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THE AR-PROPERTY FOR KALTON SETS

TRAN VAN AN

Abstract. *In [2] Kalton constructed compact convex sets which can not be affinely embedded into the space L_0 of all measurable functions. In this paper we prove that a lot of compact comvex sets constructed by Kalton are AR.*

1. PRELIMINARIES

Let X be a linear space over the field of complex numbers C. By a quasi-norm on X we mean a real non-negative function $x \to \|x\|_*$ such that

(i) $\|x\|_* > 0$ for every non-zero point $x \in X$;

(ii) $\|\alpha x\|_* = |\alpha|\ \|x\|_*$ for every $\alpha \in C$ and $x \in X$;

(iii) $\|x+y\|_* \leq k(\|x\|_* + \|y\|_*)$ for every $x, y \in X$,

where k is a constant independent of x, y.

The sets $\{x \in X : \|x\|_* < \varepsilon\}$ form a base of neighbourhoods of θ for a metrizable topology on X. If this topology is complete then X is called a complex quasi-Banach space. We shall say that a quasi-norm $\|\cdot\|_*$ is a p-norm $(0 < p \leq 1)$ if it satisfies

$$\|x+y\|_*^p \leq \|x\|_*^p + \|y\|_*^p$$

for every $x, y \in X$. Then $(X, \|\cdot\|_*)$ is called a p-normed space.

A well-known theorem of Aoki and Rolewicz [7] asserts that every quasi-norm is equivalent to a p-norm for a certain number p with $0 < p \leq 1$.

Therefore, from now on we shall suppose that a complex quasi-Banach space X is p-normed for some $0 < p \leq 1$ and denote $\|x\| = \|x\|_*^p$

for every $x \in X$. Then the topology induced by the metric $\|\cdot\|$ is equivalent to the original one.

Let Δ denote the open unit disc in the complex plane $\mathbf{C}$ and T the unit circle. Let X be a complex quasi-Banach space. A function $f : \Delta \to X$ is called analytic iff for every $z \in \Delta$, $f(z)$ can be represented as the sum of a power series $f(z) = \sum_{n\geq 0} a_n z^n$, where the constant coefficients a_n belong to X.

By $A_0(X)$ we denote the space of functions $f : \overline{\Delta} \to X$ which are continuous on $\overline{\Delta}$ and analytic on Δ.

Let A be a subset of a complex quasi-Banach space X. By co A we denote the convex hull of A in X and by Card A we denote the cardinality of A. We also use the following notation:

$$A^+ = \text{co}\,(A \cup \{0\});$$
$$iA = \{ia : a \in A\};$$
$$\hat{A} = \text{co}\,((A^+) \cup (-A^+) \cup (iA^+) \cup (-iA^+));$$

and if $x, y \in X$ we write

$$\|x - A\| = \inf\{\|x - y\| : y \in A\}.$$

Let L_0 denote the space of all measurable functions from $[0,1]$ into the real line $\mathbf{R}$. Then L_0 is a linear metric space with F-norm:

$$\|f\| = \int_0^1 \frac{|f(t)|}{1 + |f(t)|}\,dt$$

for every $f \in L_0$.

We say that a metric space X is AR iff for any metric space Z containing X as a closed subset there exists a continuous map $r : Z \to X$ such that $r(x) = x$ for every $x \in X$.

Let X be a complex quasi-Banach space. Then we say that $x \in X$ is an analytic needle point of X iff for any $\varepsilon > 0$ there exists $g \in A_0(X)$ such that:

(1) $g(0) = x$;

(2) $\|g(z)\|_* < \varepsilon$ for every $z \in T$;

(3) If $y \in \text{co}\, g(\overline{\Delta})$ then there exists an $\alpha \in [0,1]$ such that $\|y - \alpha x\|_* < \varepsilon$.

A complex quasi-Banach space X is called an analytic needle point space iff every non-zero point of X is an analytic needle point.

For undefined notations, we refer to [1], [3] and [7].

1.1. Lemma [2]. *Let x be an analytic needle point of X. Then given any $\varepsilon > 0$ there is a finite set $F = F(x, \varepsilon) \subset X$ and a polynomial $P \in A_0(X)$ such that:*

(4) $P(\overline{\Delta}) \subset \text{co}\, F$;

(5) $P(0) = x$;

(6) $\|P(z)\|_* < \varepsilon$ for every $z \in T$;

(7) If $y \in \text{co}\, F$ then there exists $\alpha \in [0,1]$ such that $\|y - \alpha x\|_* < \varepsilon$

(8) If $y \in F$ then $\|y\|_* < \varepsilon$.

2. KALTON SETS

In this section we describe Kalton's method of constructing compact convex sets without any extreme points.

Let X be an analytic needle point space. Let $\{\delta_n\}$ be a sequence of positive numbers such that $\sum \delta_n^p < \infty$. Let $G_0 = \{x_0\}$, where x_0 is any non-zero point of X. Assume that $G_{n-1} = \{y_1, \ldots, y_N\}$ has been selected. Let $\varepsilon_n = N^{-\frac{1}{p}} \delta_n$ and put

$$G_n = \bigcup_{j=1}^{N} F(y_j, \varepsilon_n),$$

where $F(y_j, \varepsilon_n)$ is given by Lemma 1.1. Then we have

$G_{n-1}^+ \subset G_n^+$ for every $n \in \mathbb{N}$;

(9) $\|x - G_{n-1}^+\| \leq N\varepsilon_n^p \leq \delta_n^p$ for every $x \in G_n^+$.

Denote

(10) $K_0 = \overline{\bigcup_{n=0}^{\infty} G_n^+}$ and $K = \hat{K}_0$

By Kalton's method [2] we can prove that K is a compact convex set without extreme points, see [6], and there is no affine embedding of K into L_0.

Remark. Our construction of K in Formula (10) is slightly different from that of Kalton [2]. As pointed out by Kalton in his recent letter to author, there is no reason to say that the set K (in [2]) is convex and our definition of K will replace Kalton's compact set in [2].

We shall call the set K defined as above a Kalton set.

3. THE MAIN RESULT

3.1. Theorem. *The set K corresponding to a sequence $\{\delta_n\}$ with*

$$\sum_{n=1}^{\infty} m(n-1)\delta_n^p < \infty$$

is an AR.

The proof of this theorem is based on the following facts.

3.2. Lemma [5].

$$K = \overline{\bigcup_{n=0}^{\infty} \hat{G}_n}$$

3.3. Lemma. *Suppose $m(n) = \text{Card}\, G_n$.*

(i) For every finite set $A \subset \hat{G}_n$ we have

$$\operatorname{diam} \operatorname{co} A \leq (4m(n)+1) \operatorname{diam} A;$$

(ii) There is a continuous retraction $r : X \to \hat{G}_n$ such that

$$\|r(x) - x\| \leq 22m(n)\|x - \hat{G}_n\|$$

for every $x \in X$.

Proof. (i) Let A be a finite subset of $\hat{G}_n$. Since $\operatorname{card} G_n = m(n)$, $\hat{G}_n$ lies in a real linear space Y with $\dim Y = 2m(n)$. Then $\operatorname{co} A \subset \hat{G}_n \subset Y$. By Caratheodory's theorem, every point $x \in \operatorname{co} A$ is a convex combination of at most $2m(n)+1$ affinely independent extreme points of $\operatorname{co} A$. Obviously, the set of extreme points of $\operatorname{co} A$ is a subset of A. Therefore, if $x, y \in \operatorname{co} A$ then

$$x = \sum_{i=1}^{2m(n)+1} \mu_i b_i, \qquad y = \sum_{j=1}^{2m(n)+1} \lambda_j a_j$$

with $a_j, b_i \in A$; $\lambda_j \geq 0$, $\mu_i \geq 0$, $i, j = 1, \ldots, 2m(n)+1$ and

$$\sum_{j=1}^{2m(n)+1} \lambda_j = \sum_{i=1}^{2m(n)+1} \mu_i = 1.$$

Hence for every $x, y \in \operatorname{co} A$ we have

$$\begin{aligned}
\|x - y\| &= \left\| \sum_{i=1}^{2m(n)+1} \mu_i b_i - \sum_{j=1}^{2m(n)+1} \lambda_j a_j \right\| \\
&= \left\| \sum_{i=1}^{2m(n)+1} \mu_i b_i - \sum_{i=1}^{2m(n)+1} \mu_i a_1 + \sum_{j=1}^{2m(n)+1} \lambda_j a_1 - \sum_{j=1}^{2m(n)+1} \lambda_j a_j \right\| \\
&\leq \sum_{i=1}^{2m(n)+1} \|\mu_i (b_i - a_1)\| + \sum_{j=2}^{2m(n)+1} \|\lambda_j (a_1 - a_j)\| \\
&\leq \sum_{i=1}^{2m(n)+1} \|b_i - a_1\| + \sum_{j=2}^{2m(n)+1} \|a_1 - a_j\| \\
&\leq (4m(n)+1) \operatorname{diam} A.
\end{aligned}$$

Consequently,

$$\operatorname{diam} \operatorname{co} A \leq (4m(n)+1) \operatorname{diam} A.$$

(ii) Let $\{U_s, a_s\}_{s\in S}$ be a Dugundji system for $X \setminus \hat{G}_n$, (see [1]) and $\{b_s\}_{s\in S}$ be a locally finite partition of unity inscribed into $\{U_s\}_{s\in S}$. We define $r : X \longrightarrow \hat{G}_n$ by Dugundji formula

$$r(x) = \begin{cases} x & \text{if } x \in \hat{G}_n \\ \sum_{s\in S} b_s(x) a_s & \text{if } x \in X \setminus \hat{G}_n \end{cases}$$

Then $r : X \longrightarrow \hat{G}_n$ is a continuous retraction (see [1]). Let us show that r satisfies the required condition.

Since $\{b_s\}_{s\in S}$ is a locally finite partition of unity of $X \setminus \hat{G}_n$, for each $x \in X \setminus \hat{G}_n$ there is a finite set $S(x) \subset S$ and an open neighbourhood $O(x)$ of x such that $b_s(x) = 0$ for all $y \in O(x)$ iff $s \in S \setminus S(x)$. Thus,

$$r(x) = \sum_{s\in S} b_s(x) a_s = \sum_{s\in S(x)} b_s(x) a_s.$$

Let $s_0 \in S(x)$. Using the property of Dungundji system and (i) we get

$$\begin{aligned}
\|r(x) - x\| &= \Big\| \sum_{s\in S(x)} b_s(x) a_s - x \Big\| \\
&= \Big\| \sum_{s\in S(x)} b_s(x) a_s - a_{s_0} + a_{s_0} - x \Big\| \\
&\le \Big\| \sum_{s\in S(x)} b_s(x) a_s - a_{s_0} \Big\| + \|x - a_{s_0}\| \\
&\le \Big\| \sum_{s\in S(x)} b_s(x) a_s - a_{s_0} \Big\| + 2\|x - \hat{G}_n\| \\
&\le \operatorname{diam} \operatorname{co} \{a_s : s \in S(x)\} + 2\|x - \hat{G}_n\| \\
&\le (4m(n) + 1) \operatorname{diam} \{a_s : s \in S(x)\} + 2\|x - \hat{G}_n\| \\
&\le (4m(n) + 1) 4\|x - \hat{G}_n\| + 2\|x - \hat{G}_n\| \\
&= 2(2(4m(n) + 1) + 1)\|x - \hat{G}_n\| \le 22m(n)\|x - \hat{G}_n\|.
\end{aligned}$$

Here we have used the obvious assumption $m(n) = \operatorname{Card} G_n \ge 1$. The lemma is proved.

We recall that a convex set M in a linear metric space is said to be admissible iff for every compact subset A of M and for every $\varepsilon > 0$

there is a continuous map f from A into a finite dimensional subset of M such that $\|f(x) - x\| < \varepsilon$ for every $x \in A$.

The following result is due to Klee [4].

3.4. Proposition. *Every admissible compact set is an AR.*

Thus in order to prove Theorem 3.1, by the Proposition 3.4, it suffices to show

3.5. Claim. *K is admissible.*

Proof. Let us prove the following more general fact. For every $\varepsilon > 0$ there exists a continuous map f from K into a finite dimensional subset of K such that $\|f(x) - x\| < \varepsilon$ for every $x \in K$.

In fact, for any $\varepsilon > 0$ we take a number $n \in \mathbb{N}$ such that

$$(11) \qquad 110 \sum_{i=n+1}^{\infty} m(i-1)\delta_i^p < \varepsilon$$

By Lemma 3.3 there exists a continuous retraction $f : X \longrightarrow \hat{G}_n$ such that

$$(12) \quad \|f(x) - x\| \leq 22m(n)\|x - \hat{G}_n\| \quad \text{for every } x \in X.$$

Let us show that $\|f(x) - x\| < \varepsilon$ for every $x \in C$. Assume that $x \in \hat{G}_{n+1}$. Then there exist $\alpha_i \geq 0$, $x_i \in G_{n+1}^+$ with $\sum_{i=1}^{4} \alpha_i = 1$ such that $x = \alpha_1 x_1 - \alpha_2 x_2 + i\alpha_3 x_3 - i\alpha_4 x_4$.

Since $x_i \in G_{n+1}^+$, there exist $\lambda_j^i \geq 0$, $i = 1, \ldots, 4$;

$j = 1, \ldots, m(n)$ with $\sum_{j=1}^{m(n)} \lambda_j^i \leq 1$ and $y_j^i \in \operatorname{co} F(a_j^n, \varepsilon_{n+1})$

such that $x_i = \sum_{j=1}^{m(n)} \lambda_j^i y_j^i$ for $i = 1, \ldots, 4$.

By (7) for every i, j with $i = 1, ..., 4$; $j = 1, \ldots, m(n)$ there exists $\mu_j^i \in [0, 1]$ such that $\|y_j^i - \mu_j^i a_j^n\|_* < \varepsilon_{n+1}$.

Putting

$$z_i = \sum_{j=1}^{m(n)} \lambda_j^i \mu_j^i a_j^n, \quad i = 1, \dots, 4,$$

we get $z_i \in G_n^+$ and

$$\|x_i - z_i\| \le \sum_{j=1}^{m(n)} \|y_j^i - \mu_j^i a_j^n\| < m(n)\varepsilon_{n+1}^p.$$

Let us put

$$y = \alpha_1 z_1 - \alpha_2 z_2 + i\alpha_3 z_3 - i\alpha_4 z_4.$$

Then $y \in \hat{G}_n$,

$$\|x - y\| \le \sum_{i=1}^{4} \|x_i - z_i\| < 4m(n)\varepsilon_{n+1}^p = 4\delta_{n+1}^p$$

and $\|x - \hat{G}_n\| < 4\delta_{n+1}^p$ for every $x \in \hat{G}_{n+1}$.

Let x be an arbitrary point of K. We take $y \in \bigcup_{n=0}^{\infty} \hat{G}_n$ such that $\|x - y\|_* < \delta_{n+1}$. Assume that $y \in \hat{G}_{n+k}$. Then we have

$$\|x - \hat{G}_n\| < \|x - y\| + \|y - \hat{G}_n\| < \delta_{n+1}^p + 4\delta_{n+1}^p + \dots + 4\delta_{n+k}^p < 5 \sum_{i=n+1}^{\infty} \delta_i^p.$$

From (11) and (12) it follows that

$$\|f(x) - x\| \le 22m(n)\|x - \hat{G}_n\| < 110m(n) \sum_{i=n+1}^{\infty} \delta_i^p$$

$$< 110 \sum_{i=n+1}^{\infty} m(i-1)\delta_i^p < \varepsilon$$

The claim is proved. This completes the proof of Theorem 3.1.

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Departement of Mathematics
Pedagogical Institute of Vinh
Vinh, Vietnam

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