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ON BALAYAGE PRINCIPLES BY INVERSE SOURCE PROBLEMS*

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Abstract. *The paper establishes some balayage principles (or sweeping-out principles) in the sense of distributions by using results of inverse source problems for partial differential operators with constant coefficients. Examples for the Laplace operator, the heat conduction operator, the Helmholtz operators, and the wave operator are given.*

1. INTRODUCTION

The balayage principle in the potential theory (see [1], [3], [6], [9]) says that for a measure ν located in a closed domain $G \subset R^3$ there exists a so-called swept-out measure μ distributed on the boundary ∂G, generating the same potential as ν exterior to $\overline{G}$, i.e.

$$E_N * \nu(x) = E_N * \mu(x) \quad \forall x \in R^3 \setminus \overline{G},$$

where $E_N(x) = 1/4\pi|x|$ is the Newtonian kernel, and $*$ is the convolution.

Some authors [3], [9] used this property to introduce the following inverse source problem to study the solution set $L(\mu)$ (or information content $L(\mu)$ [3]) of all such measures (sources) ν provided that μ is known on the boundary ∂G. For the case of Laplace operator, the solution set $L(\mu)$ is defined by:

$$L(\mu) := \left\{ \nu \in C'(\overline{G}) : E_N * \nu(x) = E_N * \mu(x) \quad \forall x \in G_1 := R^3 \setminus \overline{G} \right\}.$$

The present paper generalizes the same idea for a class of partial differential operators with constant coefficients to establish some balayage principles in the sense of distributions. We shall use the notations of [4], [9], [10], [11], [12] for distributions.

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2. THE DIRECT-INVERSE METHOD

Let

$$P := \sum_{|\alpha| \le m} a_\alpha D^\alpha, \quad a_\alpha = \text{const}$$

be a partial differential operator in R^n of order m with constant coefficients, where $\alpha = (\alpha, \dots, \alpha_1)$ are multi-indices, and E be some fundamental solution of P, according to certain physical meaning. Consider a domain $G \subset R^n$, not necessarily bounded, with its boundary ∂G and closure $\overline{G}$. Denote by $G_1 := R^n \setminus \overline{G}$ the complement of $\overline{G}$. For any subset $A \subset R^n$ we denote by $\mathcal{D}'(R^n, A)$ the set of all distributions ν of $\mathcal{D}'(R^n)$ with support on A, supp $\nu \subseteq A$. In order to pose the inverse source problem correctly, it is necessary to impose certain restrictions on $\mathcal{D}'(R^n, \overline{G})$ so that the convolution $E * f$, $f \in \mathcal{D}'(R^n, \overline{G})$, has a sense in $\mathcal{D}'(R^n)$, i.e. it exists. Introducing the set $\mathbf{E}(\overline{G})$ of all such distributions:

$$\mathbf{E}(\overline{G}) := \Big\{ \nu \in \mathcal{D}'(R^n, \overline{G}) : E * \nu \quad \text{exists in} \quad \mathcal{D}'(R^n) \Big\}$$

we have a well-defined transformation

$$T_*^E : \mathbf{E}(\overline{G}) \to \mathcal{D}'(G_1),$$

$$\nu \mapsto E * \nu\big|_{G_1},$$

where $E * \nu\big|_{G_1}$ is the restriction of $E * \nu$ on the open set G_1. It is an easy direct problem to study properties of $E * \nu\big|_{G_1}$, $\nu \in \mathbf{E}(\overline{G})$. It has the general property:

$$P\big(E * \nu\big|_{G_1}\big) = 0 \quad \text{in} \quad \mathcal{D}'(G_1).$$

Thus we have the reason to introduce the set H of all such functions (distributions) having the same property:

$$H := \Big\{ v \in \mathcal{D}'(G_1) : Pv(x) = 0 \quad \text{in} \quad \mathcal{D}'(G_1) \Big\}. \tag{1}$$

We also can write $Pv(x) = 0$, $x \in G_1$, instead of $Pv(x) = 0$ in $\mathcal{D}'(G_1)$ (see [11], [12]).

Definition 1. The inverse source problem with respect to some given distribution $v \in H$ is to find a source $\mu \in \mathcal{D}'(R^n, \overline{G})$ satisfying the condition

$$E * \mu(x) = v(x), \quad \forall x \in G_1,$$

provided that the convolution $E * \mu$ exists.

Let $L(v)$ denote the set of all solutions of the above problem

$$L(v) := \Big\{ f \in \mathbf{E}(\overline{G}) : E * f(x) = v(x), \quad \forall x \in G_1 \Big\}.$$

By definition, $\mathcal{D}'(R^n, \overline{G}_1) \subset \mathcal{D}'(R^n) \subset \mathcal{D}'(G_1)$. In view of (1), that is $H \subset \mathcal{D}'(G_1)$, we diagnose whether the inverse source problem with respect to $v \in H$ is solvable if H is reduced to $\mathcal{D}'(R^n, \overline{G}_1)$.

Theorem 1. *Assume $v \in H \cap \mathcal{D}'(R^n, \overline{G}_1)$ such that the convolution $E * v$ exists. Then there exists a solution $\nu = Pv \in L(v) \cap \mathcal{D}'(R^n, \partial G)$.*

Proof. Since $v \in H$, by definition we have $\operatorname{supp} Pv \subseteq R^n \setminus G_1 = \overline{G}$. By assumption $v \in \mathcal{D}'(R^n, \overline{G}_1)$ we obtain $\operatorname{supp} Pv \subseteq \overline{G}_1$. So we get $\operatorname{supp} Pv \subseteq \overline{G} \cap \overline{G}_1 = \partial G$ or $Pv \in \mathcal{D}'(R^n, \partial G)$. On the other hand, from the existence of the convolution $E * v$ it follows that the convolutions $E * Pv$ and $(PE) * v$ exist, and

$$\begin{aligned} P[E * v(x)] = E * Pv(x) = (PE) * v(x) = \\ = \delta(x) * v(x) = v(x), \quad \forall x \in R^n. \end{aligned}$$

In particular, we have $E * Pv(x) = v(x), \ \forall x \in G_1$, or $Pv \in L(v) \cap \mathcal{D}'(R^n, \partial G)$, completing the proof.

We are going to use this theorem to obtain some balayage principle. It is well known [12] that for each set $A \subset R^n$ and a number $\varepsilon > 0$ there exists the function $\eta_{A,\varepsilon} \in C^\infty(R^n)$ with

$$\begin{aligned} \eta_{A,\varepsilon}(x) = 1, \quad \forall x \in A, \\ \eta_{A,\varepsilon}(x) = 0, \quad \forall x \overline{\in} A_\varepsilon, \end{aligned}$$

where

$$A_\varepsilon = \Big\{ x \in R^n : |x - y| < \varepsilon, \quad \forall y \in A \Big\}.$$

Using this function we give the following.

Definition 2. Let $f \in \mathcal{D}'(R^n)$ and A be a closed set in R^n. If there exists a distribution $g \in \mathcal{D}'(R^n)$ such that for each sequence $\{\eta_{A,\varepsilon}\}$, $\varepsilon > 0$, the following condition holds

$$g = \lim_{\varepsilon \to 0} f\eta_{A,\varepsilon} \text{ in } \mathcal{D}'(R^n)$$

or

$$(g, \varphi) = \lim_{\varepsilon \to 0} (f\eta_{A,\varepsilon}, \varphi) \text{ in } R^1, \quad \forall \varphi \in \mathcal{D}(R^n),$$

then we call the distribution g the restriction of f on the closed set A and denote it by $f\big|_A$.

The above definition is correct, i.e, it is well defined if it exists. Indeed,if there exists an another restriction g' of f on A, then

$$(g, \varphi) = \lim_{\varepsilon \to 0} (f\eta_{A,\varepsilon}, \varphi) = (g', \varphi) \quad \forall \varphi \in \mathcal{D}(R^n)$$

or $g = g'$ in $\mathcal{D}'(R^n)$.

The notation $v \in H \cap \mathcal{D}'(R^n, \overline{G})$ also means that $v \in \mathcal{D}'(G_1)$ is extendable in $\mathcal{D}'(R^n)$ to the whole $\overline{G}_1$. For each continuous function the corresponding distribution has the restriction on each closed set as by a usual function. It is not difficult to show that if the restriction $f\big|_A$ exists, then it belongs to $\mathcal{D}'(R^n, A)$.

Theorem 2 (Sweeping-Out Principle). *Let $\nu \in \mathbf{E}(\overline{G})$ and suppose that there exists the restriction of $E * \nu$ on $\overline{G}_1$. Moreover, assume the existence of the convolution $E * (E * \nu\big|_{\overline{G}_1})$. Then there exists a swept-out distribution $\nu' \in \mathcal{D}'(R^n, \partial G)$ such that*

$$E * \nu(x) = E * \nu'(x), \quad \forall x \in G_1.$$

The above theorem follows directly from Theorem 1 since $E * \nu\big|_{G_1} \in H$, $E * \nu\big|_{\overline{G}_1} \in \mathcal{D}'(R^n, \overline{G}_1)$, and the convolution $E * (E * \nu\big|_{G_1})$ exists. The swept-out distribution is defined by

$$\nu' = P(E * \nu\big|_{\overline{G}_1}).$$

Examples

a) Laplace operator. For the Laplace operator described in the introduction, the set H (see [1]) is of the form $H = \{u \in \mathcal{D}'(G_1) : \Delta_3 u(x) = 0 \quad \forall x \in G_1\}$. By Theorem 1, for each $v \in H \cap D'(R^3, \overline{G}_1)$ with the existence of $E_N * v$, there exists a solution ν carried by ∂G. From Theorem 2 for each distribution ν of $\mathcal{D}'(R^3, \overline{G})$ with the existence of the restriction $E_N * \nu|_{\overline{G}_1}$ and the convolution $E_N * (E_N * \nu|_{\overline{G}_1})$, there exists a swept-out distribution $\nu' \in \mathcal{D}'(R^3, \partial G)$ such that $E * \nu(x) = E * \nu'(x), \ \forall x \in G_1$.

Under certain additional assumptions on H, one can obtain some existence and uniqueness statements with the corresponding swept-out distributions in form of simple layers and mixed double layers [7]. There can exist several sources having the same swept-out distribution. In [7] an example is given, in which the set of all such sources is of power not less than that of the continuum.

b) Helmholtz operator. For the Helmholtz operator

$$P := \Delta_3 + \lambda, \ \lambda \in R$$

we distinguish two cases

$$P^+ := \Delta_3 + k^2, \quad k > 0,$$
$$P^- := \Delta_3 - k^2, \quad k > 0,$$

and consider the inverse source problem in both cases on the same domain as for the Laplace operator.

Regarding the positive Helmholtz operator P^+ we can take two fundamental solutions (see [11], [12])

$$E(x) = -\frac{e^{ik|x|}}{4\pi|x|}, \quad \text{and} \quad \overline{E}(x) = -\frac{e^{-ik|x|}}{4\pi|x|},$$

and according to the two Sommerfeld emission conditions at infinity, we have

$$S(\infty) := \left\{ u \in C^1(G_1) : \frac{\partial u(x)}{\partial |x|} - iku(x) = 0(|x|^1) \right\},$$
$$\overline{S}(\infty) := \left\{ u \in C^1(G_1) : \frac{\partial u(x)}{\partial |x|} + iku(x) = 0(|x|^{-1}) \right\}.$$

For the above fundamental solutions E and $\overline{E}$ with the corresponding Sommerfeld emission conditions, interested reader may use Theorem 1 and Theorem 2 to deduce inverse existence statements and balayage principles.

c) The heat conduction operator. As an example for the case of unbounded domains we consider the heat conduction operator

$$P_n u(x,t) := \frac{\partial u(x,t)}{\partial t} - \Delta_n u(x,t), \quad n = 1,2,3,\dots$$

with the unbounded strip

$$\overline{G_T} := \Big\{(x,t) \in R^{n+1} : x \in R^n,\ 0 \le t \le T\Big\}, \quad T > 0.$$

The well-known fundamental solution is

$$E_n(x,t) = \frac{\theta(t)}{(4\pi t)^{n/2}} e^{-|x|^2/4t},$$

where $\theta(t)$ is the Heaviside function: $\theta(t) = 1$ for $t \ge 0$, and $\theta(t) = 0$ otherwise. From Theorem 1 for each distribution $v \in H \cap \mathcal{D}'(R^{n+1}, \overline{G}_1)$ such that $E_n * v$ exists in $\mathcal{D}'(R^{n+1})$, there exists a solution $\nu = P_n v$ carried by $R^n \times (t = T)$, where $G_1 = \{(x,t) : x \in R^n,\ t > T\}$ and $H = \{v \in \mathcal{D}'(G_1) : P_n v(x,t) = 0\ \forall t > T\}$. In view of Theorem 2, for each distribution $\nu \in \mathbf{E}(\overline{G_T})$ such that the restriction $E * \nu|_{\overline{G}_1}$ and the convolution $E_n * (E_n * \nu|_{\overline{G}_1})$ exist, there exists a swept-out distribution $\nu' \in \mathcal{D}'(R^{n+1}, R^n \times (t = T))$ so that $E_n * \nu(x,t) = E_n * \nu'(x,t)$, $\forall t > T$. One can, of course, impose additional assumptions on H in order to get inverse and sweeping-out statements.

d) The wave operator. For the wave operator

$$P_n := \frac{\partial^2}{\partial t^2} - a^2 \Delta_n, \quad a > 0, \quad n = 1,2,3,$$

with the same closed strip $\overline{G_T}$ and the domain G_1 as in the proceeding example one can take the fundamental solutions of the form (see [11], [12])

$$E_1(x,t) = \frac{1}{2}\theta(at - |x|), \quad n = 1;$$

$$E_2(x,t) = \frac{\theta(at - |x|)}{2\pi a\sqrt{a^2t^2 - |x|^2}}, \quad n = 2;$$

$$E_3(x,t) = \frac{\theta(t)}{4\pi a^2 t^2}\delta_{S_{at(x)}} \equiv \frac{\theta(t)}{4\pi a}\delta(a^2t^2 - |x|^2), \quad n = 3.$$

The set H is defined as

$$H = \left\{ v \in \mathcal{D}'(G_1) : P_n v(x,t) = 0, \quad \forall t > T \right\}, \quad n = 1,2,3.$$

By Theorem 1 the inverse statement is: For each $v \in H \cap \mathcal{D}'(R^{n+1}, \overline{G}_1)$ with the existence of $E_n * v$, $n = 1,2,3$, there exists a solution ν carried by $R^n \times (t = T)$. This yields the balayager principle (Theorem 2): For each $\nu \in \mathbf{E}(\overline{G}_T)$ with that the restriction $E_n * \nu|_{\overline{G}_1}$ and the convolution $E_n * (E_n * \nu|_{\overline{G}_1})$ exist, there exists a swept-out distribution ν' carried by $R^n \times (t = T)$ so that

$$E_n * \nu(x,t) = E_n * \nu'(x,t), \quad \forall t > T, \quad n = 1,2,3.$$

Again, one can make certain restrictions on H in order to get further inverse statements as well as sweeping-out ones [8].

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