Vietnam Journal of Mathematics
Volume 23, Number 1, 1995

A Short Communication

MULTIDIMENSIONAL QUANTIZATION AND THE DEGENERATE PRINCIPAL SERIES REPRESENTEATIONS *

DO NGOC DIEP

Let G be a connected Lie group, $\mathbf{g}$ its Lie algebra, $\mathbf{g}_{\mathbb{C}}$ the complexification of $\mathbf{g}$. For an element Z of $\mathbf{g}_{\mathbb{C}}$ we denote its conjugate by $\overline{Z}$. If $\mathbf{a}$ is a sub-set of $\mathbf{g}_{\mathbb{C}}$ we pose

$$\mathbf{a} := \{\overline{Z}; Z \in \mathbf{a}\}.$$

Let $\mathbf{g}^*$ be the dual space of Lie algebra $\mathbf{g}$, $\mathcal{O}(G)$ the orbit space of G, $\Omega \in \mathcal{O}(G)$ a K-orbit, $F \in \Omega$ a fixed point on Ω, which is admissible in the sense of Duflo [1], G_F the stabilizator of the point F, $\tilde{\sigma}\chi_F$ an irreducible unitary representation of G_F, the restriction of which on the connected component $(G_F)_0$ is a multiple of χ_F, where

$$\chi_F(\exp(.)) := \exp\left(\frac{i}{\hbar}\langle F, .\rangle\right)$$

and $\hbar := \dfrac{h}{2\pi}$ is the normalized Planck constant. Since the value of $\hbar$ does not play any role in the results of this note we may normalize $\hbar$ so that it has almost everywhere the value 1. The admissibility of F guarantees the existence of such a representation $\tilde{\sigma}\chi_F$.

In [2] we have introduced the notion of polarization as follows. We say that $(\mathbf{p}, \rho, \sigma_0)$ is a $(\tilde{\sigma}, F)$ - polarization, iff:

(a) $\mathbf{p}$ is a complex Lie subalgebra of $\mathbf{g}_{\mathbb{C}}$, containing $(\mathbf{g}_F)_{\mathbb{C}}$.

---

* This work was supported in part by the National Basic Research Program in Natural Science of Vietnam and by the Alexander von Humboldt-Stiftung of Germany

(b) The subalgebra $\mathfrak{p}$ is invariant with respect to the operators $Ad_{\mathfrak{g}_\mathbb{C}}x$, $x \in G_F$.

(c) The vector space $\mathfrak{p}+\overline{\mathfrak{p}}$ is the complexification of a real subalgebra $\mathfrak{m}$, that is $\mathfrak{m} = (\mathfrak{p}+\overline{\mathfrak{p}}) \cap \mathfrak{g}$.

(d) All the groups M_0, H_0, M, H are closed in G, where M_0 (resp., H_0) is the connected subgroup of G, of Lie algebra $\mathfrak{m}$ (resp., $\mathfrak{h} := \mathfrak{p} \cap \mathfrak{g}$) and $M := G_F \times M_0$, $H := G_F \times H_0$.

(e) σ_0 is an irreducible representation of the group H_0 in a Hilbert space V, such that: (e_1) the restriction $\sigma_0|_{G_F \cap H_0}$ is a multiple of the restriction to $G_F \cap H_0$ of $\tilde{\sigma}\chi_F$ and (e_2) the point σ_0 is fixed under the action of group G_F in the dual $\hat{H}_0$ of the group H_0

(f) ρ is a representation of the complex Lie algebra $\mathfrak{p}$ in V, which satisfies all the E. Nelson's condition for H_0, and $\rho|_{\mathfrak{h}} = d\sigma_0$.

This notion is equivalent (see [3]) to the notion tangent G-distribution L, which is

(a') integrable.

(b') AdG_F - invariant

(c') complexly integrable, i.e. $L+\overline{L}$ is integrable,

(d') closed.

(e') weakly Lagrangian.

(f') complexly extended (see [3] for details).

Theorem 1. *The polarization $(\mathfrak{p}, \rho, \sigma_0)$ with $\dim \sigma_0 = 1$, or equivalently, the corresponding to L tangent G distribution, is being Lagrangian, is maximal if and only if*

$$\sigma_0 = \chi_F, \qquad \mathrm{codim}_{\mathfrak{g}}\mathfrak{h} = \frac{1}{2}\dim \Omega_F$$

and $\mathfrak{h}$ is subordinate to the functional F, $\langle F, [\mathfrak{h},\mathfrak{h}]\rangle \equiv 0$.

In other words, $\mathfrak{h}$ is a polarization in the sense of M. Duflo.

The idea to prove is to consider the quotient $(G_F \cap H_0)\backslash H_0$, which is a group according to the condition (b). Therefore σ_0 corresponds to a projective representation of dimension 1 of $(G_F \cap H_0)\backslash H_0$, which is trivial, in passing, if necessary, to the 2-fold covering. One has therefore

$\sigma_0 = \chi_F$ and $\langle F, [\mathbf{h}, \mathbf{h}]\rangle \equiv 0$. In the case of a Lagrangian G-distribution (L, ρ, σ_0), one has obviously $\langle F, [\mathbf{h}, \mathbf{h}]\rangle \equiv 0$ and χ_F is a representation of H_0, which is extended to σ_0. By the irreductibility of σ_0, one has $\sigma_0 = \chi_F$. The argument that $\mathrm{codim}_{\mathbf{g}}\mathbf{h} = \frac{1}{2}\dim \Omega_F$ is classical one, see, for example [4]

The construction of Duflo [1] proposes a reduction to the smallest dimension and, in the last step, a particular definition,

$$T_{F,\tilde{\sigma}\chi_F} := \pi\left(\delta^F \tilde{\sigma}\chi_F, \frac{i}{\hbar}\langle F, .\rangle\big|_{\mathbf{h}}\right)$$

for the reductive connected groups. This representation is defined, according to Duflo, by a construction of Harish-Chandra. We remark that these representations can be also obtained by the procedure of multidimensional quantization in considering the polarizations $(\mathbf{p}, \rho, \sigma_0)$

Theorem 2. *For the reductive groups, the representations of the principal series of Harish - Chandra can be also obtained by the procedure of multidimensional quantization.*

In fact, for $F \in \mathbf{g}^*$ which is good polarized [1] and admidssible, and for $\tau \in \chi_G(F)$, the function δ^F over $G_F^{\mathbf{g}}$ is a character and $(\tau\delta^F, \lambda)$ (with $\lambda := \frac{i}{\hbar}F\big|_{\mathbf{h}}$, $\mathbf{h} = \mathbf{g}_F$, $H = G_F$) is a pseudo - character in the sense of Vogan and

$$\pi(\tau\delta^F, \lambda) = \mathrm{Ind}_{FM_0N}^G(\tau\delta^F \otimes \pi^{M_0}),$$

where

$$\pi^{M_0} \in (\widehat{M_0})_{\mathrm{disc(mod\ cent}(M_0))},$$
$$G_F = H, \quad (G_F)_0 = G_F \cap H_0, \quad H = FH_0.$$

Following the method of Harish - Chandra, the representation π^{M_0} is characterized as follows. Let

$$\mathbf{t}_m := \left(\mathbf{h}_c + \sum_{\alpha\in\Delta_{m,c}} \mathbf{g}^\alpha\right) \cap \mathbf{g}$$

and K_{M_0} be the corresponding analytic subgroup, then π^{M_0} is defined by the restriction $\pi^{M_0}\big|_{K_{M_0}}$, which is a multiple of the representation of K_{M_0} with the dominant weight $\lambda + \delta^\lambda$, with respect to the subset $\Delta^+_{\mathfrak{m},c}$ of compact roots, i.e.

$$\pi^{M_0}\big|_{H_0} = \text{mult}(\chi_F.\delta^F)\big|_{H_0}.$$

We have the restriction $\tau\delta^F\big|_{G_F\cap M_0} = \tau\delta^F\big|_{H_0} = \text{mult}(\delta^F.\chi_F)\big|_{H_0}$. Then $\tau\delta^F \otimes \pi^{M_0}$ is trivial on the kernel $G_F \cap M_0 = H_0$ of the projection

$$FH_0 \times M_0 \to FH_0M_0,$$

and it induces a representation, denoted by the same symbol $\tau\delta^F \otimes \pi^{M_0}$, of FM_0. Finally, $((\tau\delta^F \otimes \pi^{M_0}) \otimes Id_N)$ gives us a (τ, F) - polarization of the orbit Ω_F and

$$\text{Ind}^G_{FM_0\times N}((\tau\delta^F \otimes \pi^{M_0}) \otimes Id_N)$$

is obtained by the procedure of multidimensional quantification.

When the reductive groupe G is not connected, one considers the extension of π^{E_0} to a representation $S(\tilde{x})\pi^{E_0}(y)$ of $G^{\mathfrak{g}}_F \times E_0$. Also, one considers then

$$\pi(xy) := \tau'(\tilde{x}) \otimes S(\tilde{x})\pi^{E_0}(y),$$

where $\tilde{x}$ is a preimage of x with respect to the two-fold covering, τ' is an odd representation of the two-fold covering $G^{\mathfrak{g}}_F$; and finally one induces to have $\text{Ind}^G_E(\pi \otimes Id_N)$. This is also the representation obtained from the procedure of multidimensional quantification.

Corollary. *The procedure of multidimensional quantization gives us a complete geometric illustration of Duflo's construction.*

Duflo's construction proposes an extension of the character χ_F of $(G_F)_0$ to an irreducible representation τ of $G_F \times H_0$ or, if necessary to its two-fold covering $G^{\mathfrak{g}}_F \times H_0$ which is a multiple of χ_F on restricting to $(G_F)_0$ and to H_0. Actually,

$$\tau \in \widehat{G^{\mathfrak{g}}_F}.$$

$$\tau\big|_{(G^{\mathfrak{g}}_F)_0} = \text{mult}\, \chi^{\mathfrak{g}}_F,$$

$$\chi_F^{\mathfrak{g}}(e) = -1,$$

where

$$1 \to \{1, e\} \to G_F^{\mathfrak{g}} \to G_F \to 1,$$

τ corresponds bijectively to a projective representation $\tilde{\sigma} \in \widehat{(G_F)_0 \setminus G_F}$.

The group $(G_F)_0 \setminus G_F$ is discrete and it was proved in [1], that $T_{F,\tau}$ is of type I if and only if $\dim \tau$ is finite, and the orbit Ω_F is locally closed.

The procedure of multidimensional quantization proposes not only an extension of $\chi_F|_{(G_F)_0}$ to a representation τ of G_F, but also an extension of $\chi_F|_{G_F \cap H_0}$ to an irreducible representation σ_0 of H_0, an extension to a representation $S\sigma_0$ of $G_F \times H_0$ and finally tensor product $\tau \otimes S\sigma_0$. We have therefore a similar result:

Theorem 3. *The representation* $\mathrm{Ind}(G; \mathfrak{p}, \rho, \sigma_0)$ *obtained by the procedure of multidimensional quantization is of type I if and only if* σ_0 *is of type I,* $\dim \tau < \infty$ *and the orbit* Ω_F *is locally closed.*

The author would like to thank the Alexander von Humbold - Stiftung for the fellowship grant and the Fakultät für Mathematik and Sonderforschungbereich 343 der Universität Bielefeld, Germany, for the hospitality and the provided perfect condition of work, and especially Professor Anthony Bak for his genereous encouragement.

REFERENCES

1. M. Duflo, Proc. Conf. "Lie group representations III" Lecture Notes in Math., Vol. 1077 (1984) 101-165, Springer - Verlag Berlin - Heidelberg - New York.
2. Do Ngoc Diep, *Construction des représentations unitaires par les K-orbites et quantification*, C. R. Acad. Sci. Paris, 291, Série A (1980), 295-298.
3. Do Ngoc Diep, *Quantification des systèmes Hamiltoniens à l'action plate d'un groupe de Lie*, C. R. Acad. Sci. Paris, 295, Série I (1982), 345 - 348.
4. A. Kirillov, *Elements of the theory of representations*, Springer - Verlag, Berlin - Heidelberg - New York, 1976.

Institute of Mathematics
P.O.Box 631, Bo Ho,
Hanoi, Vietnam

Received July 6, 1994