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A Short Communication

A NOTE ON QUASI–FROBENIUS RINGS

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1. This is a brief report on a recent paper of the authors [6] under the same title.

A ring R is called a *quasi–Frobenius* ring (briefly, a QF–ring), if R is right (or left) artinian, right (or left) self–injective. The class of QF–rings is an interesting generalization of semisimple rings, and the number of papers devoted to them is so large that we are unable to give all the references here. Instead we refer to Faith [2] and Kasch [7] for the basic properties of QF–rings.

Now let R be a QF–ring. Then every projective right R–module P is injective (see Faith [2, Theorem 24.20]). Hence any *closed* submodule U of P is an injective direct summand of P. In particular, U is a *non-small* module. From this it is natural to ask the question: *Which rings R have the property that all closed submodules of any projective right R–module are non–small?*

In this note we give an answer for a part of this question by proving the following theorem.

Theorem 1. *Let R be a semiperfect ring and let $R(\omega)$ denote the direct sum of ω copies of the right R–module R, where ω is the first infinite ordinal. Then the following statements are equivalent:*

(*a*) *R is a QF–ring.*

(*b*) *Every closed submodule of $R(\omega)$ is non–small.*

(*c*) *R has finite right uniform dimension and every closed uniform submodule of $R(\omega)$ is non–small.*

(*d*) *R has finite right uniform dimension, no non–zero projective right ideal of R is contained in the Jacobson radical $J(R)$ of R and every closed uniform submodule of $R(\omega)$ is a direct summand.*

The main part of the proof of Theorem 1 is to establish (d) $\Rightarrow$ (a). This has been done by showing first that $R(\omega)$ has a direct decomposition that complements direct summands. Then it follows that every local direct summand of $R(\omega)$ is a direct summand. Using this we show in the next step that $R(\omega)$ is a CS–module. Thus R is a QF–ring by [5, Corollary 2].

Since a right *continuous* semiperfect ring R is the direct sum of finitely many uniform right ideals and $J(R)$ is a *singular* right R–module which can not contain non–zero projective submodules (cf. Goodearl [4]), the following result is an immediate consequence of Theorem 1.

Corollary 2. *A right continuous semiperfect ring R is QF if and only if every closed uniform submodule of $R(\omega)$ is a direct summand.*

2. Corollary 2 improves a part of [1, Theorem 1]. Concerning Corollary 2 we should mention that there is a commutative self–injective semiperfect non–QF–ring and any right and left self–injective right perfect ring is QF (see Osofsky [8]). The question on one–sided self–injectivity for perfect rings remains open, even assuming that the ring is semiprimary. This now known as Faith's Conjecture:

A right self–injective semiprimary ring is QF.

This conjecture motivates several investigations in the area. For more about this and related questions we refer to Faith [3].

In [1, Lemma 6] is was shown that *a right quasi–continuous semiperfect ring with nil Jacobson radical is right continuous.* From this and Corollary 2 it follows that *a right quasi–continuous semiperfect ring R is QF if and only if $J(R)$ is nil and any closed uniform submodule of $R(\omega)$ is a direct summand.*

We would like to notice further that the following equivalences have been essentially established in [1] for a semiperfect ring R:

(i) *R is QF;*

(ii) *R is right self–injective and each uniform submodule of $R(\omega)$ is contained in a finitely generated submodule (of $R(\omega)$);*

(iii) *R is right continuous, $R_R \oplus R_R$ is CS and each uniform submodule of $R(\omega)$ is contained in a finitely generated submodule.*

It is easy to see that (i) $\Leftrightarrow$ (iii) *is a consequence of Corollary 2.*

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