theorem. Let M be a real-valued random measure. If $f : S \to X$ is a M-integrable function in the definition of Rosinski then it is not difficult to show that the mapping F defined by

$$F(E) = \int_E f \, dM$$

is a X-valued random measure, which is dominated by M in the sense that: If (E_n) is a sequence in $\mathcal{S}$ such that $P - \lim_n M(E_n) = 0$ then $P - \lim_n F(E_n) = 0$. Now question is: Given a X-valued random measure

F and a real-valued random measure M such that F is dominated by M. Suppose that the Banach space X has the R-N property. Is there a M-integrable function $f : S \to X$ such that the X-valued random measure G defined by

$$G(E) = \int_E f\, dM$$

is a modification of F?

By a analogous manner as in the case of real-valued random measures, we can define the stochastic integral of real-valued functions w.r.t. vector random measures.

Definition. Let F be a X-valued random measure with the control measure μ.

a. Let $f = \sum_{i=1}^{n} t_i 1_{E_i}$ be a real simple function on S. Then for every $A \in S$ we define

$$\int_A f dF = \sum_{i=1}^{n} t_i F(E_i \cap A).$$

b. A measurable function f is said to be F-integral if there exists a sequence (f_n) of simple functions such that $f_n \to f$ μ - a.s. and for every A , the sequence $\left\{ \int_A f_n dF \right\}$ converges in probability.

In this case we put

$$\int_A f dF = P - \lim_n \int_A f_n dF.$$

By using the random version of the Vitali-Hahn-Saeks theorem we can prove that the definition is well-defined i.e. $\int_A f\, dF$ does not depend on the approximating sequence $\{f_n\}$.

The following results give the condition for passing to the limit under the stochastic integral sign and the characterization of F-integrable functions.

Theorem (Thang [26]). *Let F be a X-valued symmetric random measure with the control measure μ. Suppose that $\{f_n\}$ is a sequence of real-valued F-integrable functions such that $\lim_n f_n(t) = f(t)$ for μ-almost surely. Then the equality*

$$P - \lim \int_S f_n \, dF = \int_S f \, dF$$

holds if and only if

$$\lim_{\mu(E)\to 0} \sup_n P\Big\{\Big\| \int_E f_n \, dF \Big\| > \varepsilon\Big\} = 0.$$

In particular, we can pass to the limit under the stochastic integral sign if the sequence (f_n) is dominated by a F-integrable function.

It should be noted that the dominated convergence theorem does not hold for the stochastic integral of Banach space-valued function w.r.t. a real random measure.

The following theorem provides a necessary and sufficient condition for the existence of $\int f \, dF$ in terms of certain parameters of F.

Theorem (Thang [26]). *Let F be a X-valued random measure such that the ch.f. of $F(A)$ is given by*

$$\Phi_A(a) = \exp\Big\{ - Q(A,a) + \int_X (\cos(x,a) - 1) H(A, dx) \Big\},$$

where $Q(A,a)$ is a positive measure on S for each $a \in X'$ and $H(dt, dx)$ is a σ-finite measure on $\sigma(S) \times \mathcal{B}(\mathbf{R})$. Then a function $f : S \to \mathbf{R}$ is F-integrable if and only if

i) For each $a \in X'$

$$g(a) = \int_S |f(t)|^2 Q(dt, a) + \int_S \int_X \{1 - \cos(xf(t), a)\} H(dt, dx) < \infty$$

ii) The function $\exp\{-g(a)\}$ *is the ch.f. of a probability measure on* X. *In this case,* $\exp\{-g(a)\}$ *is the ch.f. of* $\int_S f\,dF$.

The conditions in definition (b) can be restated in an following equivalent form which is more useful in applications.

A function $f : S \to \mathbf{R}$ *is F-integrable if and only if*

a. *The function*

$$\psi(a) = \int_S |f(t)|^2 Q(dt, a)$$

is a Gaussian covariance function.

b. *The measure* μ *on* X *given by*

$$\mu(B) = H\Big\{(t,x) \in S \times X : f(t)x \in B \setminus 0\Big\}$$

is a Levy measure on X.

4. ITO STOCHASTIC INTEGRAL AND THE PREDICTABLE STOCHASTIC INTEGRAL

In 1944, in order to provide a powerful method for the explicit construction of the paths of diffusion processes, Ito [5] introduced a very important generalization of the Wiener stochastic integral by obmiting the restriction that the integrand was a deterministic function. Consider the class of random functions $u = u(t)$ satisfying

i) $P\Big\{\int_0^1 u^2(t,\omega)dt < \infty\Big\} = 1$

ii) u is adapted w.r.t. the Wiener process i.e. for $s \le t$, $u(s)$ is independent of the increaments $u(v) - u(t)$ for $v > t$.

Ito's idea for constructing the stochastic integral $\int_0^1 udW$ is the following. First, if u is adapted simple random function of the form

$$\sum_{i=0}^{n-1} \alpha_i 1_{[t_i, t_{i+1})} ,$$

where α_i is $\mathcal{F}_{t_i}$ - measurable random variable for every $i = 0,..,n$ then $\int_0^1 u\,dW$ is defined by

$$\int_0^1 u\,dW = \sum_{i=0}^{n-1} \alpha_i [W(t_{i+1}) - W(t_i)].$$

By the independence of α_i and $W(t_{i+1}) - W(t_i)$ we get

$$\mathbf{E}\left|\int_0^1 u\,dW\right|^2 = \int_\Omega \int_0^1 |u|^2 dt\,dP.$$

Denote by $\mathcal{L}_2$ the Hilbert space of adapted random functions u such that $\int_\Omega \int_0^1 |u|^2 dt dP < \infty$. It is shown that the set of simple adapted random functions is dense in $\mathcal{L}_2$. Because the mapping $u \to \int_0^1 udW$ is a linear isometry from the dense subspace of $\mathcal{L}_2$ it can be extended to a linear isometry from $\mathcal{L}_2$ into $L_2(\Omega)$. For $u \in \mathcal{L}_2$, we define $\int_0^1 udW$ as the image under this mapping. Finally, the extension to the case where u is an adapted random function such that $\int_0^1 |u(t)|^2\,dt < \infty$ a.s. is achieved by using the so-called technique of localization.

The Ito stochastic integral can be defined as the limit in probability of Riemann integral sums with the left end-points approximation points i.e. the sums of the form

$$\sum_{i=0}^{n-1} u_{t_i} [W(t_{i+1}) - W(t_i)]$$

under the condition that $\max(t_{i+1} - t_i)$ tends to zero. If we choose the mid-point approximation points i.e. the sums are of the form

$$\sum_{i=0}^{n-1} u_{s_i}[W(t_{i+1}) - W(t_i)]$$

where $s_i = \dfrac{t_{i+1} + t_i}{2}$, we get the so-called Stratonovich stochastic integral (see [21]).

The Ito stochastic integral is uselful in analysis because the process

$\left\{ \int_0^t u_s dW,\ 0 \le t \le 1 \right\}$ is a martingale and various good estimates are available. It does not, however, behave so nicely under a transformation and is subject to a strange calculus. For example, if f is a smooth function then we have the Ito formula

$$f(W_t) = f(W_0) + \int_0^t f'(W_s)dW_s + \frac{1}{2}\int_0^t f''(W_s)\,ds,$$

often written instead in the differential form

$$d(f(W_t)) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt.$$

The Stratonovich stochastic integral has the advantage of leading to ordinary chain rule formulas under a transformation, i.e. there are no second order terms in the Stratonovich analogue of the Ito formula. This property makes the Stratonovich integral natural to be used in connection with stochastic differential equations on manifolds (see Ikeda and Watanabe [6]). However, the indefinite Stratonovich integral is not martingale, so it does not give good estimates.

Equipped with the notion of the Ito stochastic integral one can consider stochastic differential equations. For example, given smooth functions A, B with bounded derivatives and a random starting point x_0 find a process X_t satisfying

$$dX_t = A(X_t)dW_t + B(X_t)dt, \quad X_0 = x_0.$$

This is a shorthand for the integral equation

$$X_t = x_0 + \int_0^t A(X_s)\, dW_s + \int_0^t B(X_s)\, ds.$$

It is shown that the solution X_t is a strong Markov process even a diffusion. Hence, stochastic differential equations provide an effective mean of constructing diffusions with given infinitesimal generators.

The Ito stochastic integral for the Wiener process is insufficient for applications as well as for mathematical questions. A more general stochastic integral in which the integrator M is a semimartingale has been developed. We provide here an outline of several stages in the definition of the stochastic integral of the form $\int_0^\infty X\, dM$ only in the case where M is a right continuous (cadlag) local L_2-martingale and X is a process satisfying certain conditions about measurability and integrability.

i) The definition of predictable sets and predictable processes: The family of subsets of $\mathbf{R}^+ \times \Omega$ containing all sets of the form $\{0\} \times F_0$ and $(s,t] \times F$ where $F_0 \in \mathcal{F}_0$ and $F \in \mathcal{F}_s$ for $s < t$ is called the class of predictable rectangles and we denote it by $\mathcal{R}$. The σ-field $\mathcal{P}$ of subsets of $\mathbf{R}^+ \times \Omega$ generated by $\mathcal{R}$ is called the predictable σ-field and sets in $\mathcal{P}$ are called predictable sets. A process X considered as a function on $\mathbf{R}^+ \times \Omega$ is called predictable if it is $\mathcal{P}$-measurable. It can be shown that any predictable process is adapted and any left continuous adapted process is predictable.

ii) Measure on the predictable sets: Let $M = \{M(t), t \in \mathbf{R}^+\}$ be a cadlag L_2-martingale. Define a set function Z_M on $\mathcal{R}$ by

$$Z_M\big((s,t] \times F\big) = \mathbf{E}\big\{1_F[M(t) - M(s)]^2\big\}.$$

Then by the assumption that M is a cadlag L_2-martingale, Z_M can be extended to a measure on $\mathcal{P}$ which is denoted by μ_M and it is called the Dolean measure of M.

iii) Let $\mathcal{E}$ denote the class of all $\mathcal{R}$-simple fuctions. If $X \in \mathcal{E}$ is of the form

$$X = \sum_{i=1}^n c_i 1_{(s_i,t_i] \times F_i}$$

then $\int\limits_{\mathbf{R}^+} X\,dM$ is defined by

$$\int\limits_{\mathbf{R}^+} X\,dM = \sum_{i=1}^{n} c_i 1_{F_i}[M(t_i) - M(s_i)].$$

It can be shown that the following isometry holds

$$\mathbf{E}\Big[\int\limits_{\mathbf{R}^+} X\,dM\Big]^2 = \int\limits_{\mathbf{R}^+}\int\limits_{\Omega} |X|^2\,d\mu_M.$$

iv) This isometry is used to extend the definition of $\int\limits_{\mathbf{R}^+} X\,dM$ to any random process X belonging the space $\mathcal{L}_2 = L_2(\mathbf{R}^+ \times \Omega, \mathcal{P}, \mu_M)$ since the set of $\mathcal{R}$-simple random functions is dense in $\mathcal{L}_2$.

v) Finally, the extension to the case when M is a cadlag local L_2-martingale and X is "locally" in $\mathcal{L}_2$ is achieved by using a sequence of optional times tending to infinity.

Brooks and Dunculean [1] extended the stochastic integral for processes with valued in Banach spaces. Let E, F and G be Banach spaces, X be a process with valued in $E \in L(F,G)$ and H be a process with values in F. Suppose that X is cadlag, adapted and $\mathbf{E}\|X(t)\|^p < \infty$ for every t. Define a set function I_X from $\mathcal{R}$ into $L^p_E(\Omega)$ by

$$I_X\{(s,t] \times F\} = 1_F[X(t) - X(s)].$$

The process X is called summable if I_X can be extended to a $L^p_E(\Omega)$-valued measure with finite semivariation on $\mathcal{P}$. In this case, the stochastic integral $\int H\,dX$ is defined as the bilinear vector integral of H with respect to the vector measure with finite semivariation I_X. The summable processes play in this theory the role played by L_2-martingales in the classical theory. It turns out that every Hilbert space-valued L_2-martingale is summable but for any infinitely dimensional Banach space E there exists a E-valued summable process which is not even a semimartingale.

Thang [29] constructed the stochastic integral $\int_0^1 u\,dZ_p$ in which the integrator Z_p is a vector p-stable random measure taking values in a sufficiently smoothable Banach space X. The procedure for constructing this type of stochastic integral is following: A random function $\{u = u_t, 0 \le t \le 1\}$ is said to be simple adapted (w.r.t. Z_p) if there exists a finite partition $0 = t_0 < ... < t_n = 1$ and the random variables α_i $(i = 0, ..., n)$ such that α_i is $\mathcal{F}_{t_i}$-measurable and

$$u_t = \sum_{i=0}^{n-1} \alpha_i 1_{[t_i, t_{i+1})},$$

where $\mathcal{F}_t$ denotes the σ-algebra generated by the X-valued random variables
$\{Z_p(A),\ A \in [0,t]\}$. The stochastic integral of such a simple adapted u is defined as

$$\int_0^1 u\,dZ_p = \sum_{i=0}^{n-1} \alpha_i Z_p([t_i, t_{i+1})).$$

We associate to Z_p a non-negative measure $|Q_p|$ called the control measure of Z_p. A random function u is said to belong to the class $\mathcal{V}(Z_p)$ if there exists a sequence (u_n) of simple adapted random function such that $u_n \in L_p(|Q_p| \times P)$ and u_n converges to u in $L_p(|Q_p| \times P)$. Notice that when $|Q_p|$ is continuous, $\mathcal{V}(Z_p)$ is precisely the class of adapted random functions in $L_p(|Q_p| \times P)$. If $|Q_p|$ is any measure with $|Q_p|\{0\} = 0$ than the class $\mathcal{V}(Z_p)$ is still large enough to contain all the predictable random functions in $L_p(|Q_p| \times P)$. Under the assumption that the Banach space X is q-smoothable, where $q > p$ if $p < 2$ and $p = 2$ if $q = 2$, by using the Assoud-Pisier inequality for martingale differences taking values in smoothable Banach spaces, it is shown that the mapping $u \to \int_0^1 u\,dZ_p$ is a linear continuous operator from the set of simple adapted random functions into the space $L_0^X(\Omega)$. Hence it admits an extension to the whole space $\mathcal{V}(Z_p)$. For $u \in \mathcal{V}(Z_p)$, $\int_0^1 u\,dZ_p$ is defined as the image of u under this mapping.

5. THE NON-ADAPTED STOCHASTIC INTEGRALS

The measurability conditions which prescribes that the integrand should be independent of future increament of the Wiener process is a very restrictive one. Whereas it is a natural condition in many situations, where the filtration represents the evolution of the available information, it is in many cases a limitation both for developing the theory as well as in application of stochastic calculus. Because in applications the random function to be integrated is not always adapted (or non anticipating) there arises the need to weaken the adaptedness requirement for the integrand of Ito stochastic integrals. Different definitions of the stochastic integral of a non-adapted with respect to the Wiener process have been proposed by several authors. Below we briefly mention some kinds of non-adapted stochastic integrals and the relationship between them. For more details we refer the readers to [9].

Let u_t be a Borel measurable random function such that

$$\int_0^1 u_t^2 dt < \infty \quad \text{a.s.}$$

and let $\pi = \{0 = t_0 < \dots < t_n = 1\}$ denote a partition of the interval $[0, 1]$.

1) The smoothed Stratonovich integral (see [9,10]):

The random function $u = u_t$ is said to be smoothed Stratonovich integrable if the integral sum of the form

$$S_\pi = \sum_{i=0}^{n-1} \overline{u_i}(W(t_{i+1}) - W(t_i)),$$

where

$$\overline{u_i} = \frac{\int_{t_i}^{t_{i+1}} u_s\, ds}{t_{i+1} - t_i}$$

converges in probability as the gauge $|\pi| = \max(t_{i+1} - t_i)$ tends to zero and moreover if the limit does not depend on the choice of the sequence of partitions whose gauge tends to zero. When u is smoothed Stratonovich integrable we denote by $\int_0^1 u_t o dW$ the above limit.

2) The Ogawa integral (see [10]):

A random function $u = u_t$ is said to be Ogawa integral if for any orthonormal system (e_i) in $L_2[0,1]$ the series

$$\sum_{i=1}^{\infty}\left(\int_0^1 u_t e_i(t)\,dt\right)\int_0^1 e_i(t)dW_t$$

converges in probability and if the limit does not depends on the choice of the particular basis (e_i). When u is Ogawa integrable we denote by $\int_0^1 u_t * dW$ the sum of the above series.

It was shown [2] that if for some $p > 2$,

$$\sum_{i=1}^{\infty}\left(\mathbf{E}|u_i|^p\right)^{1/p} < \infty,$$

where $u_k = \int_0^1 e_k(t)u_t\,dt$ and (e_i) is any continuous, uniformly bounded orthonormal base then u_t is Ogawa integrable. Moreover, Nualart and Zakai [10] proved that the existence of the Ogawa integral implies that the smoothed Stratonovich integral exists and these two integrals are equal.

3) The Pardoux-Protter two-sided integral [13]:

Let $\mathcal{F}_t$ denote the σ-algebra generated by $\{W_s, 0 \le s \le t\}$ and $\mathcal{F}^t$ denote the σ-algebra generated by $\{W_1 - W_s, t \le s \le 1\}$. In the filtering theory, there arises the need to evaluate integrals with respect to a Wiener process of random functions of the form

$$u_t = \Phi(t, X_t, Y^t),$$

where X_t is a $\mathcal{F}_t$-adapted random function and Y^t is a $\mathcal{F}^t$-adapted random function. The stochastic integral of a random function of the above form is defined as the limit in probability of the integral sum of the form

$$S_\pi = \sum_{i=0}^{n-1} \Phi(t_i, X_{t_i}, Y^{t_{i+1}})[W(t_{i+1}) - W(t_i)]$$

as the gauge of the partition $|\pi| = \max(t_{i+1} - t_i)$ tends to zero and the limit does not depend on the choice of the sequence of partitions whose gauge tends to zero.

4) The Skorokhod integral [19]:

The most general non-adapted stochastic integral is the Skorokhod integral. Let (T, S, μ) be a measurable space with a finite measure μ and M be a Gaussian symmetric random measure on with the control measure μ. Suppose that $u : T \times \Omega \to \mathbf{R}$ be a random function having finite second moment $\mathbf{E} \int_T |u(t,\omega)|^2 \, d\mu < \infty$. For each fixed $t \in T$ the random variable u_t can be represented as a sum of multiple Ito integral

$$u_t = g_0 + \sum_{k=1}^{\infty} \int_{T^k} g_k(t, t_1, .., t_k) M(dt_1)...M(dt_k).$$

For each $k \geq 0$ the non-random function g_k belongs to $L_2(T^{k+1}, \mu^{k+1})$ and is symmetric with respect to the last k arguments. We denote by $\tilde{g}_k$ the symmetrization of g_k with respect to all $k+1$ arguments. Then the Skorokhod integral of u with respect to M is defined by the equality

$$\delta(u) = \sum_{k=0}^{\infty} \int_{T^{k+1}} \tilde{g}_k(t_1, ..., t_{k+1}) M(dt_1)...M(dt_{k+1}),$$

provided that the series on the right-hand side converges in $L_2(\Omega)$. We note that if u is adapted then the Ito integral and the Skorokhod integral coincide; If $u_t = \Phi(t, X_t, Y^t)$ is Skorokhod integrable then it is also integrable in the sense of Pardoux-Protter and both integrals coincide. Under a slight assumption, the existence of the Skorokhod integral implies that the Ogawa integral exists. In addition if u is Skorokhod integrable then it is also Stratonovich integrable. However, two integrals are different in general. For example, if X_t is a continuous adapted semimartigale and $f \in C^1(\mathbf{R})$ then $u_t = f(X_t)$ is Skorokhod integrable and

$$\int_0^1 u_t o\, dW = \delta(u) + \frac{1}{2}\int_0^1 f'(X_t)d\langle X, W\rangle_t.$$

6. RANDOM OPERATORS : A NATURAL FRAMEWORK OF STOCHASTIC INTEGRALS

Let X and Y be two Fréchet spaces. By a random mapping from X into Y we mean a mapping from X into $L_0^Y(\Omega)$. We may think of the mapping from X into Y as an action which transforms each input $x \in X$ into an output $Ax \in Y$. It might happen that Ax is not completely known but subject to some random noise so that we can only hope to be able to say about the probability distribution of the output. In other words, instead of considering Ax as an element of Y we have to think of it as an Y-valued random variable.

A random linear mapping from X into Y is called a random linear operator. Mathematically, by a random operator (we omit the word "linear" since we only consider random linear operators from now on) we mean a linear continuous mapping from X into $L_0^Y(\Omega)$. Random series and stochastic integrals are most important examples of random operators.

Example. 1. Let (ξ_i) be a sequence of real-valued i.i.d. Gaussian random variables and H be a Hilbert space. It was known that if (x_n) is a sequence in H such that $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$ then the series $\sum_{n=1}^{\infty} x_n \xi_n$ converges a.s. in the norm topology of H. Define a mapping from $\ell_2(H)$ into $L_0^H(\Omega)$ by

$$Ax = \sum_{i=1}^{\infty} x_i \xi_i \quad \text{ if } x = (x_n) \in \ell_2(H).$$

we get a random operator from $\ell_2(H)$ into H.

2. Let M be a infinitely divisible random measure. It was shown [15] that the mapping $f \to \int_S f\, dM$ is linear continuous from the space

$\mathcal{L}(M)$, which is a certain Musielak-Orlicz space, into $L_0(\Omega)$. Consequently the stochastic integral mapping is a random operator.

3. Let $K(t,s)$ be a function defined on the square $[0,1]^2$. Defined a mapping on $C[0,1]$ by

$$Ax(t) = \int_0^1 K(t,s)x(s)dW(s).$$

It can be shown that this mapping is a random operator from $C[0,1]$ into $L_2[0,1]$ and it is called a random integral operator with the kernel $K(t,s)$.

Consequently, random operators can be considered as a natural framework for stochastic integrals. In other words, stochastic integrals are prototypes of random operators. Moreover, it is shown that every symmetric p-stable random operators has a representation of the form of a stochastic integral or a random series.

Theorem (Thang [24]). *Suppose that A is a symmetric Gaussian random operator from X into Y. Then there exist a sequence (ξ_n) of real-valued i.i.d. Gaussian random variables and a sequence (B_n) of non-random linear operators from X into Y such that for each $x \in X$*

$$Ax = \sum_{n=1}^{\infty} \xi_n B_n x,$$

where the series is convergent a.s. in the norm topology of Y.

Theorem (Thang[27]). *Suppose that A is a symmetric p-stable random measure $(p < 2)$. Then there exist an p-stable random measure M on some measurable space $(S, \mathcal{S}, \mu)$ and a linear continuous mapping from X into the space $\mathcal{L}_Y(M)$ of Y-valued M-integrable functions such that*

$$Ax \simeq \int_S Gx(t)dM(t).$$

Under the original definition, a random operator with the domain X can not be applied to X-valued random variable. Taking into account many circumstances in which the inputs are also subject to the influence of a random enviroment, there arises the need to give a reasonable meaning to the action of the random operator on some X-valued random variables. Mathematically, given a random operator A with domain X, the problem is to extend the domain of A to some class $\mathcal{V}$ of X-valued random variables. Of course, different procedures may be proposed but the aim will be that the class $\mathcal{V}$ must be as wide as possible and at the same time the extension of A should enjoy many good properties similar to those of A. This problem is also motivated by the following consideration. Let A be a random operator from $L_2[0,1]$ into $\mathbf{R}$ defined by the Wiener stochastic integral

$$Ax = \int_0^1 x_t \, dW \quad \text{if} \quad x \in L_2[0,1].$$

For a measurable random function $u = u_t$ with sample paths in $L_2[0,1]$, as we have seen in Section 4, the different definitions of stochastic integral $\int_0^1 u_t \, dW$ have been proposed. Now if u is a random variable with values in $L_2[0,1]$ we can define the action of A on u as the stochastic integral $\int_0^1 u_t \, dW$ if it exists in some sense. The problem of defining the stochastic integral of a random function with respect to a Wiener process turns out to be equivalent to that of extending the domain of the random operator generated by the Wiener stochastic integral.

It should be noted that it is not always possible to define the extension of A by direct substitution $\tilde{A}u(\omega) = A\big(u(\omega)\big)(\omega)$. Indeed, for each $x \in X$, Ax is defined on some set D_x of probability one, but then $\tilde{A}u(\omega)$ is defined (by direct substitution) only on the set $\bigcap D_{u(\omega)}$, which can be empty.

In [30] a reasonably large class $\mathcal{V}$ of X-valued random variables was introduced on which the extension of a random operator A with domain X is defined in a natural way. In the case where A is the random operator generated by the Wiener integral, it is shown that if u is Skorokhod integrable then by choosing a suitable approximating sequence (u_n) of X-valued r.v.'s in the class $\mathcal{V}$ the sequence $\tilde{A}u_n$ converges to the Skorokhod stochastic integral of u. Motivated by the notion of Ogawa integral, another procedure of extension was proposed in [30] for the case $X = \ell_s (1 \le s < \infty)$. Namely, an X-valued random variable u is said to be A-applicable if the series $\sum_{n=1}^{\infty} u_n A e_n$ converges in $L_0^Y(\Omega)$, where (e_n) is the standard basis in l_s and u_n stands for the n-th coordinate of u. It is not difficult to show that if the random operator A viewed as a Y-valued random field indexed by the parameter set X admits a modification with sample paths in the space $L(X,Y)$ of linear continuous operators then each X-valued random variable is A-applicable. (For various conditions ensuring the existence of a modification of A whose sample paths belong to $L(X,Y)$ we refer the readers to Thang [28]). One of the main results of [30] is the following

Theorem (Thang [30]). *1. Let Y be a Hilbert space and the random variables (Ae_n) are independent. Then each X-valued random variable u such that u_n is $\mathcal{F}_{n-1}$-measurable for every $n > 1$ is A-applicable. Here $\mathcal{F}_n$ is the σ-algebra generated by $Ae_1, \ldots, Ae_n$.*

2. Let A be a symmetric p-stable random operator and Y be a q-smoothable Banach space, where $q = 2$ if $p = 2$ and $q > p$ if $p < 2$. Suppose that the random variables (Ae_n) are independent. Then each X-valued random variable u such that u_n is $\mathcal{F}_{n-1}$-measurable for each $n > 1$ and $\sum_{n=1}^{\infty} |u_n|^p < \infty$ a.s. is A-applicable.

Other topics of the theory of random operators in Hilbert spaces can be found in [20].

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Department of Mathematics,
University of Hanoi,
90 Nguyen Trai,
Dong da, Hanoi

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