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A Short Communication

ON THE REAL STABILITY RADIUS OF POSITIVE LINEAR SYSTEMS *

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In this note we consider the problem of robust stability of linear discrete-time systems whose trajectories are invariant with respect to a closed convex cone in the state space. We are able to derive some estimates for upper bounds and lower bounds of parameter perturbations which preserve stability of the system. In the case the constraint cone is the positive orthant R^n_+ , the obtained bounds yield a simple formula for real stability radius of the system. Our proofs are based on the state space approach to robustness analysis of stability developed by Hinrichsen and Pritchard, (see e.g. [2]) and spectral theory of positive matrices founded by Perron and Frobenius [1]. It is worth noticing that the problem of deriving a computable explicit formula of *real stability radius*, even for a simple autonomous linear system, was a difficult problem. Only recently, a general formula for the real stability radius has been found by Qiu et al. [7]. Its computation, however, requires the solution of a complicated global optimization problem.

Consider a linear system described by the difference equation

$$x_{k+1} = Ax_k, k = 0, 1, \dots, \tag{1}$$

subject to the state constraint

$$x_k \in K \subset \mathbf{R}^n, \tag{2}$$

where $A \in \mathbf{R}^{n \times n}$ and K is a nonempty closed convex cone in $\mathbf{R}^n$. The cone K is invariant with respect to System (1) if every solution $x_k(x_0)$ starting at an arbitrary point $x_0 \in K$ remains in K or, equivalently, if and only if

$$AK \subset K. \tag{3}$$

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In such a case, the system (1) is said to be *positive w.r.t.* K. If $K = \mathbf{R}^n_+$, the positive orthant, then (1) is simply called *positive*. Positive systems arise frequently from the modeling of real processes in such fields as economics, population dynamics, ecology, etc. where the state variables may represent quantities which do not have meaning unless they are nonnegative.

We recall that the system (1) is said to be asymptotically stable or *Schur stable* if the spectrum $\sigma(A)$ of A lies in the open unit disk $\mathbf{C}_1 = \{s \in \mathbf{C} : |s| < 1\}$, or equivalently, iff $\rho(A) < 1$ where $\rho(A)$ is the spectral radius of A : $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$. The basic problem under consideration that we address in this paper is the following. Given a Schur stable linear discrete-time system (1) satisfying (3) and given a set of perturbations $\mathcal{D} \subset \mathbf{C}^{n\times n}$ such that $(A+D)K \subset K$ for any $D \in \mathcal{D}$, determine the largest value $\gamma > 0$ for which the perturbed system

$$x_{k+1} = (A+D)x_k, \|D\| < \gamma$$

remains Schur stable for each $D \in \mathcal{D}$. For this purpose we introduce the *$\mathcal{D}$-radius of stability* of A by defining

$$r_{\mathcal{D}}(A) = \inf\{\|D\| : \quad D \in \mathcal{D}, \quad \rho(A+D) \geq 1\}. \tag{4}$$

Here and in what follows the norms of matrices are defined as operator norms induced by some vector norms on $\mathbf{R}^n$. If the norm of D in the above definition need to be specified then we shall use the notation $r_{\mathcal{D}}(A; \|\cdot\|)$ instead of $r_{\mathcal{D}}(A)$.

If $\mathcal{D} = \mathbf{C}^{n\times n}$(resp., $\mathbf{R}^{n\times n}$) the above definition is reduced to the one of complex (resp., real) *unstructured stability radius* $r_{\mathbf{C}}(A)$ (resp., $r_{\mathbf{R}}(A)$) (see e.g. [2], [4]).

In this paper we shall restrict ourselves to the case of *unstructured nonnegative perturbations*, i.e. when

$$\mathcal{D} = \mathcal{D}_+ := \{D \in \mathbf{R}^{n\times n} : DK \subset K\}.$$

The corresponding $\mathcal{D}_+$-radius of stability of A will be shortly denoted by $r_+(A)$. The case of *structured perturbations of linear output feedback type* is considered in [5]. It can be shown first that for an arbitrary norm on $\mathbf{R}^n$ the following bounds for r_+ hold.

Proposition 1. *Let $K \subset \mathbf{R}^n$ be a pointed closed convex cone with* $\operatorname{int} K \neq \emptyset$ *and A be a Schur stable positive (w.r.t. K) matrix. Then*

$$\frac{1}{\|(I-A)^{-1}\|} \leq r_+(A) \leq \frac{1}{\sup\limits_{e \in K \cap B_1} \|(I-A)^{-1}e\|}, \tag{6}$$

(where B_1 is the closed unit ball in $\mathbf{R}^n$).

The proof of the above assertion is based on the Krein Theorem on extension of positive functionals, see e.g. [6].

For any matrix $P \in \mathbf{R}^{n \times n}$ let us define

$$M(P) = \{e \in B_1 : \|P\| = \|Pe\|\},$$

Then, the following result is an immediate consequence of Proposition 1.

Proposition 2. *Let $K \subset \mathbf{R}^n$ be a pointed closed convex cone with* $\operatorname{int} K \neq \emptyset$ *and A be a Schur stable positive (w.r.t. K) matrix. If*

$$K \cap M((I-A)^{-1}) \neq \emptyset, \tag{7}$$

then

$$r_+(A) = \frac{1}{\|(I-A)^{-1}\|}.$$

It is remarkable that in case $K^* = K$ (i.e. K is *self-dual*), where K^* is the nonnegative polar cone of K, the condition (7) holds for the operator norms of matrices $\|\cdot\|_p$, $p = 1, 2, \infty$, induced by the corresponding vector norms.

Proposition 3. *Let $K \subset \mathbf{R}^n$ be a self-dual pointed closed convex cone with* $\operatorname{int} K \neq \emptyset$ *and A be a Schur stable positive (w.r.t. K) matrix. Then*

$$r_+(A; \|\cdot\|_2) = s_n((I-A)^{-1}),$$

where $s_n(\cdot)$ denotes the minimal singular value of a matrix.

It is obvious that $\mathbf{R}^n_+$ is a self-dual pointed closed convex cone and the matrix A is positive w.r.t. $\mathbf{R}^n_+$ if and only if all the entries of A are nonnegative. Therefore, we have

Proposition 4. *If $A \in \mathbf{R}_+^{n\times n}$ is a Schur stable matrix, then*

$$r_+(A; \|\cdot\|_\alpha) = \frac{1}{\|(I-A)^{-1}\|_\alpha}, \tag{8}$$

where $\alpha = 1, 2, \infty$.

Now we are going to study the relationship between stability radii. Clearly, by definition, we always have

$$r_{\mathbf{C}}(A) \le r_{\mathbf{R}}(A) \le r_+(A). \tag{9}$$

It is well known that if A is an arbitrary real matrix, then $r_{\mathbf{R}}$ and $r_{\mathbf{C}}$ may be largely different. In fact, in [3] it has been shown that the ratio $r_{\mathbf{R}}/r_{\mathbf{C}}$ may be unbounded. Fortunately, this does not happen if A has all nonnegative entries, provided that $\|\cdot\| = \|\cdot\|_\alpha$, with $\alpha = 1, 2, \infty$ in the definition of these stability radii. Moreover, the following result shows that in this case all the stability measures in (9) coincide.

Proposition 5. *Let $A \in \mathbf{R}_+^{n\times n}$ be a Schur stable matrix. Then*

$$r_{\mathbf{C}}(A; \|\cdot\|_\alpha) = r_{\mathbf{R}}(A; \|\cdot\|_\alpha) = r_+(A; \|\cdot\|_\alpha),$$

provided that $\alpha = 1, 2, \infty$.

We note that in [5] it has been shown that for any Schur stable matrix $A \in \mathbf{R}^{n\times n}$ and for arbitrary operator norm $\|\cdot\|$,

$$r_{\mathbf{C}}(A; \|\cdot\|) = \frac{1}{\max_{\varphi\in[0,2\pi]} \|(e^{j\varphi}I - A)^{-1}\|}.$$

The above theorem shows that if matrix A is nonnegative and $\|\cdot\| = \|\cdot\|_\alpha$ with $\alpha = 1, 2, \infty$, then the above maximum is achieved at $\varphi = 0$ and, moreover, the real stability radius and the complex stability radius coincide.

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