Vietnam Journal of Mathematics
Volume 23, Number 1, 1995

ON GLOBALLY MINIMAL STEINER NETWORKS WITH CONVEX BOUNDARY POINTS ON THE PLANE

NGUYEN HUU QUANG

Abstract. *In this paper we investigated global minimality of Steiner networks on the plane. We prove that if a Steiner network has no growths and branch points then it is globally minimal in the class of networks with the same topological type.*

1. INTRODUCTION

Let C be a class of Steiner networks with given fixed convex boundary points on the plane. The problem of finding a length minimizing network in the class C was studied by some authors (see, for examples, [1], [2]).

In this paper we use the calibration system principle presented by Dao Trong Thi to prove the global minimality of some series of locally minimal Steiner networks on the plane.

2. PRELIMINARIES

A Steiner network on the plane is any connected one - dimensional simplicial complex, whose vertices all have degree at most three. A Steiner network without vertices of degree two is said to be nondegenerate. Henceforth we shall study only a cyclic, nondegenerate Steiner network with a convex boundary consisting of the vertices of degree one.

We recall that every locally minimal Steiner network has the following properties:

(1) The network consists of stright line segments;

(2) At every vertex the segments meet at angles of 120°.

We know that (see [1]) a minimal Steiner network is described as a dual graph of some tree tiling with the twisting number at most five.

2.1. Definition. A vertex of degree three of a given Steiner network N is called a node. We consider a tree P tiling with the dual graph N. A node is called a growth if the corresponding cell is a growth. A node is called a branch point if the corresponding cell is a branch point.

2.2. Definition. A network is said to be oriented if its sides can be oriented so that every two adjacent sides are oriented opposite to each other.

Every Steiner network is oriented.

Suppose that N is an oriented Steiner network on the plane R^2. A path in N joining two boundary points is called a maximal path. A set of maximal paths $\{P_j\}_j$ in N is said to be a basis of maximal paths if $\{P_j\}_j$ satisfies the following conditions:

(1) The union of all paths from $\{P_j\}_j$ overlaps N;

(2) The system $\{P_j\}_j$ is independent;

(3) Every maximal path in N is a combination of paths from $\{P_j\}_j$.

We note that every Steiner network with k boundary points has a basis of maximal paths consisting of $(k-1)$ maximal paths.

2.3. Definition. Let N, N' be the networks in R^2 with the same boundary points $A_1, A_2, \ldots, A_k$. We say that N and N' are of the same topological type if there is a homeomophism $F: R^2 \to R^2$ such that $F(A_i) = A_i$; $i = 1, 2, \ldots, k$ and $F(N) = N'$.

2.4. Calibration system principle. *Let N be an oriented Steiner network with k boundary points on the plane and $\{P_1, P_2, \ldots, P_{k-1}\}$ is the basis of maximal paths in N. Suppose that there is a system of close differential 1-form $\{\omega_1, \omega_2, \ldots, \omega_{k-1}\}$ on the plane such that*

1) $\sum_{j \in J_a} \omega_j(\varepsilon_j(a)\vec{N}_x) = 1, \quad \forall a;$

2) $\left\| \sum_{j \in J_a} \varepsilon_j(a)\omega_j \right\| = 1, \quad \forall a,$

where $\varepsilon_j(a)$ *is the sign of side* a *in the path* P_j, $J_a = (j | a \in P_j)$ *and* $\vec{N}_x$ *is unit tangent vector to* N *at* $x \in a$ *with the same orientation as* a.

Then N *is a length-minimizing network in the class of network with fixed topological type.*

Proof. Let N' be any Steiner network belonging to the topological type of N. Assume that $f : R^2 \to R^2$ is a homemophism such that $f(N) = N'$ and $f(A_i) = A_i$ for each i, where A_i are the vertices of N. The orientation on N' is induced by the orientation on N under f. $\{f(P_1), \dots, f(P_{k-1})\}$ is a basis of the maximal paths in N'. Denote the length of N and N' by $|N|$, $|N'|$, respectively; the length of the sides a and a', by $|a|$, $|a'|$ where $a' = f(a)$. Putting $x' = f(x)$, we have

$$|N| = \sum_{a \in N} |a| = \sum_{a \in N} \int_a \Big(\sum_{j \in J_a} \omega_j(\varepsilon_j(a)\vec{N}_x \Big)$$

$$= \sum_j \int_{P_j} \omega_j = \sum_j \int_{P'_j} \omega_j = \sum_{a'} \int_{a'} \sum_{j \in J_{a'}} \omega_j(\varepsilon_j(a')\ \vec{N}'_{x'})$$

$$\leq \sum_{a'} \Big| \int_{a'} 1 \Big| = |N'|,$$

where $P'_j = f(P_j)$. The proof is completed.

The system $\{\omega_j\}_j$ is called a calibration system on N.

3. RESULTS

3.1. Lemma. *The Steiner network shown in Fig. 1 is globally minimal in the class of networks with fixed topological type.*

Proof. Suppose that the Steiner network N is oriented as in Fig. 2.

Denote the sides of N by, a_1, a_2, a_3. We put $P_1 = a_1 - a_3$ and $P_2 = a_2 - a_3$. Clearly, the system $\{P_1, P_2\}$ is the basis of maximal paths of N.

Let $\vec{a}_1$, $\vec{a}_2$, $\vec{a}_3$ be the unit tangent vectors to N on a_1, a_2, a_3 respectively and a_1^*, a_2^*, a_3^* are the unit co-vectors dual to them. Assume that ω_1, ω_2 are constant differential 1-forms induced by a_1^* and a_2^*, respectively . We have $\omega_i(\vec{N}_x) = 1$ for $x \in a_i$, $i = 1,2$ and $(\omega_1 - \omega_2)(\vec{N}_x) = 1$ for $x \in a_3$ and $\|\omega_1\| = \|\omega_2\| = 1$. By (1.3), N is length-minimizing network.

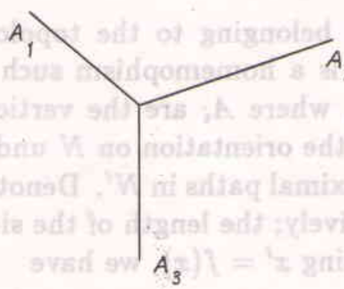

Fig. 1

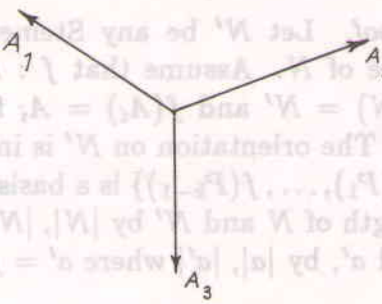

Fig. 2

3.2. Theorem. *A Steiner network N without growths and branch points is globally minimal in the class of the networks with fixed topological type.*

Proof. The Steiner network N is of the form shown in Fig. 3.

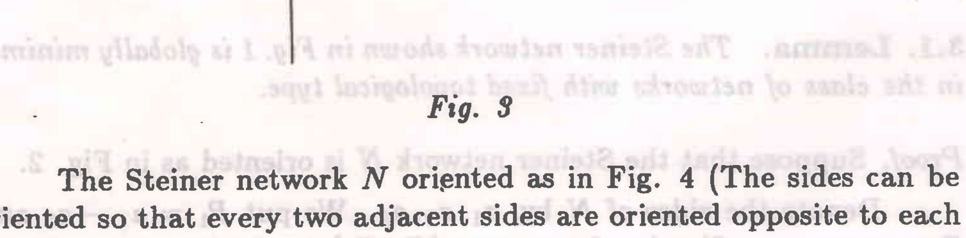

Fig. 3

The Steiner network N oriented as in Fig. 4 (The sides can be oriented so that every two adjacent sides are oriented opposite to each other).

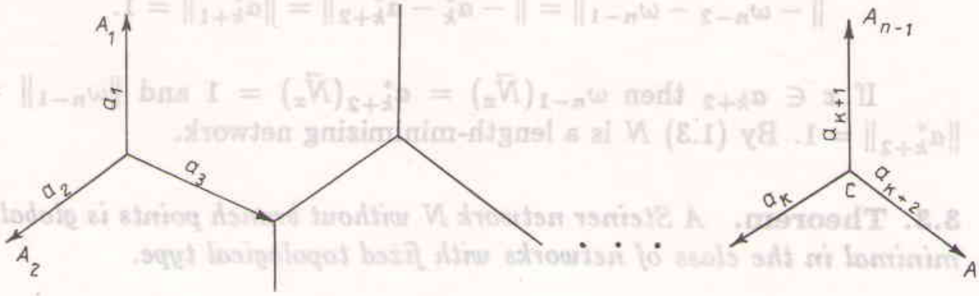

Fig. 4

Denote the boundary points of N by $A_1, \ldots, A_n$. We shall prove the theorem by induction on n. For $n = 3$, by Lemma 2.1, we have the basis of maximal paths $\{P_1, P_2\}$ and the calibration system $\{\omega_1, \omega_2\}$. Suppose that the Steiner network $N_1 \subset N$ with $(n-1)$ boundary points $A_1, \ldots, A_{n-2}, C$ has the basis of maximal paths $\{P_1, \ldots, P_{n-2}\}$. Without loss of generality, we can assume that the side a belongs to only one path P_{n-2} in N_1. We put

$$\overline{P}_i = P_i, \quad i = 1, \ldots, n-3,$$
$$\overline{P}_{n-2} = P_{n-2} - a_{k+1},$$
$$\overline{P}_{n-1} = a_{k+2} - a_{k+1},$$

where a_{k+1}, a_{k+2} are the sides of N joining C with A_{n-1}, A_{n-2}, respectively.

To prove that the system $\{\overline{P}_1, \ldots, \overline{P}_{n-1}\}$ is the basis of minimal paths in N, we need only to check the condition (3) of Definition 1.1. Let $\overline{P}$ be a path joining two boundary point A_i and A_j in N. Here we only consider the case where A_i belongs to N_1 and A_j is the end of a_{k+2}. We suppose that P is the maximal path joining A_i, C in N_1. Then P is the combination of $P_{\alpha_1}, \ldots, P_{\alpha_t}$ from $\{P_1, \ldots, P_{n-2}\}$ and $\overline{P}$ is the combination of $\{\overline{P}_{\alpha_1}, \ldots, \overline{P}_{\alpha_t}, \overline{P}_{n-1}\}$ in N. Now, we construct a calibration system on N.

Denote by ω_{n-1} the 1-form induced by a^*_{k+2}. Here a^*_{k+2} is the co-vector dual to the unit tangent $\vec{a}_{k+2}$ on the side a_{k+2}. Then, the system $\{\omega_1, \ldots, \omega_{n-2}, \omega_{n-1}\}$ is the calibration system on N. Indeed if $x \in a_{k+1}$ then

$$(\omega_{n-2} + \omega_{n-1})(-\vec{N}_x) = (a^*_k + a^*_{k+2})(-\vec{N}_x) = -a^*_{k+1}(-\vec{N}_x) = 1,$$

and

$$\| - \omega_{n-2} - \omega_{n-1}\| = \| - a_k^* - a_{k+2}^*\| = \|a_{k+1}^*\| = 1.$$

If $x \in a_{k+2}$ then $\omega_{n-1}(\vec{N}_x) = a_{k+2}^*(\vec{N}_x) = 1$ and $\|\omega_{n-1}\| = \|a_{k+2}^*\| = 1$. By (1.3) N is a length-minimizing network.

3.3. Theorem. *A Steiner network N without branch points is globally minimal in the class of networks with fixed topological type.*

Proof. The network N is of the form shown in fig. 5.

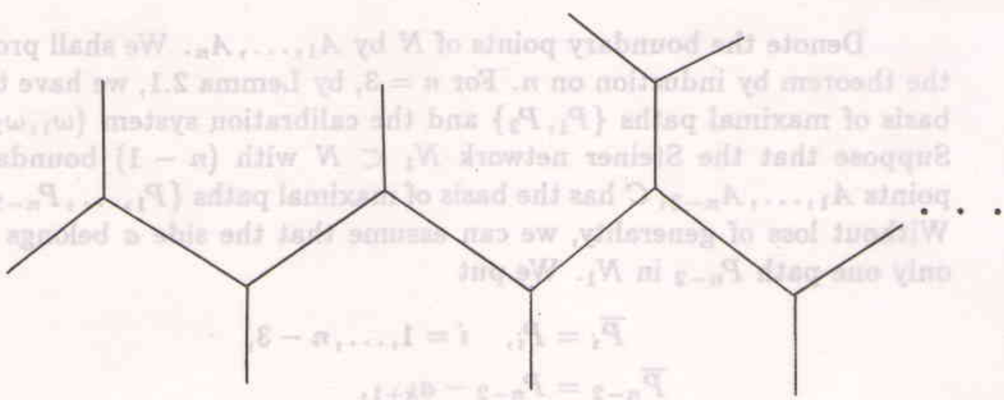

Fig. 5

We choose the orientation on N such that every two adjacent sides are oriented opposite to each other (see Fig. 6).

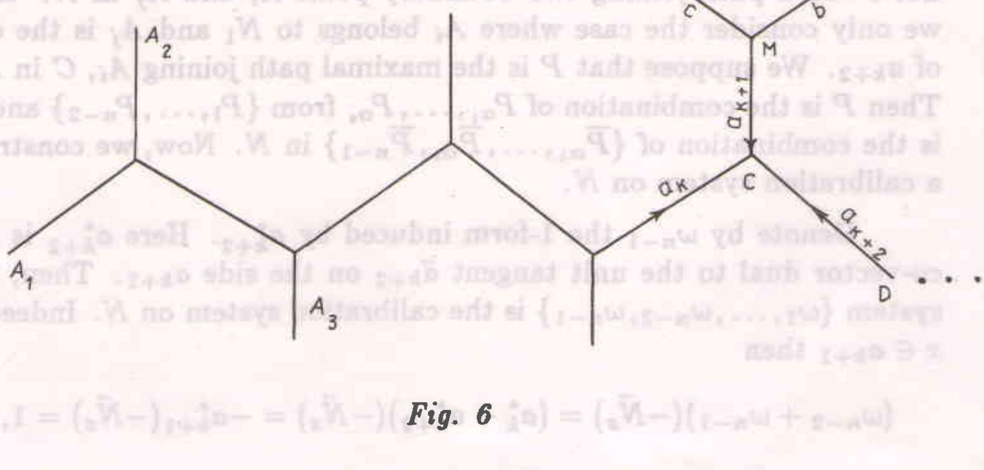

Fig. 6

Suppose that C is the first growth of the Steiner network N with boundary points $A_1, A_2, A_3, \dots, C$. We assume that $\{P_1, P_2, \dots, P_t\}$ and $\{\omega_1, \omega_2, \dots, \omega_t\}$ are respectively, the basis of maximal paths and the calibration system on the network $(A_1, A_2, A_3, \dots, C)$ (by Theorem 2.2). We can assume that the side a_k belongs only to one path P_t. Put

$$\overline{P}_i = P_i; \quad i = 1, 2, \dots, t-1,$$
$$\overline{P}_t = -P_t + a_{k+2},$$
$$\overline{P}_{t+1} = -P_t + a_{k+1} - c,$$
$$\overline{P}_{t+2} = b - c.$$

Then the system $\{\overline{P}_1, \overline{P}_2, \dots, \overline{P}_{t+2}\}$ is a basis of maximal paths of the Steiner network $(A_1, A_2, \dots, D)$. Denote by $\overline{\omega}_t$, $\overline{\omega}_{t+1}$, $\overline{\omega}_{t+2}$ the 1-forms induced by a^*_{k+2}, a^*_{k+1}, b^*, respectively. Here a^*_{k+2}, a^*_{k+1}, a^*_k, b^*, c^* are respectively the co-vectors dual to unit tangent on the sides a_{k+2}, a_{k+1}, a_k, b, c. Put

$$\overline{\omega}_i = \omega_i; \quad i = 1, 2, \dots, t-1.$$

then $\{\overline{\omega}_1, \dots, \overline{\omega}_{t+2}\}$ is the calibration system in $(A_1, A_2, \dots, D)$. Indeed,

If $x \in a_{k+2}$ then $\overline{\omega}_t(\vec{N}_x) = a^*_{k+2}(\vec{N}_x) = 1$, and $\|\overline{\omega}_t\| = \|a^*_{k+2}\| = 1$.

If $x \in a_{k+1}$ then $\overline{\omega}_{t+1}(\vec{N}_x) = a^*_{k+1}(\vec{N}_x) = 1$, and $\|\overline{\omega}_{t+1}\| = \|a^*_{k+1}\| = 1$.

If $x \in b$ then $\overline{\omega}_{t+2}(\vec{N}_x) = b^*(\vec{N}_x) = 1$, and $\|\overline{\omega}_{t+2}\| = \|b^*\| = 1$.

If $x \in c$ then $(\overline{\omega}_{t+1} + \overline{\omega}_{t+2})(-\vec{N}_x) = (a^*_{k+1} + b^*)(-\vec{N}_x) = -c^*(-\vec{N}_x) = 1$, and $\| - \overline{\omega}_{t+1} - \overline{\omega}_{t+2}\| = \| - a^*_{k+1} - b^*\| = \|c^*\| = 1$.

Let $x \in m$, where m is any side belonging to P_t. We can assume that m also belongs to $P_{\alpha_1} \dots P_{\alpha_k}$. Then we have

$$\begin{aligned}
&(\overline{\omega}_{\alpha_1} + \dots + \overline{\omega}_{\alpha_k})(\vec{N}_x) + (\overline{\omega}_t + \overline{\omega}_{t+1})(-\vec{N}_x) \\
&= (\omega_{\alpha_1} + \dots + \omega_{\alpha_k})(\vec{N}_x) + (a^*_{k+2} + a^*_{k+1})(-\vec{N}_x) \\
&= (\omega_{\alpha_1} + \dots + \omega_{\alpha_k})(\vec{N}_x) + (-a^*_k)(-\vec{N}_x) \\
&= (\omega_{\alpha_1} + \dots + \omega_{\alpha_k} + a^*_k)(\vec{N}_x) \\
&= (\omega_{\alpha_1} + \dots, \omega_{\alpha_k} + \omega_t)\ (\vec{N}_x) \\
&= 1.
\end{aligned}$$

Therefore,

$$\begin{aligned}&\|\overline{\omega}_{\alpha_1}+\cdots+\overline{\omega}_{\alpha_k}-\overline{\omega}_t-\overline{\omega}_{t+1}\|\\&=\|\omega_{\alpha_1}+\cdots+\omega_{\alpha_k}+a_k^*\|\\&=\|\omega_{\alpha_1}+\cdots+\omega_{\alpha_k}+\omega_t\|\\&=1.\end{aligned}$$

Proceeding in this way, we construct at the end the basis of maximal paths and the calibration system on N, satisfying the conditions of Definition 1.3. The theorem is proved.

REFERENCES

1. Dao Trong Thi and A. T. Fomenko, *Minimal surfaces, atratified mutivarifolds and the Plateau problem*, ASM, Providence, Rhode Island, 1991.

2. Ivanov - Iuzhilin, *Solution to the Steiner problem for convex boundaries*, Vestnik Moskow Univ. Ser. Math. Mech., No 2, 1989.

Department of Mathematics
Pedagogical Institute of Vinh
Vinh, Vietnam

Received June 24, 1993