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THE ADAPTIVE ALGORITHM FOR MINIMIZING THE SUM OF ABSOLUTE VALUES OF LINEAR FUNCTIONS[1]

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Abstract. *This paper is devoted to developing the adaptive algorithm for minimizing a sum of linear functions. The algorithm is based on the notion of support plans which was firstly proposed by R. Gabasov and F. M. Kirillova in linear programming. In this paper for the problem of minimizing the sum of linear functions a support plan is defined, the optimal criterion of a support plan is proved and the algorithm is described in great detail.*

1. INTRODUCTION

We consider the following problem

$$f(x) = \sum_{k\in K} |c_k' x + \alpha_k| \longrightarrow \min, \qquad (1)$$

$$b_* \leq Ax \leq b^*, \;\; d_* \leq x \leq d^*,$$

where A is $m \times n$ matrix; $x, d_*, d^* \in R^n; b_*, b^* \in R^m$. Problem (1) is nonsmooth extremum problem. Problems of this sort arise frequently in many questions of approximation, optimization and decision making. By using the auxiliary variables we can transform (1) into a traditional linear programming problem. But in doing so we can increase dimensions and lose characteristics of the problem. In this paper, using approaches previously studied in [2 3] we shall build an algorithm to solve directly the problem (1).

Define the index sets $I = \{1, \ldots, m\}, J = \{1, \ldots, n\}$. Pair $\{I_{0n}, J_{0n}\} \subseteq \{I, J\}$ is called a support of conditions if $det A(I_{0n}, J_{0n}) \neq 0$.

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Denote $C(K,J) = \{c'_k, k \in K\}$ and $D(K,J) = C(K,J_{0n})A_{0n}^{-1}A(I_{0n},J) - C(K,J)$. Pair $\{K_{0x}, J_{0x}\} \subseteq \{K, J\backslash J_{0n}\}$ is called a support of aim functional if $det D(K_{0x}, J_{0x}) \neq 0$. A plan x with a support $F_{0z} = \{I_{0n}, J_{0n}, K_{0x}, J_{0x}\}$ is called a support plan and denoted by $\{x, F_{0z}\}$.

The support plan $\{x, F_{0z}\}$ is said to be non-degenerated if $d_{*j} < x_j < d_j^*$ for all $j \in J_{0n} \cup J_{0x}$; $b_{*i} < A_i x < b_i^*$ for all $i \in I_n = I\backslash I_{0n}$ and $f_k(x) = c'_k x + \alpha_k \neq 0$ for all $k \in K_n = K\backslash K_{0x}$.

Consider a disintegration: $K_n = K_n^+ \cup K_n^-$ where $K_n^+ = \{k \in K_n : f_k(x) = c'_k x + \alpha_k \geq 0\}, K_n^- = \{k \in K_n : f_k(x) = c'_k x + \alpha_k \leq 0\}, K_n^- \cap K_n^+ = \emptyset$. (If $f_k(x) = 0$ then we can put k in K_n^+ or K_n^- arbitrarily).

Let

$$\begin{aligned} e(K_n) &= \{e_k, k \in K_n : e_k = 1 \text{ if } k \in K_n^+ \text{ and } e_k = -1 \text{ if } k \in K_n^-\}, \\ u'(K_{0x}) &= e'(K_n)D(K_n, J_{0x})D_{0x}^{-1}, \\ v'(I_{0n}) &= [e'(K_n), -u'(K_{0x})]C(K, J_{0n})A_{0n}^{-1}, \\ \Delta'(J_n) &= -e'(K_n)D(K_n, J_n) + u'(K_{0x})D(K_{0x}, J_n), \\ J_n &= J\backslash(J_{0n} \cup J_{0x}). \end{aligned}$$

The support plan $\{x, F_{0z}\}$ is said to be adequate if $f_k(x) = 0$, $-1 \leq u_k \leq 1$ for all $k \in K_{0x}$. We note that the support plan $\{x, F_{0z}\}$ with $K_{0x} = J_{0x} = \emptyset$ is an adequate support plan.

Let x^0 is an optimal plan of (1). A plan x is said to be $\varepsilon-$ optimal of (1) if $f(x) - f(x^0) < \varepsilon$.

2. OPTIMAL CRITERIA

Define

$$\beta = \sum_{\Delta_j > 0} \Delta_j(x_j - d_{*j}) + \sum_{\Delta_j < 0} \Delta_j(x_j - d_j^*) + \sum_{v_i > 0} v_i(A_i x - b_{*i}) + \sum_{v_i < 0} v_i(A_i x - b_i^*). \quad (2)$$

Theorem 1. *Assume that $\{x, F_{0z}\}$ is an adequate support plan. If $\beta \leq \varepsilon$ then $\{x, F_{0z}\}$ is a ε-optimal plan.*

Proof. We consider the following problem which is dual to problem (1):

$$\varphi(\lambda) = -b^{*'}s + b'_*t - d^{*'}w + d'_*r + \alpha'\xi \longrightarrow \max, \tag{3}$$
$$A'(s-t) + w - r + C'\xi = 0,$$
$$-1 \leq \xi_k \leq 1 \quad \text{for all } k \in K,$$
$$s_i \geq 0, t_i \geq 0 \quad \text{for all } i \in I,$$
$$w_j \geq 0, r_j \geq 0 \quad \text{for all } j \in J,$$

where $\xi \in R^K; w, r \in R^n; s, t \in R^m$.

A plan $\lambda = \{s, t, w, r, \xi\}$ of the problem (3) may be chosen as folls. We set, for all $i \in I_{0n}$,

$$t_i = 0, \quad s_i = -v_i \quad \text{if } v_i \leq 0,$$
$$t_i = v_i, \quad s_i = 0 \quad \text{if } v_i > 0,$$
$$t(I_n) = s(I_n) = 0,$$

for all $j \in J_n$,

$$r_j = \Delta_j, \quad w_j = 0 \qquad \text{if } \Delta_j \geq 0,$$
$$r_j = 0, \quad w_j = -\Delta_j \qquad \text{if } \Delta_j < 0,$$
$$r(J_{0n} \cup J_{0x}) = 0, \quad w(J_{0n} \cup J_{0x}) = 0,$$

and, finally, for all $k \in K$:

$$\xi_k = \begin{cases} e_k & \text{if } k \in K_n, \\ -u_k & \text{if } k \in K_{0x}. \end{cases}$$

It follows from (2) that

$$\begin{aligned}
\beta &= \Delta' x - \sum_{\Delta_j > 0} \Delta_j d_{*j} - \sum_{\Delta_j < 0} \Delta_j d_j^* - \sum_{v_i > 0} v_i b_{*i} - \sum_{v_i > 0} v_i b_i^* + v'Ax \\
&= (\Delta' + v'A)x - b'_*t + b^{*'}s - d'_*r + d^{*'}w \\
&= e'(K_n)(C(K_n, J)x(J) + \alpha(K_n)) - u'(K_{0x})(C(K_{0x}, J)x(J) \\
&\qquad + \alpha(K_{0x})) - \varphi(\lambda) \\
&= f(x) - \varphi(\lambda).
\end{aligned}$$

Let x^0, λ^0 be optimal plans of problems (1) and (3). Then

$$\beta = (f(x) - f(x^0)) + (\varphi(\lambda^0) - \varphi(\lambda)) \tag{4}$$

Therefore, from $\beta \le \varepsilon$ it easily implies that $f(x) - f(x^0) \le \varepsilon$, completing the proof.

Assume $\{x, F_{0z}\}$ is an adequate support plan. Denote $K_n^0 = \{k \in K_n : f_k(x) = 0\}$. Let Δx be so small that $x + \Delta x$ is a plan of (1) and $f_k(x + \Delta x) \ge 0$ for $k \in K_n^+ \backslash K_n^0$ and $f_k(x + \Delta x) \le 0$ for $k \in K_n^- \backslash K_n^0$. We can then write the increment Δf of the aim function f as

$$\begin{aligned} \Delta f &= f(x + \Delta x) - f(x) \\ &= \Delta' \Delta x + v' A \Delta x + \sum_{k \in K_n^0} (|c_k' \Delta x| - e_k c_k' \Delta x) \\ &\quad + \sum_{k \in K_{0z}} (|c_k' \Delta x| - u_k c_k' \Delta x) \end{aligned} \tag{5}$$

Theorem 2. *If $\{x, F_{0z}\}$ is an adequate support plan and*

$$\begin{cases} v_i \ge 0 & \text{when } A_i x = b_{*i}, \\ v_i = 0 & \text{when } b_{*i} \le A_i x \le b_i^*, \\ v_i \le 0 & \text{when } A_i x = b_i^*, \quad \forall i \in I_{0n}, \end{cases} \tag{6}$$

$$\begin{cases} \Delta_j \ge 0 & \text{when } x_j = d_{*j}, \\ \Delta_j = 0 & \text{when } d_{*j} \le x_j \le d_j^*, \\ \Delta_j \le 0 & \text{when } x_j = d_j^*, \quad \forall j \in J_n, \end{cases} \tag{7}$$

then x is an optimal plan of problem (1). Conversely, if $\{x, F_{0z}\}$ is an adequate support optimal non-degenerated plan then the conditions (6), (7) are satisfied.

Proof. Assuming the conditions (6), (7) fulfilled, implies together with (2) that $\beta = 0$. We invoke Theorem 1 to deduce that $f(x) - f(x^0) = 0$, which means that x is an optimal plan of problem (1).

We now turn to the proof of the necessity. Given a non-degenerated adequate support optimal plan, we can choose Δx so small that $\{x + \Delta x, F_{0z}\}$ is an adequate support plan. Set $z(I_{0n}) = A(I_{0n}, J)\Delta x(J)$, where

$$b_*(I_{0n}) - A(I_{0n}, J)x \le z(I_{0n}) \le b^*(I_{0n}) - A(I_{0n}, J)x \tag{8}$$

and choose $\Delta x(J_n)$ so that

$$d_*(J_n) - x(J_n) \le \Delta x(J_n) \le d^*(J_n) - x(J_n)\,. \tag{9}$$

Choose

$$\begin{aligned}\Delta x(J_{0x}) &= D_{0x}^{-1}\{C(K_{0x}, J_{0n}).A_{0n}^{-1}.Z_{0n} - D(K_{0x}, J_n)\Delta x(J_n)\},\\ \Delta x(J_{0n}) &= A_{0n}^{-1}\{Z_{0n} - A(I_{0n}, J\backslash J_{0n})\Delta x(J\backslash J_{0n})\}.\end{aligned} \tag{10}$$

Since $\{x, F_{0z}\}$ is a support non-degenerate plan, it follows that $K_n^0 = \emptyset$. By (10), we get $c'_k\Delta x = 0$ for all $k \in K_{0x}$. Hence

$$\Delta f = f(x + \Delta x) - f(x) = \sum_{j\in J_n} \Delta_j \Delta x_j + \sum_{i\in I_{0n}} v_i z_i \,.$$

Assuming the condition (6) is violated, we shall choose $\Delta x(J_n) = 0$ and $z(I_{0n})$ from (8) so small that $\Delta f = \sum v_i z_i < 0$. Assuming the condition (7) is violated, we shall choose $z(I_{0n}) = 0$ and $\Delta x(J_n)$ from (9) so small that $\Delta f = \sum \Delta_j \Delta x_j < 0$. Hence, $f(x + \Delta x) < f(x)$, contradicting the hypothesis. It follows that x is an optimal plan of problem (1) and completes the proof.

3. ALGORITHM

Assuming a $\{x, F_{0z}\}$ is an adequate support plan and $\beta > \varepsilon$, we shall build a new adequate support plan $\{\overline{x}, \overline{F_{0z}}\}$ so that $\overline{\beta} < \beta$. It follows from (4) that we can decrease β by decreasing function $f(\overline{x}) \leq f(x)$ and increasing dual function $\varphi(\overline{\lambda}) \geq \varphi(\lambda)$. A new support $\overline{F_{0z}}$ will be built by solving the dual problem (3).

The new plans $\overline{x}$ and $\overline{\lambda}$ are defined by

$$\overline{x} = x + \theta\ell, \ \ \overline{\lambda} = \lambda + \sigma\tau, \tag{11}$$

where ℓ is a decreasing direction of function $f(x)$, and τ is an increasing direction of dual function $\varphi(\lambda)$.

Direction ℓ and step θ will be found as follows: for all $j \in J_n$, put $\ell_j = d_{*j} - x_j$ if $\Delta_j > 0$; $\ell_j = d_j^* - x_j$ if $\Delta_j < 0$ and $\ell_j = 0$ if $\Delta_j = 0$. Then, we choose vector $wd(I_{0n})$ by setting for all $i \in I_{0n}$, $wd_i = b_{*i} - A_i x$ if $v_i > 0$; $wd_i = b_i^* - A_i x$ if $v_i < 0$ and $wd_i = 0$ if $v_i = 0$. Finally, one put

$$\begin{aligned}\ell(J_{0x}) &= D_{0x}^{-1}\{C(K_{0x}, J_{0n}).A_{0n}^{-1}.wd_{0n} - D(K_{0x}, J_n)\ell(J_n)\},\\ \ell(J_{0n}) &= A_{0n}^{-1}\{wd_{0n} - A(I_{0n}, J\backslash J_{0n})\ell(J\backslash J_{0n})\}.\end{aligned}$$

The step θ is chosen by

$$\theta = \min\{\theta_{k_0},\ \theta_{i_0},\ \theta_{j_0}\},$$

where

$$\theta_{i_0} = \min\{\theta_i, i \in I_{0n}\},$$

$$\theta_i = \begin{cases} (b_i^* - A_i x)/A_i\ell & \text{if } A_i\ell > 0, \\ (b_{*i} - A_i x)/A_i\ell & \text{if } A_i\ell < 0, \\ \infty & \text{if } A_i\ell = 0; \end{cases}$$

$$\theta_{j_0} = \min\{\theta_j, j \in J_{0n} \cup J_{0x}\},$$

$$\theta_j = \begin{cases} (d_j^* - x_j)/\ell_j & \text{if } \ell_j > 0, \\ (d_{*j} - x_j)/\ell_j & \text{if } \ell_j < 0, \\ \infty & \text{if } \ell_j = 0; \end{cases}$$

$$\theta_{k_0} = \min\{\theta_k, k \in K_n\},$$

$$\theta_k = \begin{cases} -f_k(x)/c_k'\ell & \text{if } e_k c_k'\ell < 0\,, \\ \infty & \text{if } e_k c_k'\ell \geq 0. \end{cases}$$

Then, for a new adequate support plan $\{\overline{x}, F_{0x}\}$, $\overline{x} = x + \theta\ell$, we have $f(x) - f(\overline{x}) = \theta\beta$. It follows from Theorem 1 that if $\theta = 1$ then $\overline{x} = x + \theta\ell$ is an optimal plan of problem (1) and if $\beta(1-\theta) \leq \varepsilon$ then $\overline{x}$ is an ε-optimal plan of (1).

Assume that $\beta(1-\theta) > \varepsilon$. Find now a new support and the new evaluating vectors. The new evaluating vectors will be found as follows:

$$\overline{\Delta}(J) = \Delta(J) + \sigma\tau(J),\quad \overline{u}(K_{0x}) = u(K_{0x}) + \sigma\tau(K_{0x}),$$
$$\overline{v}(I) = v(I) + \sigma\tau(I),\qquad \overline{e}(K_n) = e(K_n) + \sigma\tau(K_n),$$

where directions τ will be found by increasing function $\varphi(\lambda)$ of the dual problem (3) and σ is a dual step.

The directions τ will be found, according to the followings cases:

1) If $\theta = \theta_{k_0}$, $k_0 \in K_n$, then

$$\tau_{k_0} = -e_{k_0},\quad \tau(K_n \backslash k_0) = 0,\quad \tau(J_{0x}) = 0,$$
$$\tau(J_{0n}) = 0,\quad \tau(I_n) = 0,\quad \tau(K_{0x}) = \tau_{k_0} D(k_0, J_{0x}) D_{0x}^{-1},$$
$$\tau(I_{0n}) = \tau_{k_0}\left[C(k_0, J_{0n}) - D(k_0, J_{0x}) D_{0x}^{-1} C(K_{0x}, J_{0n})\right] A_{0n}^{-1},$$
$$\tau(J_n) = \tau_{k_0}\left[D(k_0, J_{0x}) D_{0x}^{-1} D(K_{0x}, J_n) - D(k_0, J_n)\right],$$

$$\Delta^{(q)}(J) = \Delta^{(q-1)}(J) + \varrho_{q-1}\tau(J),$$
$$v^{(q)}(I) = v^{(q-1)}(I) + \varrho_{q-1}\tau(I),$$
$$u^{(q)}(K_{0x}) = u^{(q-1)}(K_{0x}) + \varrho_{q-1}\tau(K_{0x}),$$
$$e^{(q)}(K_n) = e^{(q-1)}(K_n) + \varrho_{q-1}\tau(K_n),$$

$$\gamma_0 = \begin{cases} (1-\theta)|c_{k_0}\ell| & \text{in case 1,} \\ (1-\theta)|A_{i_0}\ell| & \text{in case 2,} \\ (1-\theta)|\ell_{j_0}| & \text{in case 3, 4.} \end{cases}$$

The step ϱ_{q-1} is a minimal value of $\varrho > 0$, so that one of the following conditions will happen:

$$\Delta_j^{(q-1)} + \varrho\tau_j = 0, \qquad v_i^{(q-1)} + \varrho\tau_i = 0; \tag{13}$$

$$u_k^{(q-1)} + \varrho\tau_k = \begin{cases} \text{-sign}u_k & \text{if } u_k\tau_k < 0, \\ \text{sign}u_k & \text{if } u_k\tau_k > 0, \\ \text{sign}\tau_k & \text{if } u_k = 0; \end{cases} \tag{14}$$

$$e_{k_0}^{(q-1)} + \varrho\tau_{k_0} = -2e_{k_0} \quad \text{if} \quad \theta = \theta_{k_0}, k_0 \in K_n \tag{15}$$

Searching along the directions τ will be stoped when for some s we have $\gamma_s \le 0, \gamma_{s-1} > 0$ or one of the conditions (14), (15) happens. Then, for the dual step σ, we have

$$\sigma = \sum_{q=0}^{s-1} \varrho_q$$

and the marking index is one of the indexes $j_* \in J_n, i_* \in I_{0n}, k_* \in K$ and so one of the conditions (13), (14), (15) is fulfilled. After the dual step, the aim function φ of (3) increases in the value

$$\sum_{q=0}^{s-1} \varrho_q\gamma_q$$

and the new evaluating norm is

$$\overline{\beta} = \beta(1-\theta) - \sum_{q=0}^{s-1} \varrho_q\gamma_q$$

2) $i_* \in I_{0n}$. If $A_{0n}^{-1}(j_0, i_*) \neq 0$ then $\overline{I}_{0n} = I_{0n} \backslash i_*, \overline{J}_{0n} = J_{0n} \backslash j_0$. Conversely, choose $j_+ \in J_{0n}$ so that $A_{0n}^{-1}(j_+, i_*) \neq 0$ and choose $j_- \in J_{0x}$ so that $A_{0n}^{-1}(j_0, I_{0n})A(I_{0n}, j_-) \neq 0$ and $D_{0x}^{-1}(j_-, K_{0x}) \times C(K_{0x}, J_{0n})A_{0n}^{-1}(J_{0n}, i_*) \neq 0$. A new support will be build in the following order: $\tilde{I}_{0n} = I_{0n} \backslash i_*, \tilde{J}_{0n} = J_{0n} \backslash j_+, \overline{J}_{0x} = (J_{0x} \backslash j_-) \cup j_+, \overline{J}_{0n} = (\tilde{J}_{0n} \backslash j_0) \cup j_- \overline{I}_{0n} = \tilde{I}_{0n}$.

3) $k_* \in K_{0x}$. Choose $j_+ \in J_{0x}$ so that $A_{0n}^{-1}(j_0, I_{0n})A(I_{0n}, j_+) \neq 0$ and $D_{0x}^{-1}(j_+, k_*) \neq 0$. Then $\overline{K}_{0x} = K_{0x} \backslash k_*, \overline{J}_{0x} = J_{0x} \backslash j_+, \overline{J}_{0n} = (J_{0n} \backslash j_0) \cup j_+$.

(iv) The cases $\theta = \theta_{j_0}, j_0 \in J_{0x}$. The construction of the new supports is most simplified in this case:

1) If $j_* \in J_n$ we put $\overline{J}_{0x} = (J_{0x} \backslash j_0) \cup j_*$.

2) If $i_* \in I_{0n}$, we choose $j_+ \in J_{0n}$ so that $A_{0n}^{-1}(j_+, i_*) \neq 0$ and then we put $\overline{I}_{0n} = I_{0n} \backslash i_*, \overline{J}_{0n} = J_{0n} \backslash j_+, \overline{J}_{0x} = (J_{0x} \backslash j_0) \cup j_+$.

3) If $k_* \in K_{0x}$ we put $\overline{K}_{0x} = K_{0x} \backslash k_*, \overline{J}_{0x} = J_{0x} \backslash j_0$.

Thus, in each of cases, the new supporting matrices $\overline{A_{0n}^{-1}}, \overline{D_{0x}^{-1}}$ can be constructed by using old supporting matrices A_{0n}^{-1}, D_{0x}^{-1} or intermediary matrices $\tilde{A}_{0n}^{-1}, \tilde{D}_{0x}^{-1}$ and by the formulas analogous to the formula (16).

The iterative step $\{\overline{x}, \overline{F}_{0x}\}$ of algorithm is said to be non-degenerate if $\theta + \sigma > 0$ and degenerate if otherwise. We can prove the following theorem (Ref. [2], [3]).

Theorem 3. *After a finite number of interative steps, the stated algorithm will give an optimal plan (or ε-optimal plan) if the number of degenerate interative steps is finite.*

4. COMPUTATIONAL EXPERIMENTS

The described algorithm was coded in Turbo Pascal and has been run on IBM PC AT 80286. In the following two tables we present the results of our experiments.

Table 1.

m	n	K	f_0	f_*	Iteration	Time
15	25	15	321.70	69.42	7	1.21
15	25	25	147.24	21.70	4	0.93
15	25	35	196.25	34.00	5	1.70
25	25	15	175.92	46.12	6	0.93
25	25	25	9250.25	19.07	17	5.87
25	25	35	12961.55	22.40	22	10.48
35	25	15	5553.02	9.28	12	2.80
35	25	25	9251.15	17.65	17	6.09
35	25	35	12966.35	24.28	22	10.75
35	35	15	5758.06	6.46	12	3.95
35	35	25	9590.57	17.21	17	8.67
35	35	35	13423.37	6.21	22	15.30
35	45	15	2927.96	6.62	11	3.29
35	45	25	4881.79	2.03	16	6.91
35	45	35	6838.13	23.13	21	12.23

The meanings of some symbols used in Table 1 are as follows:

– f_0: the value of the aim function f on the start plan

– f_*: the value of the aim function f on the $\varepsilon-$ optimal plan with $\varepsilon = 10^{-8}$.

Table 2.

No	m	n	K	Iteration	Time
1	10	15	10	3.0	0.34
2	10	25	15	2.4	0.66
3	10	30	15	2.2	0.72
4	10	45	15	2.0	1.00
5	10	15	30	2.0	1.00

No	m	n	K	Iteration	Time
6	15	20	15	2.5	1.00
7	20	25	30	3.0	1.05
8	25	30	30	3.3	1.27
9	25	30	35	4.7	1.30
10	25	35	35	4.8	1.39
11	30	35	35	5.0	1.48
12	30	40	35	5.0	1.88
13	30	45	35	5.0	3.20
14	35	40	35	5.2	3.74
15	35	45	35	11.2	7.98

The elements of the matrices C, A and vectors d^*, d_*, b^*, b_* are randomly generated in the interval $[-100, 100]$. For each size, five problems were tested.

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