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THE DUAL ALGORITHM FOR MINIMIZING THE SUM OF ABSOLUTE VALUES OF LINEAR FUNCTIONS [1]

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Abstract. *This paper is devoted to develop the dual method for minimizing a sum of absolute linear functions. The algorithm is based on the support co-plan notion that is firstly proposed by R. Gabasov and F. M. Kirillova in linear programming. In this paper a support co-plan is defined, the optimal criterions are proved and the algorithm is described in great detail.*

1. INTRODUCTION

We consider the following problem

$$\begin{cases} f(x) = \sum\limits_{k\in K} |c'_k x + \alpha_k| \longrightarrow \min \\ b_* \le Ax \le b^*, \qquad d_* \le x \le d^*, \end{cases} \tag{1}$$

where $A = A(I,J)$, $I = \{1,...,m\}$, $J = \{1,...,n\}$, is a $m \times n$-matrix; $x, d_*, d^* \in R^n$; $b_*, b^* \in R^m$, $K = \{1,...,K\}$. By using the auxiliary variables $z(K) = \{z_k | k \in K\}$ we can transform Problem (1) into the following linear programming problem

$$\begin{cases} \sum\limits_{k\in K} z_k \longrightarrow \min \\ b_* \le Ax \le b^*, \qquad d_* \le x \le d^*, \\ -z \le Cx + \alpha \le z, \qquad z \ge 0. \end{cases} \tag{1'}$$

Then, by using well-known methods in the linear programming

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we can solve Problem (1'). This way, unfortunately, is not always effective, because dimensions of the problem may be increased and its characteristic may be lost.

In [1], [2] Gabasov R., Kirilova F. M. and others considered the problem of minimizing the sum of absolute values of linear functions with equality constraints:

$$\begin{cases} \sum_{k\in K} |c'_k x + \alpha_k| \longrightarrow \min \\ Ax = b, \qquad d_* \le x \le d^*, \end{cases} \tag{2}$$

of course, we can also transform (1) into (2) by using the auxiliary variables. But then dimensions of the problem will be also increased. In this paper, we shall build an algorithm to solve Problem (1) directly. Aiming at this purpose we consider the dual problem of (1):

$$\begin{cases} \varphi(\lambda) = -b'^* s + b'_* t - d'^* w + d'_* v + \alpha' \xi \ \longrightarrow\ \max, \\ A'y + w - v + C'\xi = 0, \quad y = s - t; \\ s_i \ge 0, \quad t_i \ge 0, \quad i \in I; \qquad w_j \ge 0, \quad v_j \ge 0, \quad j \in J; \\ -1 \le \xi_k \le 1, \quad k \in K; \end{cases} \tag{3}$$

where $\lambda = \{y, w, v, \xi\}$, $\xi \in R^K$, $w, v \in R^n$, $s, t, y \in R^m$, $C = C(K, J) = \{c'_k, k \in K\}$. In fact, Problem (3) is the dual problem of (1').

Let $\{I_{on}, J_{on}\} \subset \{I, J\}$ with $|I_{on}| = |J_{on}|$ and $\det A(I_{on}, J_{on}) \ne 0$. Then we build the matrix $D(K, J) = C(K, J_{on}) A_{on}^{-1} A(I_{on}, J) - C(K, J)$. Let $\{K_{oz}, J_{oz}\} \subseteq \{K, J \setminus J_{on}\}$ with $|K_{oz}| = |J_{oz}|$ and $\det D(K_{oz}, J_{oz}) \ne 0$.

The set $F_{oz} = \{I_{on}, J_{on}, K_{oz}, J_{oz}\}$ is called the ***support*** of Problem (3).

Let λ be a plan of Problem (3). The vector $\delta = A'y + C'\xi$ is called the *co-plan* of Problem (3). A co-plan δ with a support F_{oz} is called the ***support co-plan*** and is denoted by $\{\delta, F_{oz}\}$.

The support co-plan $\{\delta, F_{oz}\}$ is called non-degenerated if

$$\begin{cases} \delta_j \ne 0, \quad \forall\, j \in J_n = J \setminus (J_{oz} \cup J_{on}), \\ y_i \ne 0, \quad \forall\, i \in I_{on}, \qquad |\xi_k| < 1, \quad \forall\, k \in K_{oz}. \end{cases} \tag{4}$$

Notice that, it is easy to choose a plan λ of Problem (3). For instance, for any ξ, y, such that $|\xi_k| \leq 1, \ \forall\, k \ \in \ K$, we define the vectors $t, \ s, \ w, \ v$ as follows:

$$\forall\, i \in I: \quad \begin{cases} t_i = 0, & s_i = y_i \quad \text{if} \quad y_i \geq 0, \\ t_i = -y_i, & s_i = 0 \quad \text{if} \quad y_i < 0; \end{cases} \tag{5}$$

$$\forall\, j \in J: \quad \begin{cases} v_j = \delta_j, & w_j = 0, \quad \text{if} \quad \delta_j \geq 0, \\ v_j = 0, & w_j = -\delta_j \quad \text{if} \quad \delta_j < 0. \end{cases} \tag{6}$$

Then $\lambda = \{y, \ w, \ v, \ \xi\}$, satisfies the constraints of Problem (3). The co-plan δ is said to be adequate if it satisfies (5)-(6).

Let λ be a plan of Problem (3) and $\{\delta, F_{oz}\}$ is an adequate support co-plan. We define a *pseudo-plan* χ which satisfies following conditions:

1. $\forall\, j \in J_n : \chi_j = d_{*j}$ if $\delta_j > 0$, $\chi_j = d^*_j$ if $\delta_j < 0$ and $d_{*j} \leq \chi_j \leq d^*_j$ if $\delta_j = 0$.

2. $f_k(\chi) = c'_k\chi + \alpha_k = \eta_k, \forall\, k \in K_{oz}, \ \ A'_i\chi = \omega_i, \forall\, i \in I_{on},$

where η_k and ω_i are chosen as follows:

a) $\forall k \in K_{oz} : \eta_k = 0$, if $|\xi_k| < 1$; $\eta_k = \varepsilon\xi_k$, if $|\xi_k| = 1$ where $\varepsilon \in [0,1]$;

b) $\forall i \in I_{on} : \ \omega_i = b^*_i$, if $y_i > 0$; $\omega_i = b_{*i}$, if $y_i < 0$; $b_{*i} \leq \omega_i \leq b^*_i$, if $y_i = 0$.

Then

$$\chi_{oz} = D^{-1}_{oz}\Big\{\alpha_{oz} - \eta_{oz} + C(K_{oz}, J_{on})A^{-1}_{on}\omega_{on} - D(K_{oz}, J_n)\chi_n\Big\},$$
$$\chi_{on} = A^{-1}_{on}\Big\{\omega_{on} - A(I_{on}, J \setminus J_{on})\chi(J \setminus J_{on})\Big\}.$$

2. OPTIMAL CRITERIONS

Let $\overline{\delta} = \delta + \Delta\delta, \ \ \overline{\lambda} = \{\overline{y}, \overline{w}, \overline{v}, \overline{\xi}\} = \big\{y + \Delta y, w + \Delta w, v + \Delta v, \xi + \Delta\xi\big\}$, where

$$\begin{cases} \Delta\xi'_{ox} &= \Big\{\Delta\delta'_{on}A_{on}^{-1}A(I_{on},J_{ox}) - \Delta\delta'_{ox} - \Delta y'_n B(I_n,J_{ox}) \\ & \qquad -\Delta\xi'_n D(K_n,J_{ox})\Big\}D_{ox}^{-1}, \\ \Delta y'_{on} &= \Big\{\Delta\delta'_{on} - \Delta y'_n A(I_n,J_{on}) = \Delta\xi' C(K,J_{on})\Big\}A_{on}^{-1}, \\ \Delta\delta'_n &= \Delta y' A(I,J_n) + \Delta\xi' C(K,J_n), \end{cases} \tag{7}$$

and $\quad B(I,J) = A(I,J_{on})A_{on}^{-1}A(I_{on},J) - A(I,J)$.

Then $\Delta\delta' = \Delta y'A + \Delta\xi'C$, and $\alpha'\Delta\xi = \eta'\Delta\xi - \Delta\delta'\chi + \Delta y'A\chi$. Let $\overline{\lambda}$ be a plan of Problem (3) and $\overline{\delta}$ is an adequate co-plan. Then the following implications hold:

$$\begin{array}{llll}
\delta_j > 0, \ \overline{\delta}_j \geq 0 \implies \chi_j = d_{*j}, & \Delta w_j = 0, & \Delta v_j = \Delta\delta_j; \\
\delta_j > 0, \ \overline{\delta}_j < 0 \implies \chi_j = d_{*j}, & \Delta w_j = -\overline{\delta}_j, & \Delta v_j = -\delta_j; \\
\delta_j < 0, \ \overline{\delta}_j \geq 0 \implies \chi_j = d_j^*, & \Delta w_j = \delta_j, & \Delta v_j = \overline{\delta}_j; \\
\delta_j < 0, \ \overline{\delta}_j < 0 \implies \chi_j = d_j^*, & \Delta w_j = -\Delta\delta_j, & \Delta v_j = 0; \\
\delta_j = 0, \ \overline{\delta}_j \geq 0 \implies d_{*j} \leq \chi_j \leq d_j^*, & \Delta w_j = 0, & \Delta v_j = \overline{\delta}_j; \\
\delta_j = 0, \ \overline{\delta}_j < 0 \implies d_{*j} \leq \chi_j \leq d_j^*, & \Delta w_j = -\overline{\delta}_j, & \Delta v_j = 0;
\end{array}$$

$$\begin{array}{llll}
y_i > 0, \ \overline{y}_i > 0 \implies A_i\chi = b_i^*, & \Delta s_i = \Delta y_i, & \Delta t_i = 0; \\
y_i > 0, \ \overline{y}_i \leq 0 \implies A_i\chi = b_i^*, & \Delta s_i = -y_i, & \Delta t_i = -\overline{y}_i; \\
y_i < 0, \ \overline{y}_i > 0 \implies A_i\chi = b_{*i}, & \Delta s_i = \overline{y}_i, & \Delta t_i = y_i; \\
y_i < 0, \ \overline{y}_i \leq 0 \implies A_i\chi = b_{*i}, & \Delta s_i = 0, & \Delta t_i = -\Delta y_i; \\
y_i = 0, \ \overline{y}_i > 0 \implies b_{*i} \leq A_i\chi \leq b_i^*, & \Delta s_i = \overline{y}_i, & \Delta t_i = 0; \\
y_i = 0, \ \overline{y}_i \leq 0 \implies b_{*i} \leq A_i\chi \leq b_i^*, & \Delta s_i = 0, & \Delta t_i = -y_i;
\end{array}$$

$$\begin{array}{lll}
\xi_k = 1, \quad -1 \leq \overline{\xi}_k \leq 1 \implies & \Delta\xi_k = \overline{\xi}_k - 1, & \eta_k \geq 0; \\
\xi_k = -1, \quad -1 \leq \overline{\xi}_k \leq 1 \implies & \Delta\xi_k = \overline{\xi}_k + 1, & \eta_k \leq 0; \\
-1 < \xi_k < 1 \implies \eta_k = 0, & -1 - \xi_k \leq \Delta\xi_k \leq 1 - \xi_k. &
\end{array}$$

Let

$$\begin{aligned}
\Delta\varphi &= \varphi(\overline{\lambda}) - \varphi(\lambda) \\
&= -b'^*\Delta s + b'_*\Delta t - d'^*\Delta w + d'_*\delta v + \alpha'\delta\xi \\
&= -b'^*\Delta s + b'_*\Delta t + \Delta y' A\chi - d'^*\Delta w + d'_*\delta v - \Delta\delta'\chi \\
&\quad + \eta'_{ox}\Delta\xi_{ox} + \sum_{k\in K_n} \Delta\xi_k f_k(\chi).
\end{aligned}$$

Denote $\Delta\varphi = \Delta\varphi_1 + \Delta\varphi_2 + \Delta\varphi_3$, where

$$\begin{aligned}
\Delta\varphi_1 &= \Delta s'(A\chi - b^*) - \Delta t'(A\chi - b_*) \\
&= \sum_{i\in I_n} \Delta s_i(A_i\chi - b_i^*) - \sum_{i\in I_n} \Delta t_i(A_i\chi - b_{*i}) + \sum_{\substack{y_i>0,\\ \overline{y_i}\le 0}} \overline{y_i}(b_i^* - b_{*i}) \\
&\quad + \sum_{\substack{y_i<0,\\ \overline{y_i}\ge 0}} \overline{y_i}(b_{*i} - b_i^*) + \sum_{\substack{y_i=0,\\ \overline{y_i}>0}} \overline{y_i}(\omega_i - b_i^*) + \sum_{\substack{y_i=0,\\ \overline{y_i}<0}} \overline{y_i}(\omega_i - b_{*i}),
\end{aligned}$$

$$\begin{aligned}
\Delta\varphi_2 &= \Delta w'(\chi - d^*) - \Delta v'(\chi - d_*) \\
&= \sum_{j\in J\setminus J_n} \Delta w_j(\chi_j - d_j^*) - \sum_{j\in J\setminus J_n} \Delta v_j(\chi_j - d_{*j}) + \sum_{\substack{\delta_j>0,\\ \overline{\delta_j}\le 0}} \overline{\delta_j}(d_j^* - d_{*j}) \\
&\quad + \sum_{\substack{\delta_j<0,\\ \overline{\delta_j}\ge 0}} \overline{\delta_j}(d_{*j} - d_j^*) + \sum_{\substack{\delta_j=0,\\ \overline{\delta_j}>0}} \overline{\delta_j}(d_{*j} - \chi_j) + \sum_{\substack{\delta_j=0,\\ \overline{\delta_j}<0}} \overline{\delta_j}(d_j^* - \chi_j),
\end{aligned}$$

$$\Delta\varphi_3 = \sum_{k\in K_n} \Delta\xi_k f_k(\chi) + \sum_{\substack{\xi_k=1,\\ -1\le\overline{\xi_k}\le 1}} (\overline{\xi_k} - 1)\eta_k + \sum_{\substack{\xi_k=-1,\\ -1\le\overline{\xi_k}\le 1}} (\overline{\xi_k} + 1)\eta_k.$$

Assume that $\{\delta, F_{oz}\}$ is an adequate co-plan and χ is pseudo-plan. Then we have the following

Theorem 1. *If χ satisfies the following conditions*

$$\begin{cases} \chi_j = d_j^*, & \text{if } \delta_j < 0, \\ \chi_j = d_{*j}, & \text{if } \delta_j > 0, \\ d_{*j} \le \chi_j \le d_j^*, & \text{if } \delta_j = 0; \end{cases} \tag{8}$$

$$\begin{cases} A_i\chi = b_i^*, & \text{if } y_i > 0, \\ A_i\chi = b_{*i}, & \text{if } y_i < 0, \\ b_{*i} \le A_i\chi \le b_i^*, & \text{if } y_i = 0; \end{cases} \tag{9}$$

$$\begin{cases} \xi_k = \operatorname{sign} f_k(\chi), & \text{if } f_k(\chi) \ne 0, \\ -1 \le \xi_k \le 1, & \text{if } f_k(\chi) = 0, \quad k \in K_n = K \setminus K_{ox}, \end{cases} \tag{10}$$

then χ is an optimal plan of Problem (1). Conversely, if λ is an optimal plan of Problem (3) and the pseudo-plan χ, associated with a non-degenerate adequate support co-plan $\{\delta, F_{oz}\}$ is an optimal plan of Problem (1) then conditions (8), (9) and (10) are satisfied.

Proof. Let a pseudo-plan χ satisfy (8), (9), (10). From (10) we obtain $\xi_k = \operatorname{sign} f_k(\chi)$, for all $k \in K$ satisfying $f_k(\chi) \neq 0$. Now

$$\begin{aligned} f(\chi) &= \sum_{k\in K} |f_k(\chi)| = \sum_{k\in K} |c'_k\chi + \alpha_k| = \sum_{k\in K} \xi_k\,(c'_k\chi + \alpha_k) \\ &= \xi' C\chi + \xi'\alpha = (v' - w' - y'A)\chi + \xi'\alpha \\ &= v'(\chi - d_*) - w'(\chi - d^*) - s'(A\chi - b^*) + t'(A\chi - b_*) \\ &\qquad\qquad + v'd_* - w'd^* - s'b^* + t'b_* + \xi'\alpha \\ &= \mu + \varphi(\lambda), \end{aligned}$$

where $\mu = v'(\chi - d_*) - w'(\chi - d^*) - s'(A\chi - b^*) + t'(A\chi - b_*)$. It follows from (5), (6), (8) and (9) that $\mu = 0$. Hence

$$f(\chi) = \varphi(\lambda). \tag{11}$$

By (8), (9) the pseudo-plan χ satisfies the constraints of Problem (1). In view of the dual theory of mathematical programming, by (11) we deduce that χ is an optimal plan of (1).

Now we turn to the proof of necessity. Let λ be an optimal plan of Problem (3) and let χ be an optimal plan of Problem (1) corresponding with a non-degenerate adequate support co-plan $\{\delta, F_{oz}\}$. By contradiction, we shall prove that χ satisfies (8), (9), (10).

Notice that, by (7), for any $\{\Delta y, \Delta w, \Delta v, \Delta\xi\}$ small enough we always have

$$\begin{aligned} \Delta\varphi_1 &= \sum_{i\in I_n} \Delta s_i(A_i\chi - b_i^*) - \sum_{i\in I_n} \Delta t_i(A_i\chi - b_{*i}), \\ \Delta\varphi_2 &= \sum_{j\in J\setminus J_n} \Delta w_j(\chi_j - d_j^*) - \sum_{j\in J\setminus J_n} \Delta v_j(\chi_j - d_{*j}), \\ \Delta\varphi_3 &= \sum_{k\in K_n} \Delta\xi_k f_k(\chi). \end{aligned}$$

Let us consider the following three cases.

Case 1. If the condition (8) is violated, for instance, $\delta_{j_0} < 0$ and $\chi_{j_0} < d^*_{j_0}$, then we choose $\Delta\delta_{j_0} = \rho$ where $0 < \rho \le |\delta_{j_0}|$, $\Delta\delta(J_{on} \cup J_{ox} \setminus j_o) = 0$, $\Delta y(I_n) = 0$, $\Delta\xi(K_n) = 0$. Then $\Delta\varphi_1 = 0$, $\Delta\varphi_3 = 0$ and $\Delta\varphi_2 = -\rho(\chi_{j_0} - d^*_{j_0}) > 0$, that is $\Delta\varphi > 0$.

Case 2. If the condition (9) is violated, for instance $y_{i_o} = 0$ and $A_{i_0}\chi > b^*_{i_0}$, $i_o \in I_n$, then we choose $\Delta y_{i_0} = \rho$, with $\rho > 0$ (or $\rho < 0$), $\Delta y(I_n \setminus i_o) = 0$, $\Delta\delta(J\setminus J_n) = 0$, $\Delta\xi(K_n) = 0$. Then $\Delta\varphi_2 = 0$, $\Delta\varphi_3 = 0$, $\Delta\varphi_1 = -\rho(A_{i_0}\chi - b^*_{i_0}) > 0$ (or $\Delta\varphi_1 = -\rho(A_{i_0}\chi - b_{*i_0}) > 0$), that is $\Delta\varphi > 0$.

Case 3. If the condition (10) is violated, for instance $\xi_{k_0} \ne \operatorname{sign} f_{k_0}(\chi)$, then we choose $\Delta\xi_{k_0} = \rho \operatorname{sign} f_{k_0}(\chi)$, $\rho > 0$, $\Delta\xi(K_n \setminus k_o) = 0$, $\Delta\delta(J \setminus J_n) = 0$, $\Delta y(I_n) = 0$. Then $\Delta\varphi_1 = 0$, $\Delta\varphi_2 = 0$, $\Delta\varphi_3 = \rho \mid f_{k_0}(\chi) \mid > 0$, that is $\Delta\varphi > 0$.

Thus, if one of conditions (8), (9) and (10) is not satisfied, then we can build the plan $\overline{\lambda}$ of Problem (3) such that $\varphi(\overline{\lambda}) > \varphi(\lambda)$, contradicting the assumption that λ is an optimal plan of Problem (3). This completes the proof of the theorem.

Let us define

$$\beta = \sum_{\substack{\delta_j > 0,\\ j \in J\setminus J_n}} \delta_j(\chi_j - d_{*j}) + \sum_{\substack{\delta_j < 0,\\ j \in J\setminus J_n}} \delta_j(\chi_j - d^*_j) - \sum_{\substack{y_i > 0,\\ i \in I_n}} y_i(A_i\chi - b^*_i)$$
$$- \sum_{\substack{y_i < 0,\\ i \in I_n}} y_i(A_i\chi - b_{*i}) + \sum_{\substack{f_k(\chi) > 0,\\ k \in K_n}} (1 - \xi_k) f_k(\chi) - \sum_{\substack{f_k(\chi) < 0,\\ k \in K_n}} (1 + \xi_k) f_k(\chi).$$

Theorem 2. (ε-optimal criterion). *If χ satisfies the constraints of Problem (1) and $\beta \le \varepsilon$ then χ is an ε-optimal plan of Problem (1).*

Proof. By denoting $e_k = \operatorname{sign} f_k(\chi)$, we have

$$\begin{aligned} f(\chi) &= \sum_{k\in K} \mid c'_k\chi + \alpha_k \mid \\ &= \sum_{k\in K} (e_k - \xi_k)(c'_k\chi + \alpha_k) + \sum_{k\in K} \xi_k(c'_k\chi + \alpha_k) \\ &= \sum_{k\in K_n} (e_k - \xi_k) f_k(\chi) + \xi' C\chi + \xi'\alpha \\ &= \sum_{k\in K_n} (e_k - \xi_k) f_k(\chi) + \mu + \varphi(\lambda) = \beta + \varphi(\lambda). \end{aligned}$$

Since $\beta \le \varepsilon$ it follows readily that

$$(f(\chi) - f(\chi^o)) + (\varphi(\lambda^0) - \varphi(\lambda)) = f(\chi) - \varphi(\lambda) = \beta \le \varepsilon,$$

where χ^o is an optimal plan of Problem (1) and λ^o is an optimal plan of Problem (3). Thus, we have $f(\chi) - f(\chi^o) \le \varepsilon$, and hence, χ is an ε-optimal plan of Problem (1).

3. ALGORITHM

Let $\{\delta, F_{oz}\}$ be an adequate support co-plan with a plan λ of Problem (3) and the optimal conditions are not satisfied. Then we shall build a new adequate support co-plan $\{\overline{\delta}, \overline{F}_{oz}\}$ and new plan $\overline{\lambda} = \{\overline{y}, \overline{w}, \overline{v}, \overline{\xi}\}$ of (3) so that $\varphi(\overline{\lambda}) > \varphi(\lambda)$. Each step of the algorithm consists of two parts. The first is finding a new adequate co-plan and a new plan of (3). The second is determining a new support.

Denote by K^o, I^o, J^o the sets of indexes which do not satisfy optimal conditions: $K^o \subset K_n$, $I^o \subset I_n$, $J^o \subset J_{on} \cup J_{oz}$, where $K^o = \{k : \xi_k \ne \operatorname{sign} f_k(\chi),\ f_k(\chi) \ne 0\}$, and

$$\begin{aligned} I^o &= \{i : A_i\chi \ne b_i^*, y_i > 0\} \cup \{i : A_i\chi \ne b_{*i}, y_i < 0\} \\ &\quad \cup \{i : A_i\chi \notin [b_{*i}, b_i^*], y_i = 0\}; \\ J^o &= \{j : \chi_j \ne d_j^*, \delta_j < 0\} \cup \{j : \chi_j \ne d_{*j}, \delta_j > 0\} \\ &\quad \cup \{j : \chi_j \notin [d_{*j}, d_j^*], \delta_j = 0\}. \end{aligned}$$

Let $h_o \in K^o \cup J^o \cup I^o$, be chosen arbitrarily. In practice, we can choose h_o from indexes $\{k_o, j_o, i_o\}$ such that the quantity $\max\{u_k, |\delta_j|, |y_i|\}$ equals either u_{k_0}, $|\delta_{j_0}|$ or $|y_{i_0}|$, where

$$\begin{aligned} u_{k_0} &= \max\{|\xi_k - \operatorname{sign} f_k(\chi)|,\ k \in K^o\}, \\ |y_{i_0}| &= \max\{|y_i|,\ i \in I^o\}, \qquad |\delta_{j_0}| = \max\{|\delta_j|,\ j \in J^o\}. \end{aligned}$$

3.1. New co-plan and new plan of Problem (3)

The new plan $\overline{\lambda}$ of Problem (3) and the new co-plan $\overline{\delta}$ will be found so that $\overline{\delta}$ is adequate. The vectors $\overline{\xi}$, $\overline{y}$ and $\overline{\delta}$ will be found as follows:

$$\overline{\xi}(K) = \xi(K) + \sigma\tau(K), \quad \overline{y}(I) = y(I) + \sigma\tau(I), \quad \overline{\delta}(J) = \delta(J) + \sigma\tau(J).$$

a. Find vectors τ. Set

$$\begin{aligned}\tau'(K_{ox}) &= \{\tau'(J_{on})A_{on}^{-1}A(I_{on},J_{ox}) - \tau'(J_{ox}) - \tau'(I_n)B(I_n,J_{ox});\\ &\qquad - \tau'(K_n)D(K_n,J_{ox})\}D_{ox}^{-1};\\ \tau'(I_{on}) &= \{\tau'(J_{on}) - \tau'(I_n)A(I_n,J_{on}) - \tau'(K)C(K,J_{on})\}A_{on}^{-1};\\ \tau'(J_n) &= \tau'(I)A(I,J_n) + \tau'(K)C(K,J_n)\end{aligned}$$

where $\tau(K_n)$, $\tau(I_n)$, $\tau(J_{on})$, $\tau(J_{ox})$ are found according to the following cases:

a1. $k_o \in K_n$: Set $\gamma = f_{k_o}(\chi)$, $\tau_{k_o} = \operatorname{sign}\gamma$, $\tau(K_n \setminus k_o) = 0$, $\tau(I_n) = 0$, $\tau(J_{on}) = 0$, $\tau(J_{ox}) = 0$;

a2. $i_o \in I_n$: If $y_{i_o} \geq 0$ then set $\gamma = A_{i_o}\chi - b^*_{i_o}$, else set $\gamma = A_{i_o}\chi - b_{*i_o}$. Set $\tau_{i_o} = -\operatorname{sign}\gamma$, $\tau(I_n \setminus i_o) = 0$, $\tau(K_n) = 0$, $\tau(J_{on}) = 0$, $\tau(J_{ox}) = 0$;

a3. $j_o \in J_{on} \cup J_{ox}$: If $\delta_{j_o} \geq 0$ then set $\gamma = \chi_{j_0} - d^*_{j_0}$ else set $\gamma = \chi_{j_0} - d_{*j_0}$. Set $\tau_{j_0} = -\operatorname{sign}\gamma$, $\tau(J_{on} \cup J_{ox} \setminus j_o) = 0$, $\tau(I_n) = 0$, $\tau(K_n) = 0$.

b. Find step-length σ. For some $q \geq 0$, set

$$\rho^{(q)} = \min\{\rho^{(q)}_{k_*}, \rho^{(q)}_{i_*}, \rho^{(q)}_{j_*}\}, \tag{12}$$

where

$$\begin{aligned}\rho^{(q)}_{i_*} &= \min\left\{-y^{(q)}_i/\tau_i : i \in I_{on}, y^{(q)}_i.\tau_i < 0\right\},\\ \rho^{(q)}_{j_*} &= \min\left\{-\delta^{(q)}_j/\tau_j : j \in J_n, \delta^{(q)}_j.\tau_j < 0\right\},\\ \rho^{(q)}_{k_*} &= \min\left\{\rho^{(q)}_{k_*}(1), \rho^{(q)}_{k_*}(2)\right\}\end{aligned}$$

with

$$\rho^{(q)}_{k_*}(1) = \min\left\{\rho^{(q)}_k, k \in K_{ox}\right\};$$

$$\rho^{(q)}_k = \begin{cases}(1-\xi^{(q)}_k)/\tau_k, & \text{if } \tau_k > 0,\\ (-1-\xi^{(q)}_k)/\tau_k, & \text{if } \tau_k < 0;\end{cases}$$

$$\rho^{(q)}_{k_*}(2) = \begin{cases}\infty, & \text{if } h_o \notin K^o,\\ -(2\xi_{k_o} + \xi^{(q)}_{k_o})/\tau_{k_o}, & \text{if } h_o = k_o \in K^o.\end{cases}$$

Set

$$\begin{aligned}\gamma_{q+1} = \gamma_q + \sum_{\substack{\delta_j^{(q)}=0,\\ \tau_j<0, j\in J_n}} \tau_j(d_j^* - \chi_j) + \sum_{\substack{\delta_j^{(q)}=0,\\ \tau_j>0, j\in J_n}} \tau_j(d_{*j} - \chi_j)\\ - \sum_{\substack{y_i^{(q)}=0,\\ \tau_i<0, i\in I_{on}}} \tau_i(b_{*i} - \omega_i) - \sum_{\substack{y_i^{(q)}=0,\\ \tau_i>0, i\in I_{on}}} \tau_i(b_i^* - \omega_i)\end{aligned} \tag{13}$$

where

$$\delta_j^{(q)} = \delta_j^{(q-1)} + \rho_{q-1}\tau_j,\ j\in J;\quad y_i^{(q)} = y_i^{(q-1)} + \rho_{q-1}\tau_i,\ i\in I;$$
$$\xi_k^{(q)} = \xi_k^{(q-1)} + \rho_{q-1}\tau_k,\ k\in K;\quad \delta^{(o)} = \delta,\ y^{(o)} = y,\ \xi^{(o)} = \xi,$$
$$\text{and } \gamma^{(o)} = \gamma.$$

It follows from (13) that the cost function φ of Problem (3) increases by the value $\Delta\varphi_{(q)} = \sum_{i=0}^{q-1} \rho^{(i)}\gamma_i$, where $\rho^{(i)}$ is computed in the same way as in (12)-(13).

The procedure for finding σ starts with $q = 0$ and terminates at the first q_o which satisfies $\gamma_{q_o} \le 0$, or in (12) $\rho^{(q_o)} = \rho_{k_*}^{(q_o)}$, $k_* \in K$. Then, we have $\sigma = \sum_{i=0}^{q_o-1} \rho^{(i)}$.

Moreover, in this case an active index is one of the indexes $i_* \in I_{on}$, $j_* \in J_n$, or $k_* \in K_{ox}$ which are chosen by (12).

3.2. New support

The new support $\overline{F}_{oz}$ will be built in the following cases:

If $k_* = k_o$, $k_o \in K_n$ then $\overline{F}_{oz} = F_{oz}$.

a. $h_o = k_o$, $k_o \in K_n$.

a1) $j_* \in J_n$: $\overline{K}_{ox} = K_{ox} \cup k_o$, $\overline{J}_{ox} = J_{ox} \cup j_*$.

a2) $i_* \in I_{on}$: Choose $j_+ \in J_{on}$ so that $A_{on}^{-1}(j_+, i_*) \neq 0$ and $\overline{I}_{on} = I_{on}\backslash i_*$, $\overline{J}_{on} = J_{on}\backslash j_+$, $\overline{J}_{ox} = J_{ox} \cup j_+$, $\overline{K}_{ox} = K_{ox} \cup k_o$.

a3) $k_* \in K_{ox}$: $K_{ox} = (K_{ox} \setminus k_*) \cup k_o$

b. $h_o = i_o, i_o \in I_n$.

b1) $j_* \in J_n$: If $B(i_o, j_*) \neq 0$ then $\overline{I}_{on} = I_{on} \cup i_o$, $\overline{J}_{on} = J_{on} \cup j_*$. Conversely, if $B(i_o, j_*) = 0$ then choose $j_+ \in J_{ox}$ so that $B(i_o, j_+) \neq 0$ and $D_{ox}^{-1}(j_+, K_{ox})D(K_{ox}, j_*) \neq 0$. A new support is defined by $\overline{J}_{ox} = (J_{ox} \setminus j_+) \cup j_*$, $\overline{I}_{on} = I_{on} \cup i_o$, $\overline{J}_{on} = J_{on} \cup j_+$.

b2) $i_* \in I_{on}$: If $A(i_o, J_{on})A_{on}^{-1}(J_{on}, i_*) \neq 0$ then $\overline{I}_{on} = (I_{on} \setminus i_*) \cup i_o$. Conversely, choose $j_+ \in J_{on}$ so that $A_{on}^{-1}(j_+, i_*) \neq 0$ and choose $j_- \in J_{ox}$ so that $B(i_o, j_-) \neq 0$, and $D_{ox}^{-1}(j_-, K_{ox})$ $C(K_{ox}, J_{on})A_{on}^{-1}(J_{on}, i_*) \neq 0$. A new support will be built in following order: $\tilde{I}_{on} = I_{on} \setminus i_*$, $\tilde{J}_{on} = J_{on} \setminus j_+$; $\overline{J}_{ox} = (J_{ox} \setminus j_-) \cup j_+$; $\overline{I}_{on} = \tilde{I}_{on} \cup i_o$, $\overline{J}_{on} = \tilde{J}_{on} \cup j_-$.

b3) $k_* \in K_{ox}$: Choose $j_+ \in J_{ox}$ so that $B(i_o, j_+) \neq 0$ and $D_{ox}^{-1}(j_+, k_*) \neq 0$. $\overline{K}_{ox} = K_{ox} \setminus k_*$, $\overline{J}_{ox} = J_{ox} \setminus j_+$, $\overline{I}_{on} = I_{on} \cup i_o$, $\overline{J}_{on} = J_{on} \cup j_+$.

c. $h_o = j_o$, $j_o \in J_{on}$.

c1) $j_* \in J_n$: If $A_{on}^{-1}(j_o, I_{on})A(I_{on}, j_*) \neq 0$ then $\overline{J}_{on} = (J_{on} \setminus j_o) \cup j_*$. Conversely, choose $j_+ \in J_{ox}$ so that $A_{on}^{-1}(j_o, I_{on})A(I_{on}, j_+) \neq 0$ and $D_{ox}^{-1}(j_+, K_{ox})D(K_{ox}, j_*) \neq 0$. A new support is $\overline{J}_{ox} = (J_{ox} \setminus j_+) \cup j_*$, $\overline{J}_{on} = (J_{on} \setminus j_o) \cup j_+$.

c2) $i_* \in I_{on}$: If $A_{on}^{-1}(j_o, i_*) \neq 0$ then $\overline{I}_{on} = I_{on} \setminus i_*$, $\overline{J}_{on} = J_{on} \setminus j_o$. Conversely, choose $j_+ \in J_{on}$ so that $A_{on}^{-1}(j_+, i_*) \neq 0$ and choose $j_- \in J_{ox}$ so that $A_{on}^{-1}(j_o, I_{on})A(I_{on}, j_-) \neq 0$ and $D_{ox}^{-1}(j_-, K_{ox})C(K_{ox}, J_{on})A_{on}^{-1}(J_{on}, i_*) \neq 0$. A new support will be built in following order: $\tilde{I}_{on} = I_{on} \setminus i_*$, $\tilde{J}_{on} = J_{on} \setminus j_+$, $\overline{J}_{ox} = (J_{ox} \setminus j_-) \cup j_+$, $\overline{J}_{on} = (\tilde{J}_{on} \setminus j_o) \cup j_-$, $\overline{I}_{on} = \tilde{I}_{on}$.

c3) $k_* \in K_{ox}$: Choose $j_+ \in J_{ox}$ so that $A_{on}^{-1}(j_o, I_{on})A(I_{on}, j_+) \neq 0$ and $D_{ox}^{-1}(j_+, k_*) \neq 0$. $\overline{K}_{ox} = K_{ox} \setminus k_*$, $\overline{J}_{ox} = J_{ox} \setminus j_+$, $\overline{J}_{on} = (J_{on} \setminus j_o) \cup j_+$.

d. $h_o = j_o$, $j_o \in J_{ox}$.

d1) $j_* \in J_n$: $\overline{J}_{ox} = (J_{ox} \setminus j_o) \cup j_*$.

d2) $i_* \in I_{on}$: Choose $j_+ \in J_{on}$ so that $A_{on}^{-1}(j_+, i_*) \neq 0$. $\overline{I}_{on} = I_{on} \setminus i_*$, $\overline{J}_{on} = J_{on} \setminus j_+$, $\overline{J}_{ox} = (J_{ox} \setminus j_o) \cup j_+$.

d3) $k_* \in K_{ox}$: $\overline{K}_{ox} = K_{ox} \setminus k_*$, $\overline{J}_{ox} = J_{ox} \setminus j_o$.

In every case, the new supporting matrices $\overline{A}_{on}^{-1}$, $\overline{D}_{ox}^{-1}$ may be found by using old supporting matrices A_{on}^{-1}, D_{ox}^{-1}.

It is worth remarking that the idea of the above algorithm is rather similar to the one of the support dual method in linear programming [2]. Its distinction from the latter is that at each iterative step of the algorithm we need only work with two support matrices of dimensions $|I_{on}| \times |J_{on}|$ and $|K_{ox}| \times |J_{ox}|$, while the support dual method [2] when applied to the linear programming problem (1'), requires to deal, in general, with matrices of dimension $(|I_{on}| + |K_{ox}| \times |J_{on}| + |J_{ox}|)$.

4. COMPUTATIONAL EXPERIMENTS

The described algorithm was coded in and has been run on IBM AT 80286. We solved Problem (1) where all elements a_{ij} of A, c_{ij} of C, are randomly generated in the interval $[-50, 50]$ and all elements of vectors b^*, b_*, d^*, d_*, α are randomly generated in the interval $[-250, 250]$. In the program we used subroutine GENER in order to generate the start values.

Results of numerical experiments are summarized in the following three tables.

Table 1

$N°$	Iteration	Values of function φ		Time	Time for one
		φ start	φ optimal	(second)	iteration
m = 20,		n = 25,	K = 20		
1	17	-3.371,00	38.225,00	1.54	0.090
2	12	51,00	122.275,00	0.99	0,082
3	12	-16,00	49.938,00	0,99	0,082
4	16	-1.439,00	14.491,00	1,43	0,089
5	12	-23,00	122.519,00	0,99	0,082
m = 30,		n = 35,	K = 35		
1	26	-1.957,00	34.929,00	3,51	0,135
2	27	-1.993,00	35.665,00	3,84	0,142
3	20	-4.216,00	122.466,00	2,69	0,135
4	20	-12.616,00	396.866,00	2,69	0,135
5	27	-2.048,00	36.700,00	3,84	0,142

Table 2

N°	m	n	K	Iteration (average)	Time (s) (average)
1	25	15	15	6.80	0.21
2	25	15	25	9.80	0.43
3	25	15	35	12.80	0.75
4	25	25	15	11.20	0.59
5	25	25	25	17.00	1.30
6	25	25	35	22.70	2.22
7	25	35	15	11.00	0.81
8	25	35	25	17.00	1.79
9	25	35	35	22.70	3.07
10	25	45	15	6.80	0.60
11	25	45	25	9.80	1.26
12	25	45	35	12.80	2.17
13	35	15	15	6.80	0.23
14	35	15	25	9.80	0.47
15	35	15	35	12.80	0.78
16	35	25	15	11.20	0.64
17	35	25	25	17.00	1.37
18	35	25	35	22.70	2.34
19	35	35	15	11.10	0.88
20	35	35	25	17.00	1.90
21	35	35	35	22.60	3.17
22	35	45	15	6.80	0.66
23	35	45	25	9.80	1.35
24	35	45	35	12.80	2.27

(In Table 2, for each size, ten problems were tested).

Table 3 gives some computational results to compare the described algorithm in solving Problem (1) with the dual simplex method, which is described in Ref. 2, for solving Problem (1'). For each problem, both algorithms start from the same plan.

Table 3

N°	Iteration		Time (second)	
	SM	DM	SM	DM
1	2	2	0,658	0,001
2	16	9	8,886	0,274
3	3	2	1,207	0,001
4	17	10	9,435	0,219
5	8	7	4,224	0,110
6	16	11	9,270	0,384
7	3	3	1,207	0,055
8	5	4	2,414	0,055
9	18	13	10,038	0,384
10	27	16	15,469	0,823
	11,5	7,7	6,280	0,230

* SM: Dual simplex method,
DM: Dual support method which described in this paper.

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