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A SURVEY OF REGULARITY CONDITIONS AND THE SIMPLICITY OF PRIME FACTOR RINGS

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Abstract. *In this paper we survey the research, past and present, which investigates the relations between various generalizations of von Neumann regularity and the condition that all prime factor rings of a ring are simple (equivalently, every proper prime ideal is maximal).*

1. INTRODUCTION

Throughout all rings are associative, but may not necessarily have a unity. All prime ideals are assumed to be proper and $P(R)$ denotes the prime radical of a ring R. A ring is said to satisfy ***pm*** if every prime ideal is maximal.

The following problem has attracted some interest over a period of at least twenty-five years:

What are the connections between the ***pm*** *condition and various generalizations of von Neumann regularity?*

In this paper we will discuss this problem from its origins to recent results. Since this paper is expository in nature we will emphasize the historical development of the problem through theorems and examples, however few proofs will be included. The proofs can be found in the referenced literature.

We will need the following definitions throughout our discussion. Let R be a ring. Then R is said to be:

(i) *regular* if $x \in xRx$, for all $x \in R$;

(ii) π*-regular* if for every $x \in R$ there exists a positive integer n, depending on x, such that $x^n \in x^nRx^n$;

(iii) *strongly π-regular* if for every $x \in R$ there exists a positive integer n, depending on x, such that $x^n \in x^{n+1}R$;

(iv) *biregular* if $< x >= eR$ for all $x \in R$, where $< x >$ denotes the ideal in R generated by x and e is a central idempotent;

(v) *right (left) weakly regular* if $x \in x < x > (x \in < x > x)$ for all $x \in R$;

(vi) *right (left) weakly π-regular* if for every $x \in R$ there exists a positive integer n, depending on x, such that $x^n \in x^n < x^n >$ $(x^n \in < x^n > x^n)$.

An excellent reference for regular rings is [14]. Any strongly π-regular ring is π-regular. In [11] the strongly π-regular condition is shown to be left-right symmetric. Also, any ring with d.c.c. on principal right (left) ideals is strongly regular. Any simple ring with unity or any commutative regular ring is biregular. Right (left) weakly regular rings were introduced in [19] and are also called right (left) fully idempotent. Note if a commutative ring is right (or left) weakly regular, then it is regular. All biregular rings are left and right weakly regular. Recently, Andruszkiewicz and Puczylowski [1] have shown that right weakly regular rings need *not* be left weakly regular. Right (left) weakly π-regular rings were introduced in [15].

It can be shown that the class of ***pm*** rings is closed under homomorphisms, ideals, and direct sums [8]. Although every nonzero prime ideal is maximal in the integers, this ring does not satisfy ***pm*** since zero is a prime ideal. However, a straightforward argument yields that every ring which is biregular or has d.c.c. on right ideals satisfies ***pm***.

2. EARLY RESULTS

In 1968, the first clearly established equivalence between ***pm*** and regularity seems to have been made by Storrer [22] in the following result.

Theorem 2.1. *If R is a commutative ring with unity, then the following conditions are equivalent:*

(i) *R is π-regular;*

(ii) *$R/P(R)$ is regular;*

(iii) *R satisfies **pm**.*

Now there are two ways of generalizing the commutative condition: (i) one can consider extending the properties which depend on the commutative identity to the class of polynomial identity, PI, rings; or (ii) one can attempt to extend the structural properties of the ideals of a commutative ring. In 1974, Fisher and Snider [12], generalized Storrer's result to PI rings with the following theorem.

Theorem 2.2. *Let R be a PI ring. Then, the following are equivalent:*

(i) *R is π-regular;*

(ii) *each prime ideal of R is primitive;*

(iii) *R satisfies **pm**;*

(iv) *R is left (right) π-regular;*

(v) *$R/P(R)$ is π-regular;*

(vi) *each prime factor ring of R is regular.*

Recall a ring is *left (right) duo* if every left (right) ideal is an ideal. A ring is *duo* if it is both left and right duo. In 1977, using the second approach to generalizing commutativity, Chandran [10] extended Storrer's result to duo rings with the next theorem.

Theorem 2.3. *Let R be a duo ring with unity. Then R is π-regular if and only if R satisfies **pm**.*

Within a year, Hirano [17] had extended Chandran's result to one-sided duo rings not necessarily having a unity. He obtained the following result as a corollary of a more general theorem.

Theorem 2.4. *If R is a right (or left) duo, then the following are equivalent:*

(i) *R is strongly π-regular;*

(ii) *R is π-regular;*

(iii) *R is right weakly π-regular;*

(iv) *J is nil and R/J is π-regular, where J is the Jacobson radical of R;*

(v) *R/I is π-regular for some nil ideal I;*

(vi) *$R/P(R)$ is strongly regular;*

(vii) *R satisfies* **pm**;

(viii) *every completely prime ideal of R is maximal;*

(ix) *$(R)_n$, the full n-by-n matrix ring over R, is strongly π-regular $(n = 1, 2, \ldots)$.*

In Hirano's paper [17], he also considered the condition when the prime ideals of a ring are maximal one-sided ideals.

In 1985, Yao [24] called a ring R *weakly right duo* if for every $a \in R$ there is a natural number n, depending on a, such that the right ideal generated by a^n is a two-sided ideal. A ring R is called a *bounded weakly right duo* if there is a number M such that $n \leq M$ for all $a \in R$. He shows by examples that these definitions are nontrivial generalizations of the definition of a right duo ring. He proves the following results.

Theorem 2.5. *Let R be a weakly right duo ring with unity. If R satisfies* **pm**, *then R is π-regular.*

Theorem 2.6. *A bounded weakly right duo ring with unity is π-regular if and only if R satisfies* **pm**.

3. RECENT RESULTS

In 1991, Belluce [6, Theorem 7] defined a condition (qc) for rings with unity which is equivalent to biregularity when the ring is semiprime and satisfies **pm**. He uses this result to prove the following theorem. (Recall a reduced ring is one with no nonzero nilpotent elements).

Theorem 3.1. *Let R be a reduced ring with unity. Assume R/P is a simple ring for every minimal prime ideal P. Then R is biregular.*

The following corollary is an immediate consequence of Theorem 3.1.

Corollary 3.2. *Let R be a reduced ring with unity. Then R satisfies* **pm** *if and only if R is biregular.*

In 1992, Beidar and Wisbauer, apparently unaware of Belluce's paper, defined properly semiprime rings with unity [4]. Using this concept they announced in [3] the following result as a corollary to a more general theorem.

Theorem 3.3. *Let R be a reduced ring with unity. Then the following are equivalent:*

(i) R *is biregular;*

(ii) R *is left weakly regular;*

(iii) R *satisfies* **pm**.

The proof of Theorem 3.3 and other related results appear in [5].

Also in 1993, Birkenmeier, Kim, and Park [7], unaware of Belluce's paper [6] or the announcement of Beidar and Wisbauer, proved the following results with methods different from those of [5] or [6].

Theorem 3.4. *Let R be a reduced ring with unity. Then the following are equivalent:*

(i) R *is weakly regular;*

(ii) R *is right weakly π-regular;*

(iii) R *satisfies* **pm**;

(iv) *every prime factor ring of R is a simple domain.*

Note that Theorem 3.4 generalizes the well known result that when R is a reduced ring with unity then R is regular if and only if every prime factor ring is a division ring.

Corollary 3.5. *Let R be a ring with unity such that $\mathbf{P}(R)$ equals the set of nilpotent elements of R. Then the following are equivalent:*

(i) $R/\mathbf{P}(R)$ *is weakly regular;*

(ii) $R/\mathbf{P}(R)$ *is right weakly π-regular;*

(iii) R *satisfies* **pm**.

A routine argument shows that Storrer's result, Theorem 2.1, is a special case of Corollary 3.5.

In comparing the recent results with the earlier ones, we observe that a simple domain with unity which is not a division ring (e.g., a Weyl algebra over a field of characteristic zero [20]) is biregular and hence ***pm***. This fact is not accounted for in the earlier results since such a ring is neither PI, nor weakly right (or left) duo, nor π-regular. However simple domains with unity provide examples for Corollary 3.2, Theorem 3.3, and Theorem 3.4.

Furthermore the earlier results seem to emphasize the π-regular condition whereas the recent results emphasize the weakly regular condition.

The following examples will give some indication of the limits of the current theory.

Example 3.6. [12, Example 1]. Let R consist of all sequences of 2-by-2 matrices over a field which are eventually strictly upper triangular. This ring is semiprime and it satisfies ***pm***, but R is not regular.

Example 3.7. The ring of endomorphisms over a countably infinite dimensional vector space provides a ring which is regular but does not satisfy ***pm***.

Example 3.8. [6, p.1865]. Let R be the ring of all sequences of 2-by-2 matrices over a field which are eventually diagonal. Then R is regular and satisfies ***pm***, but R is not biregular.

The next example is a generalization of [7, Example 12]. It shows that the regularity properties of $R/\boldsymbol{P}(R)$ in Corollary 3.5 cannot necessarily be lifted to R.

Example 3.9. Let W be a simple domain with unity which is not a division ring. Let R be the 2-by-2 upper triangular matrix ring over W. Clearly, $\boldsymbol{P}(R)$ is the set of nilpotent elements of R and $R/\boldsymbol{P}(R) \simeq W \oplus W$ is a biregular (hence weakly regular) ring. We claim that R is neither left nor right weakly π-regular. To see this let xW be a nonzero proper right ideal of W and assume R is right weakly π-regular. Then there exists n such that

$$\begin{bmatrix} x & 1 \\ 0 & 0 \end{bmatrix}^n \in \begin{bmatrix} x & 1 \\ 0 & 0 \end{bmatrix}^n R \begin{bmatrix} x & 1 \\ 0 & 0 \end{bmatrix}^n R.$$

Observe

$$\begin{bmatrix} x & 1 \\ 0 & 0 \end{bmatrix}^n = \begin{bmatrix} x^n & x^{n-1} \\ 0 & 0 \end{bmatrix}, \text{ and}$$

$$\begin{bmatrix} x^n & x^{n-1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \begin{bmatrix} x^n & x^{n-1} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u & v \\ 0 & w \end{bmatrix} \in \begin{bmatrix} x^nWx^nW & x^nWx^{n-1}W \\ 0 & 0 \end{bmatrix},$$

for $a, b, c, u, v, w \in W$ and $n > 1$. Therefore

$$\begin{bmatrix} x & 1 \\ 0 & 0 \end{bmatrix} \in \begin{bmatrix} x^nWx^nW & x^nWx^{n-1}W \\ 0 & 0 \end{bmatrix}.$$

Hence $x^{n-1} = x^n\alpha$, where $\alpha \in Wx^{n-1}W$. So $x^{n-1}(1 - x\alpha) = 0$. Then $1 \in xW$, a contradiction! Therefore R is not right weakly π-regular. Similarly, R is not left weakly π-regular.

Example 3.6 shows that if a ring R is semiprime and it satisfies ***pm*** it is not necessarily regular. However, Corollary 3.5 gives some support that such a ring may be one-sided weakly π-regular. Our next example, which is a generalization of [7, Example 13], shows that this, in general, is not the case.

Example 3.10. Let W be a simple domain with unity which is not a division ring. Let R be the ring of all sequences of 2-by-2 matrices over W which are eventually constant upper triangular. A routine argument shows that R is semiprime. The proof that R satisfies ***pm*** is the same as that given in [7, Example 13].

To see that R is neither right nor left weakly π-regular, let $s \in R$ such that the first component is $\begin{bmatrix} x & 1 \\ 0 & 0 \end{bmatrix}$, where xW is a nonzero proper right ideal of R, and zero in all other components of s. Then as in the previous example $s^m \notin s^mRs^mR$, for any positive integer m. Similarly, R is not left weakly π-regular.

In spite of the above examples, the work of Belluce and Beidar and Wisbauer on semiprime rings suggests that further results are possible. The author with Jin Yong Kim and Jae K. Park have decided to attempt to extend the current results to rings which are not necessarily semiprime. Corollary 3.5 and Example 3.9 suggest that one needs to find conditions which are stronger than the condition that the prime radical equals the set of nilpotent elements. The following result is a corollary of our main result in [8, Theorem 3.4].

Theorem 3.11. *Let R be a ring with unity such that $\boldsymbol{P}(R)$ equals the set of nilpotent elements of R, and every idempotent element of R is central. Then the following are equivalent:*

(i) R *is weakly π-regular;*

(ii) R *is right weakly π-regular;*

(iii) $R/\boldsymbol{P}(R)$ *is biregular;*

(iv) $R/\boldsymbol{P}(R)$ *is right weakly π-regular;*

(v) R *satisfies* ***pm**;*

(vi) *every prime factor ring of R is a simple domain;*

(vii) *for each prime ideal P of R, $P = \{a \in R \mid a^n b = 0$ for some positive integer n and some $b \in R \backslash P\}$;*

(viii) *for each $a \in R$ there exists a positive integer m such that $R = Ra^m R + rt.annih(a^m)$.*

Our last example is a ring R with unity in which $\boldsymbol{P}(R)$ equals the set of nilpotent elements, every idempotent is central, and it satisfies **pm**; but it is not π-regular and is neither right nor left weakly regular. However, by Theorem 3.11, it is weakly π-regular.

Example 3.12 [8]. Let W be a simple domain with unity which is not a division ring, and let

$$R = \left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \middle| a, b \in W \right\}.$$

Clearly, $\boldsymbol{P}(R)$ equals the set of nilpotent elements of R; and since 0 and 1 are the only idempotents of R, all idempotents are central.

Claim 1: R satisfies **pm**, since $\boldsymbol{P}(R)$ is the unique maximal ideal of R.

Claim 2: R is *not* π-regular.

Proof. Let $a \neq 0$ and $a \neq 1$. Then

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}^n \notin \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}^n R \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}^n, \quad \text{for any positive integer } n.$$

Claim 3: R is neither right nor left weakly regular, since $[\boldsymbol{P}(R)]^2 = 0$.

In [21], Shin introduces the class of almost symmetric rings. This class provides interesting examples of rings in which the prime radical equals the set of nilpotents and every idempotent is central. The above example is shown to be almost symmetric in [8].

Finally we note that Sun in [23] has mentioned the problem of:

$$\text{biregular} = (\boldsymbol{pm})+?$$

He has also looked at conditions similar to the ***pm*** condition. Armendariz has announced some results in [2] which concern the basic problem discussed in this paper. Menal has considered π-regular rings whose primitive factor rings are artinian in [18]. Yue Chi Ming has written extensively on regular rings and his recent paper [25] has some interesting results on biregular rings. Recently, Camillo and Xiao have obtained a result [9, Theorem 6] which extends Theorems 3.3 and 3.4. In turn, Theorem 3.11 extends their result [9, Theorem 6]. Finally, the author has recently found that in [16] Gupta introduced the concept of a s-weakly regular ring and used it to obtain connections between weak regularity and the ***pm*** condition.

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