Vietnam Journal of Mathematics
Volume 23, Number 1, 1995

A Short Communication

REPRESENTATION OF LIPSCHITZ GLOBAL SOLUTIONS OF THE CAUCHY PROBLEM FOR HAMILTON-JACOBI EQUATIONS [1]

TRAN DUC VAN and NGUYEN HOANG

This note is a continuation of [5]. Consider the Cauchy problem for Hamilton-Jacobi equations of the form

$$u_t + H(t, \nabla_x u) = 0\,, \quad (t,x) \in \Omega = (0,T) \times \mathbf{R}^n, \qquad (1)$$

$$u(0,x) = \sigma(x)\,, \quad x \in \mathbf{R}^n. \qquad (2)$$

We shall establish the representation of Lipschitz global solutions of Problem (1)-(2) under the assumptions that $H(t,q)$ is a continuous function of $(t,q) \in (0,T) \times \mathbf{R}^n$ and $\sigma(x)$ is a d.c. function (i.e., $\sigma(x)$ is the difference of two convex functions). The class of d.c. functions is rather rich, for example, every twice continuously differentiable function on $\mathbf{R}^n$ or every continuous piece-wise linear function on $\mathbf{R}$ is d.c.. For the readers who are interested in d.c. functions, we refer to [2].

We use the following notations. Let $\Omega = (0,T) \times \mathbf{R}^n$; $\|.\|$ and $\langle .,. \rangle$ be the Euclidean norm and the scalar product in $\mathbf{R}^n$, respectively. Denote by $\mathrm{Lip}\,(\Omega)$ the set of all locally Lipschitz continuous functions u defined on Ω and set $\mathrm{Lip}\,([0,T) \times \mathbf{R}^n) = \mathrm{Lip}\,(\Omega) \cap C([0,T) \times \mathbf{R}^n)$. Furthermore, if V is subset of Ω, we denote by $\mathrm{Lip}^K(V)$ the set of all Lipschitz continuous functions in V with Lipschitz constant K.

Definition 1. A function $u(t,x)$ in $\mathrm{Lip}\,([0,T) \times \mathbf{R}^n)$ is called a Lipschitz global solution of Problem (1)-(2) if $u(t,x)$ satisfies (1) almost everywhere in Ω and $u(0,x) = \sigma(x)$ for all $x \in \mathbf{R}^n$.

---

* supported in part by the National Basic Research Program in Natural Science, Vietnam

Firstly we present the following theorem.

Theorem 1. *Let $(\sigma_\alpha(x))_{\alpha\in I}$ be a family of functions indexed by an crbitrary set I such that Equation (1) with initial datum $\sigma_\alpha(x)$ has a Lipschitz global solution $u_\alpha(t,x)$. Assume that for every $(t_0,x_0)\in[0,T)\times\mathbf{R}^n$ there exist a neighborhood $V=V(t_0,x_0)$, a constant $K>0$, a set $W_V\subset V$ with $\operatorname{mes} W_V=0$ and a set $J\subset I$ such that for all $\alpha\in J$, $u_\alpha\big|_V\in \operatorname{Lip}^K(V)$, $u_\alpha(t,x)$ satisfies (1) in $V\setminus W_V$ and $\inf_{\alpha\in I}u_\alpha(t,x)=\min_{\alpha\in J}u_\alpha(t,x)$, $(t,x)\in V$. Then the function $u(t,x)=\inf_{\alpha\in I}u_\alpha(t,x)$ is a Lipschitz global solution of (1)-(2) with $\sigma(x)=\inf_{\alpha\in I}\sigma_\alpha(x)$.*

Proof. The proof of Theorem 1 is analogous to the one of Theorem 2.1 in [5]. □

We denote by $DC(\mathbf{R}^n)$ the class of d.c. functions on $\mathbf{R}^n$. Take $\sigma(x)\in DC(\mathbf{R}^n)$, then $\sigma(x)=\varphi(x)-\psi(x)$, where φ,ψ are some convex functions. Let φ^*, ψ^* be the conjugate functions of φ, ψ respectively, and $D=\operatorname{dom}\psi^*$.

Following [4], we assume:

(A0): $H(t,q)$ satisfies the Caratheodory condition in Ω and for every positive number $\mathcal{N}$ there exists a function $g_{\mathcal{N}}\in L_\infty(0,T)$ such that for almost all $t\in(0,T)$

$$\sup_{\|q\|\le\mathcal{N}}|H(t,q)|\le g_{\mathcal{N}}(t).$$

(A1): For every $(t_0,x_0)\in[0,T)\times\mathbf{R}^n$ and $M>0$, there exist positive constants r and N such that

$$\langle x,p\rangle-\varphi^*(p+\alpha)-\int_0^t H(\tau,p)d\tau$$

$$<\max_{\|q\|\le N}\left\{\langle x,q\rangle-\varphi^*(q+\alpha)-\int_0^t H(\tau,q)d\tau\right\},$$

whenever $(t,x)\in[0,T)\times\mathbf{R}^n$, $|t-t_0|+\|x-x_0\|<r$, $\|p\|>N$ and $\|\alpha\|\le M$.

(A2): The function $L_0^\alpha(t,x)$ is single-valued for all $(t,x) \in \Omega \setminus W$, with $\operatorname{mes} W = 0$, where

$$L_0^\alpha(t,x) = \Big\{ q_0 \in \mathbf{R}^n : \max_{\|q\| \le N} \Big\{ \langle x, q\rangle - \varphi^*(q+\alpha) - \int_0^t H(\tau,q)d\tau \Big\}$$

$$= \langle x, q_0\rangle - \varphi^*(q_0+\alpha) - \int_0^t H(\tau,q_0)d\tau \Big\}, \quad \forall \alpha,\ \|\alpha\| \le M.$$

We are now able to formulate and prove the results of this note.

Theorem 2. *Let $\sigma(x) \in DC(\mathbf{R}^n)$ with $\sigma(x) = \varphi(x) - \psi(x)$. Assume (A0)-(A1)-(A2) for $H(t,q)$ and φ. Moreover, suppose that the convex function ψ is globally Lipschitz continuous on $\mathbf{R}^n$. Then the function*

$$u(t,x) = \min_{\alpha \in D} \max_{q \in \mathbf{R}^n} \Big\{ \langle x, q-\alpha\rangle - \varphi^*(q) + \psi^*(\alpha) - \int_0^t H(\tau, q-\alpha)d\tau \Big\} \quad (3)$$

is a Lipschitz global solution of Problem (1)-(2).

Proof. Let $\sigma(x) = \varphi(x) - \psi(x)$, where φ, ψ are convex functions. We put $\sigma_\alpha(x) = \varphi(x) - \langle \alpha, x\rangle + \psi^*(\alpha)$. Obviously, for all $\alpha \in \mathbf{R}^n$, $\sigma_\alpha(x)$ is a convex function. Consider the problem

$$u_t + H(t, \nabla_x u) = 0, \qquad (t,x) \in \Omega, \quad (1)$$

$$u(0,x) = \sigma_\alpha(x), \qquad x \in \mathbf{R}^n,\ \alpha \in D. \quad (2')$$

Under the assumptions (A0)-(A1)-(A2), applying Theorem 2.1 in [4], we easily see that the function

$$u_\alpha(t,x) = \max_{q \in \mathbf{R}^n} \Big\{ \langle x, q\rangle - \varphi^*(q+\alpha) - \int_0^t H(\tau,q)d\tau + \psi^*(\alpha) \Big\} \quad (4)$$

is a Lipschitz global solution of the Problem (1)-(2'). Moreover, it satisfies (1) for all $(t,x) \in \Omega \setminus W$.

Let us put $q+\alpha=y$, then (4) becomes

$$u_\alpha(t,x)=\max_{y\in\mathbf{R}^n}\left\{\langle x,y\rangle-\varphi^*(y)-\int_0^t H(\tau,y-\alpha)d\tau\right\}-\{\langle x,\alpha\rangle-\psi^*(\alpha)\}. \tag{5}$$

Since ψ is globally Lipschitz continuous on $\mathbf{R}^n$, then by [3, 13.3.3], $D=\operatorname{dom}\psi^*$ is a bounded subset of $\mathbf{R}^n$. Let $M=\sup_{\alpha\in D}\|\alpha\|$, $(t_0,x_0)\in[0,T)\times\mathbf{R}^n$, $V=V(t_0,x_0)=\{(t,x)\in[0,T)\times\mathbf{R}^n\ :\ |t-t_0|+\|x-x_0\|<r\}$, r being as in (A1). We take arbitrarily (t,x), $(t',x')\in V$ and choose $q_0\in\mathbf{R}^n$, $\|q_0\|\le N$ such that

$$u_\alpha(t,x)=\langle x,q_0\rangle-\varphi^*(q_0+\alpha)-\int_0^t H(\tau,q_0)d\tau.$$

Then

$$\begin{aligned}u_\alpha(t,x)-u_\alpha(t',x')&\le\langle x,q_0\rangle-\varphi^*(q_0+\alpha)-\int_0^t H(\tau,q_0)d\tau\\&-\ \langle x',q_0\rangle-\varphi^*(q_0+\alpha)-\int_0^{t'} H(\tau,q_0)d\tau\\&\le\|q_0\|\cdot\|x-x'\|+E|t-t'|\le K(\|x-x'\|+|t-t'|),\end{aligned}$$

with $E=\operatorname{ess.sup} g_N(t)$, and $K=\max(N,E)$. Thus $u_\alpha|_V\in\operatorname{Lip}^K(V)$.

Next, we observe that

$$\begin{aligned}\inf_{\alpha\in D}u_\alpha(t,x)&\ge\inf_{\alpha\in\overline{D}}\max_{y\in\mathbf{R}^n}\left\{\langle x,y\rangle-\varphi^*(y)-\int_0^t H(\tau,y-\alpha)d\tau\right\}\\&-\sup_{\alpha\in D}\{\langle x,\alpha\rangle-\psi^*(\alpha)\}>-\infty.\end{aligned}\tag{6}$$

Since D is bounded, we can take a sequence $(\alpha_n)\subset D$, $\alpha_n\to\alpha_0\in\overline{D}$ such that

$$\max_{y\in\mathbf{R}^n}\left\{\langle x,y\rangle-\varphi^*(y)-\int_0^t H(\tau,y-\alpha_n)d\tau\right\}$$
$$-\{\langle x,\alpha_n\rangle-\psi^*(\alpha_n)\}\to\inf_{\alpha\in D}u_\alpha(t,x).$$

If $\alpha_0 \notin D$, then by the upper semicontinuity of $\xi(\alpha) = \langle x, \alpha\rangle - \psi^*(\alpha)$ we have $\xi(\alpha_n) \to -\infty$ and therefore $\inf\limits_{\alpha\in D} u_\alpha(t,x) = +\infty$. This is a contradiction. Thus

$$\inf_{\alpha\in D} u_\alpha(t,x) = \inf_{\alpha\in \overline{D}} u_\alpha(t,x) = \min_{\alpha\in \overline{D}} u_\alpha(t,x) = \min_{\alpha\in D} u_\alpha(t,x). \qquad (7)$$

On the other hand, from the fact that [1, p.964],

$$\psi(x) = \max_{\alpha\in D}\{\langle x,\alpha\rangle - \psi^*(\alpha)\} = -\min_{\alpha\in D}\{-\langle x,\alpha\langle + \psi^*(\alpha)\},$$

we have

$$\begin{aligned}\min_{\alpha\in D}\sigma_\alpha(x) &= \min_{\alpha\in D}\{\varphi(x) - \langle x,\alpha\rangle + \psi^*(\alpha)\}\\ &= \varphi(x) - \max_{\alpha\in D}\{\langle x,\alpha\rangle - \psi^*(\alpha)\}\\ &= \varphi(x) - \psi(x) = \sigma(x).\end{aligned}$$

Applying Theorem 1, we complete the proof. □

Corollary 1. *Suppose that $H(t,q)$ is continuous in $[0,T]\times \mathbf{R}^n$, φ, ψ are globally Lipschitz continuous and convex on $\mathbf{R}^n$. Assume (A2). Then the function u given by (3) is a Lipschitz global solution of Problem (1)-(2) with $\sigma(x) = \varphi(x) - \psi(x)$.*

Example 1. Let $H(q) = (1 + |q|^3)^{1/3}$, and

$$\sigma(x) = \begin{cases} x^3/3, & x \in [-1,1],\\ x - 2/3\,\mathrm{sgn}\,x, & x \notin [-1,1].\end{cases}$$

We can rewrite $\sigma(x) = \varphi(x) - \psi(x)$, where

$$\varphi(x) = \begin{cases} 0, & x < 0,\\ x^3/3, & x \in [0,1],\\ x - 2/3, & x > 1,\end{cases}$$

and $\psi(x) = \varphi(-x)$. Then a Lipschitz global solution of (1)-(2) is

$$u(t,x) = \min_{\alpha\in[-1,0]}\ \max_{y\in[0,1]} \Big\{x(y-\alpha) - \frac{2}{3}(|y|^{3/2} - |\alpha|^{3/2}) - t(1 + |y-\alpha|^3)^{1/3}\Big\}.$$

Corollary 2. *Let $\sigma(x) \in DC(\mathbf{R}^n)$ with $\sigma(x) = \varphi(x) - \psi(x)$. Assume that (A0)-(A1)-(A2) hold for $H(t,q)$ and $\varphi(x)$ and there exists a function $h \in L_1(0,T)$ such that $H(t,q) \le h(t)$ for all $q \in \mathbf{R}^n$ and $\lim_{\|x\|\to\infty} \frac{\psi(x)}{\|x\|} = +\infty$. Then the function u given by (3) is a Lipschitz global solution of Problem (1)-(2).*

Proof. We note first that $\lim_{\|x\|\to\infty} \frac{\psi(x)}{\|x\|} = +\infty$, then so does $\lim_{\|y\|\to\infty} \frac{\psi^*(y)}{\|y\|}$ and $\operatorname{dom}\psi^* = \mathbf{R}^n$ (see [1. p. 967]). By arguing analogous to the proof of Theorem 2, we get the inequalities (6) and (7). The conclusion is then straightforward. □

The authors would like to thank Dr.N.D. Thai Son for his assistance.

REFERENCES

1. E. Hopf, *Generalized solutions of nonlinear equations of first order*, J. of Math. and Mech., **14** (1965), 951-973.
2. R. Horst, H. Tuy, *Global Optimization*, Springer-Verlag, Berlin - Heidenberg - New York, 1990.
3. T. Rockafellar, *Convex Analysis*, Prinçeton Univ. Press, Princeton, 1970.
4. Tran Duc Van and Nguyen Hoang, *On the existence of global solutions of the Cauchy problem for Hamilton-Jacobi equations*, Southeast Asian Bull. of Math. (to appear).
5. Tran Duc Van, Nguyen Hoang and Nguyen Duy Thai Son, *On the explicit representation of global solutions of the Cauchy problem for Hamilton-Jacobi equations*, Acta Math. Vietnamica, N2, **19** (1994), 111-120.

Institute of Mathematics
P.O.Box 631, Bo Ho,
10 000 Hanoi, Vietnam

Received November 12, 1994