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A Short Communication

ON MINIMAX SOLUTIONS OF FIRST ORDER NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH *t*-MEASURABLE HAMILTONIANS *

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It is well-known that there may not exist a global classical solution for a first-order nonlinear partial differential equation (PDE). In recent years, significant attention has been paid to the study of generalized solutions whose definition is based on replacing the equation by a pair of differential inequalities. A nonclassical theory of Hamilton - Jacobi equations as well as other types of first-order PDEs represents a large portion of research in which the concept of global solutions introduced by Crandall and Lions [2, 3] is used.

Another direction in the theory of generalized solutions is motivated by differential game theory and has been suggested by Subbotin [1, 5]. It leads to the notion of minimax solutions of the Cauchy problem for nonlinear PDEs.

$$\frac{\partial \varphi}{\partial t} + H(t, x, \varphi, \nabla_x \varphi) = 0,$$

where H is a continuous function of its arguments.

Our aim here is to point out that the (global) minimax solutions can be defined for certain PDEs satisfying Carathéodory's conditions. The results in this paper generalize the ones of Adiatullina and Subbotin in [1].

Let us consider the Cauchy problem of the form

$$\frac{\partial \varphi}{\partial t} + H(t, x, \varphi, \nabla_x \varphi) = 0 \quad \text{in} \quad \Omega = (0, T) \times \mathbf{R}^n, \tag{1}$$

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$$\varphi(T,x) = \sigma(x) \quad \text{on} \quad \mathbf{R}^n, \tag{2}$$

where $\nabla_x\varphi := (\partial\varphi/\partial x_1, \ldots, \partial\varphi/\partial x_n)$ denotes the gradient of φ in x. We set $S := \{s \in \mathbf{R}^n : \|s\| = 1\}$, $B := \{s \in \mathbf{R}^n : \|s\| \le 1\}$, where $\|.\|$ denotes the Euclidean norm.

Assume that $\sigma \in C(\mathbf{R}^n)$ and H has the following properties.

a) Carathéodory's conditions:

a1) For almost every (in the sense of Lebesgue measure) fixed $t \in (0,T)$, $H(t,.)$ is (totally) continuos on $\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$.

a2) For every $(x,\eta,s) \in \mathbf{R}^n \times \mathbf{R} \times S$, $H(.,x,\eta,s)$ is measurable on $(0,T)$.

b) For any bounded sets $D \subset \mathbf{R}^n$, $E \subset \mathbf{R}$ there exists a function $\Lambda(.) \in L^1(0,T)$ with

$$|H(t,x,\eta,s) - H(t,y,\eta,s)| \le \Lambda(t)\|x-y\|, \tag{3}$$

for all $x, y \in D$, $\eta \in E$, $s \in S$ and almost all $t \in (0,T)$.

c) There exists a function $k(.) \in L^1(0,T)$ such that

$$\sup\{|H(t,x,\eta,p) - H(t,x,\eta,q)| - \|p-q\|L(t,x) : p, q \in B\} \le 0, \tag{4}$$

for all $(x,\eta) \in \mathbf{R}^n \times \mathbf{R}$ and almost all $t \in (0,T)$, where $L(t,x) := k(t)(1 + \|x\|)$.

d) For all $(x,s) \in \mathbf{R}^n \times S$ and almost all $t \in (0,T)$, the function $H(t,x,.,s)$ is decreasing on $\mathbf{R}$.

e) $H(t,x,\eta,s)$ is positively homogeneous in s, i.e.,

$$H(t,x,\eta,\alpha s) = \alpha H(t,x,\eta,s), \quad \forall \alpha \ge 0, \tag{5}$$

for all $(x,\eta,s) \in \mathbf{R}^n \times \mathbf{R} \times S$ and almost all $t \in (0,T)$.

Note that the conditions a2), c), e) imply the following

a'2) $H(.,x,\eta,s) \in L^1(0,T)$ for all $(x,\eta,s) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$.

Let us denote the Euclidean scalar product in $\mathbf{R}^n$ by $\langle .,. \rangle$ and set

$$\begin{aligned} F(t,x) &:= \sqrt{2}L(t,x) \cdot B, \\ F_U(t,x,\eta,q) &:= \{f \in F(t,x) : \langle f,q \rangle \ge H(t,x,\eta,q)\}, \\ F_L(t,x,\eta,p) &:= \{f \in F(t,x) : \langle f,p \rangle \le H(t,x,\eta,p)\}. \end{aligned} \tag{6}$$

Here $(t,x) \in \Omega$, $p \in P$, $q \in Q$ with such sets P, Q that

$$\{\alpha p : p \in P, \alpha \geq 0\} = \{\alpha q : q \in Q, \alpha \geq 0\} = \mathbf{R}^n.$$

For example, we can take $P = Q = S$. It is easy to see that the nonempty convex compact valued multifunctions $\Omega \ni (t,x) \mapsto F(t,x)$, $\Omega \ni (t,x) \mapsto F_U(t,x,\eta,q)$, $\Omega \ni (t,x) \mapsto F_L(t,x,\eta,p)$ are continuous in x and measurable in t for any $\eta \in \mathbf{R}$, $p \in P$, $q \in Q$. From the monotonicity and continuity in η of H we have

$$\begin{aligned} &F_U(t,x,\eta_1,q) \subset F_U(t,x,\eta_2,q), \\ &F_L(t,x,\eta_2,p) \subset F_L(t,x,\eta_1,p), \\ &\bigcap_{\eta>\eta_1} F_U(t,x,\eta,q) = F_U(t,x,\eta_1,q), \\ &\bigcap_{\eta<\eta_2} F_L(t,x,\eta,p) = F_L(t,x,\eta_2,p), \end{aligned} \tag{7}$$

whenever $\eta_1 \leq \eta_2$. It can also be shown that

$$F_U(t,x,\eta,q) \cap F_L(t,x,\eta,p) \neq \emptyset$$

for all $p \in P$, $q \in Q$ and therefore

$$H(t,x,eta,s) = \sup_{q\in Q} \min_{f\in F_U(t,x,\eta,q)} \langle f,s\rangle = \inf_{p\in P} \max_{f\in F_L(t,x,\eta,p)} \langle f,s\rangle, \tag{8}$$

for all $x,\eta,s) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$ and almost all $t \in (0,T)$.

In the following, by $X_U(t_*,x_*,\eta,q)$ and $X_L(t_*,x_*,\eta,p)$ we denote the sets of all absolutely continuous functions $x(.) : [0,T] \to \mathbf{R}^n$ satisfying almost everywhere in $(0,T)$ the differential inclusions $\dot{x}(t) \in F_U(t,x(t),\eta,q)$ and $\dot{x}(t) \in F_L(t,x(t),\eta,p)$, respectively, subject to the same constraint $x(t_*) = x_*$. From Theorems 5.2, 7.1 in [4], it follows that $X_U(t_*,x_*,\eta,q)$ and $X_L(t_*,x_*,\eta,p)$ are nonempty compact sets in $C([0,T],\mathbf{R}^n)$ for all $(t_*,x_*) \in \overline{\Omega}$.

Definition 1. A supersolution of Prolem (1), (2) is a lower semicontinuos function $\varphi : \overline{\Omega} \to \mathbf{R}$ satisfying for all $0 \leq t < \tau \leq T$; $x \in \mathbf{R}^n$, the conditions

$$\sup_{q\in Q} \min_{x(.)\in X_U(t,x,\varphi(t,x),q)} [\varphi(\tau,x(\tau)) - \varphi(t,x)] \leq 0 \tag{9}$$

$$\varphi(T,x) \geq \sigma(x), \quad \forall x \in \mathbf{R}^n. \tag{10}$$

Definition 2. A subsolution of Problem (1)-(2) is an upper semicontinuous function $\varphi : \overline{\Omega} \to \mathbf{R}$ satisfying for all $0 \leq t < \tau \leq T$; $x \in \mathbf{R}^n$ the conditions

$$\inf_{p \in P} \max_{x(.) \in X_L(t,x,\varphi(t,x),p)} [\varphi(\tau, x(\tau)) - \varphi(t,x)] \leq 0 \tag{11}$$

$$\varphi(T,x) \leq \sigma(x), \quad \forall x \in \mathbf{R}^n. \tag{12}$$

The sets of all supersolutions and subsolutions of (1), (2) will be denoted by Sol_U and Sol_L, repectively.

Definition 3. A function $\varphi \in Sol_U \cap Sol_L$ is called a minimax solution of the Cauchy Problem (1), (2).

In [6, 7] Tran Duc Van proposed the following notion of global generalized solutions.

Definition 4. A function $\varphi \in C(\overline{\Omega})$ locally Lipschitz continuous in Ω is called a global quasi-classical solution of (1)-(2) if for all $x \in \mathbf{R}^n$ and for almost all $t \in (0,T)$, φ is (totally) differentiable and satisfies (1) at the point (t,x), and if (2) is fulfilled.

We are now able to formulate the main results in this paper.

Theorem 1. *Assume a) - e). Then*

i) Every global quasi-classical solution φ of (1)-(2), such that $\varphi(t,.) \in C^1(\mathbf{R}^n)$ for almost all $t \in (0,T)$, is also a minimax solution of the same problem.

ii) There exists a subset $A \subset (0,T)$ of the Lebesgue measure 0 such that at any point $(t,x) \in ((0,T) \setminus A) \times \mathbf{R}^n$, where a minimax solution φ of (1)-(2) is differentiable, the equation (1) must be satisfied.

Theorem 2. *Let $\sigma \in C(\mathbf{R}^n)$ and H satisfy all the conditions a) - e). Then there exists one unique minimax solution for the Cauchy Problem (1), (2).*

Our proofs are based on a sharpening of the Lebegue theorem, which states that for a function $g \in L^1_{loc}(0,T)$, $\lim_{\delta \downarrow 0}(1/\delta) \int_t^{t+\delta} |g(\tau) - g(t)|d\tau = 0$ almost everywhere in $(0,T)$, and on a new version of Gronwall's inequality. They will be published elsewhere.

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