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A Short Communication

TWO COUNTEREXAMPLES FOR THE TAUTNESS AND THE HARTOGS EXTENSIONESS[1]

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First at all we recall some definitions and relevant properties.

Definition 1. A complex space M is called taut [2] if whenever N is a complex space and $f_i : N \to M$ is a sequence of holomorphic maps, then either there exists a subsequence which is compactly divergent or a subsequence which converges uniformly on compact subsets to a holomorphic map $f : N \to M$.

It suffices that this condition should holds when $N = D$, where D is the unit disk in C, see [1].

Definition 2. A complex space X is said to have the Hartogs extension property (shortly HEP) if every holomorphic map from a Riemann domain Ω over a Stein manifold into X can be extended to the envelope of holomorphy $\widehat{\Omega}$ of Ω.

In [4] we proved the following:

Theorem A. *Let X be a complex space. Then X is taut if and only if all normalizations $\theta_i : \widetilde{S^{-i}X} \to \widetilde{S^i X}$ are taut, where $S^0(X) = X$, $S^1(X) = S(X)$ and $S(X)$ is the singular locus of X, $S^i(X) = S(S^{i-1}X)$ for all $i \geq 2$.*

In [5] we proved the following:

Theorem B. *Let $\theta : X \to Y$ be a proper holomorphic surjection between complex spaces such that $\theta^{-1}(y)$ has the HEP for all $y \in Y$. Assume that X is a Kahler space. Then X has the HEP if Y has the HEP.*

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The aim of the present paper is to give two counterexamples which are relative to the above problems.

Proposition 1. *There exists a complex space X such that its normalization $\widetilde{X}$ is taut but X is not taut.*

Proof. In $\boldsymbol{CP}^2$ we consider the smooth hyperbolic curve Γ given by the equation $x_0^5 + x_1^4 x_2 + x_2^5 = 0$ (see Zaidenberg [6]). Then the curve

$$\Gamma_0 = \Gamma \setminus \{x_2 = 0\} = \{x_0^5 + x_1^4 + 1 = 0\}$$

is complete hyperbolic. Put

$$M = \Gamma_0' \times \Gamma_0'' \subset \boldsymbol{C}^4 ,$$

where $\Gamma_0' \subset \boldsymbol{C}^2_{x',y'}$ and $\Gamma_0'' \subset \boldsymbol{C}^2_{x'',y''}$ are two copies of Γ_0. Clearly M is taut. Consider the finite proper holomorphic map $\varphi : M \to \boldsymbol{C}^5$ given by the following

$$\varphi(x', y', x'', y'') = (x', x'y', x'x'', y', y'') .$$

The image $X = \varphi(M)$ of φ is an affine subset in $\boldsymbol{C}^5$. It is easy to see that X contains the complex line

$$\{x_1 = x_2 = x_3 = 0,\ x_4 = \sqrt[4]{-1},\ x_5\} .$$

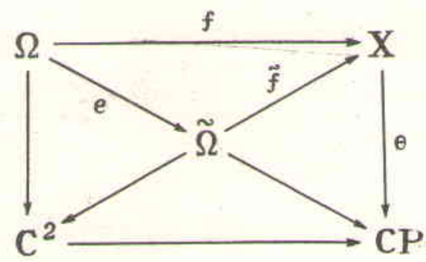
Hence X is not hyperbolic and hence X is not taut. Let $\theta : \widetilde{X} \to M$ be the normalization of M. From Theorem A it follows that $\widetilde{X}$ is taut. A direct calculation shows that the map $\varphi_0 \theta : \widetilde{X} \to X$ is the normalization of X and proves Proposition 1.

Remarks. In [3] Kobayashi stated that if a complex space X is hyperbolic then its normalization $\widetilde{X}$ is hyperbolic. At the same time he posed the question on the truth of the inverse statement. The complex space X, which is constructed in Proposition 1 is not hyperbolic but its normalization $\widetilde{X}$ is hyperbolic. Therefore the answer to Kobayashi's question is negative.

Proposition 2. *There exists a proper holomorphic surjection θ from a complex space X onto a complex space Y such that Y has the HEP and $\theta^{-1}(y)$ has the HEP for all $y \in Y$, but X has not HEP.*

Proof. Consider the canonical holomorphic map θ from the Hopf surface $X := \boldsymbol{C}^2 \setminus \{0\}/(z \sim 2z)$ onto $\boldsymbol{CP}^1$. Then $\theta^{-1}(y) \cong \boldsymbol{C} \setminus \{0\}/(z \sim 2z)$ for all $y \in \boldsymbol{CP}^1$. Since the Hopf surface contains no rational curves, the universal cover $\boldsymbol{C} \setminus \{0\}/(z \sim 2z)$ is a Stein manifold. It follows that every fibre $\theta^{-1}(y) \cong \boldsymbol{C} \setminus \{0\}/(z \sim 2z)$ has the HEP.

We check that there exists a non-empty open subset V of $\boldsymbol{CP}^1$ $(V \neq \boldsymbol{CP}^1)$ such that $\theta^{-1}(V)$ has not the HEP. In the converse case, consider the following commutative diagram.



where $\Omega = \boldsymbol{C}^2 \setminus \{0\}$ and $\tilde{f}$ is the canonical extension of f to $\tilde{\Omega}$, the locally biholomorphic envelope of Ω and $f : \Omega \to X$ is the canonical map. By the hypothesis, the map $\theta \circ \tilde{f} : \tilde{\Omega} \to \boldsymbol{CP}^1$ is locally pseudoconvex. Since X is homogeneous compact, the map $e : \Omega \to \tilde{\Omega}$ and hence $f : \Omega \to X$ is extended holomorphically to $\boldsymbol{C}^2$. This is impossible.

Remark. The counterexample in Proposition 2 showed that the assumption in Theorem B that the complex space X is a Kahler space may not be omitted.

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