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EXISTENCE AND RELAXATION OF SOLUTIONS OF FUNCTIONAL DIFFERENTIAL INCLUSIONS[1]

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Abstract. *For a general functional differential inclusions of the forms $\dot{x}(t) \in F(t, x_t)$ in a Banach space, where F a locally Lipschitzean multifunction of x_t, the set of solutions is proved to be non-empty and dense in the set of solutions of the convexified differential inclusion $\dot{x}(t) \in ClCoF(t, x_t)$. As an application, the obtained results are applied to the class of differential difference inclusions.*

1. NOTATIONS AND STATEMENT OF MAIN RESULTS

Throughout this paper X denotes a separable real Banach space with the norm $|\cdot|$ and X' denotes the topological dual of X. For a subset $A \subset X, CoA$ and ClA denote its convex hull and its closure, respectively. For a number $r > 0$ we set $B(A, r) = \{x \in X : d(x, A) \leq r\}$ where $d(x, A)$ is the distance from x to $A : d(x, A) = \inf\{\|x-a\| : a \in A\}$. We denote by $\mathcal{F}(X)$ the family of all nonempty closed subsets of X, and by H the Hausdorff distance in $\mathcal{F}(X)$, i.e. for $A, B \in \mathcal{F}(X), H(A, B) = \max\{\sup_{x\in A} d(x, B), \sup_{y\in A} d(y, B)\}$. Let I be a nonempty compact interval in the real line $\mathbb{R}$ which is assumed throughout to be endowed with the Lebesgue measure $\mu(dt) = dt$. Then, by $C_X(I)$ we mean the Banach space of all continuous functions $\varphi(\cdot)$ from I to X with the norm of uniform convergence $\|\varphi\|_I = \max\{|\varphi(t)| : t \in I\}$, and by $L^1_X(I)$ we denote the Banach space of (equivalent classes of) Bochner integrable functions $f(\cdot)$ from I to X with the norm

$$\|f\| = \int_I |f(t)|dt.$$

A multifunction $\Gamma : I \to 2^X$ is said to be measurable if it takes values

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in $\mathcal{F}(X)$ and, for every U open in X, the set $\{t \in I : \Gamma(t) \cap U \neq \emptyset\}$ is measurable. A single–valued function $g : I \to X$ such that $g(t) \in \Gamma(t)$ for every $t \in I$ is called a selection of Γ. We denote by S_Γ the set of all measurable selections of Γ. It is well–known that for any measurable multifunction $\Gamma : I \to \mathcal{F}(X), S_\Gamma \neq \emptyset$. Moreover, S_Γ contains a sequence $\{f^i\}_{i=1}^{\infty}$ such that $Cl\{f^i(t)\} = \Gamma(t)$ for every $t \in I$. Such a sequence of selections $\{f^i\}$ is called a Castaing representation of Γ. It is well-known that Γ is measurable iff it admits a Castaing representation; see [3] for details. We say that Γ is integrable if it is measurable and $S^1_\Gamma := S_\Gamma \cap L^1_X(I) \neq \emptyset$. It is worth noticing that all the above notions can be similarly introduced for a multifunction from a general measurable space $(\Omega, \mathcal{A}, \mu)$ into a Polish space (i.e. a complete separable metric space). Moreover, some of the auxiliary results proved in the Section 2 remain valid under such general assumptions. However, for the purpose of this paper, we restrict ourselves to the case of bounded Lebesgue measurable space (I, dt) with I being a nonempty compact interval in $\mathbf{R}$.

Now we are going to formulate the main results of this paper. Let us fix $t_0 \in \mathbf{R}, h \geq 0, T > 0$. Set $I = [t_0, t_0 + T], C = C_X([-h, 0]), \|\cdot\| = \|\cdot\|_{[-h,0]}$. For each $x(\cdot) \in C_X([t_0 - h, t_0 + T])$ and $t \in I$ denote by x_t the function in C defined by $x_t(\theta) = x(t + \theta), -h \leq \theta \leq 0$. We note that the function $t \to x_t$ mapping I into C is continuous and $\max_{t \in [t_0, t_0 + T_1]} \|x_t\| = \max_{t \in [t_0 - h,\, t_0 + T_1]} |x(t)|$ for any $T_1 \in (0, T]$. Let be given an open set D in C, a function $\bar{\varphi} \in D$ and a multifunction $F : I \times D \to \mathcal{F}(X)$. Consider the following Cauchy problem

$$\dot{x}(t) \in F(t, x_t), t \in I, \tag{1.1}$$

$$x_{t_0} = \bar{\varphi}. \tag{1.2}$$

A continuous function $x : [t_0 - h, t_0 + T] \to X$ satisfying (1.2) is called a solution of the above Cauchy problem on the interval $I = [t_0, t_0 + T]$ if it is absolutely continuous on I and satisfies (1.1) almost everywhere on this interval. We also consider the Cauchy problem for the convexified differential inclusion

$$\dot{x}(t) \in ClCoF(t, x_t), t \in I, \tag{1.3}$$

$$x_{t_0} = \bar{\varphi}. \tag{1.4}$$

The main results of this paper read as follows.

Theorem 1.1. *Let $I = [t_0, t_0 + T], h > 0$ and D be an open subset in $C_X([-h, 0])$. Let $F : I \times D \to \mathcal{F}(X)$ be a multifunction with closed images satisfying:*

(i) *for each $\varphi \in D$ the multifunction $F(\cdot, \varphi)$ is measurable and integrable on I,*

(ii) *$F(t, \cdot)$ is locally Lipschitz on D, i.e. for each $\varphi \in D$ there exist $\delta_\varphi > 0$ and $l_\varphi \in L^1(I)$ such that $H(F(t, \varphi^1), F(t, \varphi^2)) \leq l_\varphi(t)\|\varphi^1 - \varphi^2\|$ on I for all φ^1, φ^2 in $B(\varphi, \delta_\varphi)$.*

Then for any $\bar{\varphi} \in D$ and $T_1 \in (0, T]$ the set of solutions of the Cauchy problem (1.1) - (1.2) on $[t_0, t_0 + T_1]$ is dense for the topology of uniform convergence in the set of solutions of the Cauchy problem for the convexified differential inclusion (1.3) - (1.4).

Theorem 1.2. *Assume I, h, D are given as in Theorem 1, $\bar{\varphi} \in D$ and the multifunction $F : I \times D \to \mathcal{F}(X)$ satisfies:*

(i) *for each $\varphi \in D$ the multifunction $F(\cdot, \varphi)$ is measurable and $F(\cdot, \bar{\varphi})$ is integrable on I,*

(ii) *$F(t, \cdot)$ is Lipschitz in a neighborhood of $\bar{\varphi}$, i.e. there exists $\bar{\delta} > 0$ and $\bar{l} \in L^1(I)$ such that $H(F(t, \varphi^1), F(t, \varphi^2)) \leq \bar{l}(t)\|\varphi^1 - \varphi^2\|$ on I for every φ^1, φ^2 in $B(\bar{\varphi}, \bar{\delta})$.*

Then there exists $T_1 \in (0, T]$ such that the set of solutions of the Cauchy problem (1.1) - (1.2) on $[t_0, t_0 + T_1]$ is nonempty.

2. PROOF OF THE THEOREMS

First we state some auxiliary results which are more or less known in the theory of measurable multifunctions. For the sake of completeness we sketch out their proofs.

Lemma 2.1. *Let $\Gamma : I \to \mathcal{F}(X)$ be an integrable multifunction and $f_0 : I \to X$ an integrable single-valued function. Then the distance function $d(f_0(t), \Gamma(t))$ is integrable. Moreover, for each $\varepsilon > 0$ there exists an integrable selection $g \in S_\Gamma$ such that $|f_0(t) - g(t)| \leq (1 + \varepsilon) d(f_0(t), \Gamma(t))$ for every $t \in I$.*

Proof. By Theorem 8.2.13 in [2], $d(t) := d(f_0(t), \Gamma(t))$ is measurable. Since $\Gamma(t)$ is integrable, there exists $f(t) \in S_\Gamma$ which is integrable. Therefore, from $d(f_0, \Gamma(t)) \leq |f_0(t) - f(t)| \leq |f_0(t)| + |f(t)|$ it follows that $d(f_0(t), \Gamma(t))$ is integrable. Further, since the multifunction

$M(t) := f_0(t) + B(0, (1+\varepsilon)d(t))$ is measurable, it follows from the definition of the distance function that the multifunction $\Gamma(t) \cap M(t)$ has nonempty images and is measurable. Let g be its measurable selection, then, clearly, $|f_0(t) - g(t)| \leq (1+\varepsilon)d(t)$, and hence g is integrable.

Lemma 2.2. *Let D be an open set in a Banach space Y and I is a compact interval in $\mathbb{R}$. Suppose that the multifunction $F(t,\varphi) : I \times D \to \mathcal{F}(Y)$ is measurable in t and locally Lipschitz in φ. Then $F(t,\cdot)$ is globally Lipschitz in a neighborhood of each compact subset $K \subset D$. More precisely, for each compact K in D, there exists $\rho > 0$ and $l \in L^1(I)$ (both depending of K only) such that $K + B(0,\rho) \subset D$ and*

$$H(F(t,\varphi^1), F(t,\varphi^2)) \leq l(t)\|\varphi^1 - \varphi^2\|, \forall \varphi \in K, \forall \varphi^1, \varphi^2 \in B(\varphi,\rho). \tag{2.1}$$

Proof. Let $\{B(x_i, \frac{\delta_i}{3}) : i = 1,...,m\}$ be a finite covering of K with $x_i \in K$, and $F(t,\cdot)$ satisfies the Lipschitz condition in $B(x_i, \frac{\delta_i}{3})$ with "Lipschitz constant" $l_{x_i}(t)$. Setting $r = \inf_{x\in K} d(x, X\backslash D)$ we have obviously $r > 0$. Then to complete the proof, it suffices to put $l(t) = \max\{l_{x_i}(t) : i = 1,...,m\}$ and $\rho = \min\{r, \frac{\delta_i}{3} : i = 1,...,m\}$.

For a multifunction Γ from I to X, we define the convexified multifunction $ClCo\Gamma$ by setting $(ClCo\Gamma)(t) = ClCo(\Gamma(t))$ for each $t \in I$.

Lemma 2.3. *Let $I = [t_0, t_0 + T]$ and $\Gamma : I \to \mathcal{F}(X)$ be an integrable multifunction. Then for any $\varepsilon > 0$ and any integrable selection g of the convexified multifunction $ClCo\Gamma$ there exists an integrable selection f of Γ such that*

$$\max_{t\in I}\left|\int_{t_0}^{t}[f(s) - g(s)]ds\right| < \varepsilon.$$

Proof. This is immediate from the Theorem 1.1 in [4].

The following assertion plays a crucial role in the proof of our main results. Here, the notations $I, T, h, D, X, C, t_0, \bar{\varphi}, \varphi_t$ are of the same meaning as in the Section 1.

Lemma 2.4. *Let $T_1 \in (0,T], \varphi^0 \in C_X([t_0 - h, t_0 + T_1])$ be such that $\varphi^0_{t_0} = \bar{\varphi}$ and the compact $K = \{\varphi^0_t : t \in [t_0, t_0 + T_1]\}$ is contained in D. Suppose that the multifunction $F(\cdot,\varphi) : I \to \mathcal{F}(X)$ is integrable for every fixed $\varphi \in D$ and there exists $\rho = \rho_K > 0$ and $l(\cdot) = l_K(\cdot) \in L^1(I)$ such that $K + B(0,\rho) \subset D$ and (2.1) holds. Further, let $r > 0$ be such that*

$$\rho > r \exp\left(2\int_{t_0}^{t_0+T_1} l(t)dt\right)$$

and let f^0 *be an integrable selection of the multifunction* $t \to F(t, \varphi_t^0)$ *satisfying*

$$|\varphi^0(t) - \bar{\varphi}(0) - \int_{t_0}^{t} f^0(s)ds| < r, \ \forall\, t \in [t_0, t_0 + T_1]. \tag{2.2}$$

Then (1.1) *has a solution* φ *on* $[t_0, t_0 + T_1]$ *satisfying*

$$\max_{t\in[t_0,t_0+T_1]} |\varphi(t) - \varphi^0(t)| \le r \exp\left(2\int_{t_0}^{t_0+T_1} l(t)dt\right). \tag{2.3}$$

Proof. First, we note that for any $\varphi(\cdot) \in C_X([t_0 - h, t_0 + T_1])$ such that $\varphi_t \in K + B(0,\rho)$ for all $t \in I$, the multifunction $F(t, \varphi_t)$ is measurable. This follows from the facts that the multifunction $F(\cdot,\cdot)$ is Caratheodory on $I \times D$ and the function $t \to \varphi_t$ from I to C is continuous, see e.g. Theorem 8.2.8 in [2]. Further, it can be shown that the multifunction $t \to F(t, \varphi_t^0)$ is integrable. Indeed, let $\{t_i : i = 1, \cdots, n\}$ be chosen so that $t_0 < t_1 < \cdots < t_n = t_0 + T_1$ and $K \subset \cup_{i=1}^{n} B(\varphi_{t_i}^0, \rho)$. Defining $\hat{\varphi}_t = \varphi_{t_i}^0, t \in [t_{i-1}, t_i), i = 1, \cdots, n$, we have clearly $\|\varphi_t^0 - \hat{\varphi}_t\| \le \rho$ for any $t \in [t_0, t_0 + T_1]$. By the hypothesis, $F(t, \varphi_{t_i}^0)$ admits an integrable selection f^i. Setting $\hat{f}(t) = f^i(t)$ for $t \in [t_{i-1}, t_i), i = 1, \cdots, n$ we see that $\hat{f}$ is an integrable selection of $F(t, \hat{\varphi}_t)$. Since, by (2.1),

$$d(\hat{f}(t), F(t, \varphi_t^0)) \le H(F(t, \hat{\varphi}_t, F(t, \varphi_t^0)) \le \rho l(t)$$

it follows that $d(\hat{f}(t), F(t, \varphi_t^0))$ is integrable. Therefore, as in the proof of Lemma 2.1, there exists a measurable selection g of $F(t, \varphi_t^0)$ such that $|\hat{f}(t) - g(t)| \le (1 + \varepsilon) d(\hat{f}(t), F(t, \varphi_t^0))$. Thus $g(\cdot)$ is integrable and so is $F(t, \varphi_t^0)$.

To prove the lemma, we set

$$\varphi^1(t) = \begin{cases} \bar{\varphi}(0) + \int_{t_0}^{t} f^0(s)ds, & \text{if } t \in [t_0, t_0 + T_1] \\ \bar{\varphi}(t - t_0), & \text{if } t \in [t_0 - h, t_0]. \end{cases} \tag{2.4}$$

It is clear from (2.2) and (2.4) that $\|\varphi_t^1 - \varphi_t^0\| < r$ for all $t \in [t_0, t_0 + T_1]$. This yields, in particular, that $\varphi_t^1 \in K + B(0, \rho)$. Thus F is defined at (t, φ_t^1) and by (2.1),

$$H(F(t, \varphi_t^1), F(t, \varphi_t^0)) \le l(t)\|\varphi_t^1 - \varphi_t^0\| \le r l(t) \tag{2.5}$$

for every $t \in [t_0, t_0 + T_1]$. By Lemma 2.1 the multifunction $t \to F(t, \varphi_t^1)$ admits an integrable selection f^1 such that $|f^0(t) - f^1(t)| \leq 2d(f^0(t), F(t, \varphi_t^1))$, and therefore, by the definition of Hausdorff distance and (2.5), we have

$$|f^0(t) - f^1(t)| \leq 2rl(t)$$

for every $t \in [t_0, t_0 + T_1]$. Assume that we have defined functions

$$\varphi^0, f^0, \varphi^1, f^1, \cdots, \varphi^n, f^n, \varphi^{n+1}$$

such that for each $i \in \{0, \cdots, n\}$,

$$\varphi^i \in C_X([t_0 - h, t_0 + T_1]), \quad f^i \in L_X^1([t_0, t_0 + T_1]) \tag{2.6}$$

$$\|\varphi_t^{i+1} - \varphi_t^i\| \leq \frac{r}{i!}\Big(2 \int_{t_0}^t l(s)ds\Big)^i, \quad \forall t \in [t_0, t_0 + T_1]; \tag{2.7}$$

$$\varphi_t^i \in K + B(0, \rho), \quad \forall t \in [t_0, t_0 + T_1], \quad \text{and} \quad \varphi_{t_0}^i = \bar{\varphi}; \tag{2.8}$$

$$f^i(t) \in F(t, \varphi_t^i), \quad \text{a.e. on } [t_0, t_0 + T_1]; \tag{2.9}$$

$$\varphi^{i+1}(t) = \begin{cases} \bar{\varphi}(0) + \int_{t_0}^t f^i(s)ds, & \text{if } t \in [t_0, t_0 + T_1] \\ \bar{\varphi}(t - t_0), & \text{if } t \in [t_0 - h, t_0]. \end{cases} \tag{2.10}$$

and, for $i \in \{1, \cdots, n\}$,

$$|f^i(t) - f^{i-1}(t)| \leq \frac{2r}{(i-1)!} l(t) \Big(2 \int_{t_0}^t l(s)ds\Big)^{i-1}, \quad \forall t \in [t_0, t_0 + T_1]. \tag{2.11}$$

We then define φ^{n+2}, f^{n+1} as follows. First, since $\varphi_t^i \in K + B(0, \rho) \subset D$, the multifunction F is defined at (t, φ_t^i) for $i = 0, 1, \cdots, n+1$. As $f^n(t) \in F(t, \varphi_t^n)$, by virtue of Lemma 2.1 there exists an integrable selection f^{n+1} of the multifunction $t \to F(t, \varphi_t^{n+1})$ such that

$$|f^{n+1}(t) - f^n(t)| \leq 2d(f^n(t), F(t, \varphi_t^{n+1})).$$

Therefore, by definition of H and (2.1), (2.7) we can write

$$
\begin{aligned}
|f^{n+1}(t) - f^n(t)| &\le 2H(F(t,\varphi_t^{n+1}), F(t,\varphi_t^n)) \\
&\le 2l(t)\|\varphi_t^{n+1} - \varphi_t^n\| \\
&\le 2l(t)\frac{r}{n!}\left(2\int_{t_0}^{t} l(s)ds\right)^n, \quad \forall t \in [t_0, t_0 + T_1].
\end{aligned} \tag{2.12}
$$

Further, we define φ^{n+2} by setting $i = n+1$ in the formula (2.10). Then we have

$$
\varphi^{n+2}(t) - \varphi^{n+1}(t) = \begin{cases} \int_{t_0}^{t}[f^{n+1}(s) - f^n(s)]ds & \text{if } t \in [t_0, t_0 + T_1], \\ 0 & \text{if } t \in [t_0 - h, t_0]. \end{cases}
$$

Thus, by (2.12),

$$
|\varphi^{n+2}(t) - \varphi^{n+1}(t)| = \begin{cases} \frac{2r}{n!}\int_{t_0}^{t} l(s)(2\int_{t_0}^{s} l(\theta)d\theta)^n ds, & \text{if } t \in [t_0, t_0 + T_1], \\ 0 & \text{if } t \in [t_0 - h, t_0]. \end{cases}
$$

This yields, for every $\theta \in [-h, 0]$ and $t \in [t_0, t_0 + T_1]$,

$$
\begin{aligned}
|\varphi^{n+2}(\theta) - \varphi^{n+1}(\theta)| &= \frac{2r}{n!}\int_{t_0}^{t} l(s)\left(2\int_{t_0}^{s} l(\theta)d\theta\right)^n ds \\
&= \frac{r}{(n+1)!}\left(2\int_{t_0}^{t} l(s)ds\right)^{n+1}
\end{aligned}
$$

This shows that (2.7) holds for $i = n+1$. Hence we can deduce

$$
\|\varphi_t^{n+2} - \varphi_t^0\| \le \sum_{i=0}^{n+1} \|\varphi_t^{i+1} - \varphi_t^i\| \le r \exp\left(2\int_{t_0}^{t} l(s)ds\right) < \rho,
$$

which means $\varphi_t^{n+2} \subset K + B(0,\rho)$ for all $t \in [t_0, t_0 + T_1]$. Summarizing, we have shown that there exists two sequences of functions $\{\varphi^i\}_{i=0}^{\infty}, \{f^i\}_{i=0}^{\infty}$ which satisfy (2.6) – (2.11). In particular, from (2.11) we deduce that $\{f^i\}$ is a Cauchy sequence in $L_X^1([t_0, t_0 + T_1])$, and so converges to a function f in $L^1_X([t_0,\, t_0+T_1])$, a subsequence converging pointwisely a. e. It follows from (2.7) that $\{\varphi^i_t\}$ is a Cauchy sequence in $C_X([t_0,\, t_0+T_1])$ and so converges uniformly to a continuous function φ. It is easily seen that $\varphi_{t_0} = \bar{\varphi}, \dot{\varphi}(t) = f(t)$ for all $t \in [t_0, t_0+T_1]$, and

$$\|\varphi_t - \varphi^0_t\| \leq r \exp\left(2\int_{t_0}^{t_0+T_1} l(s)ds\right) < \rho$$

for all $t \in [t_0, t_0+T_1]$. Therefore, we can write, for almost all $t \in [t_0,\, t_0+T_1]$,

$$\begin{aligned} d(f(t), F(t,\varphi_t)) &\leq |f(t) - f^i(t)| + d(f^i(t), F(t,\varphi^i_t)) \\ &\quad + H(F(t,\varphi^i_t), F(t,\varphi_t)) \\ &\leq |f(t) - f^i(t)| + l(t)\|\varphi^i_t - \varphi^0_t\| \end{aligned}$$

Thus, we derive, by letting i tend to ∞,

$$\dot{\varphi}(t) = f(t) \in F(t,\varphi_t) \quad \text{a. e.}$$

so that $\varphi(\cdot)$ is a solution of the Cauchy problem (1.1) – (1.2). Moreover, it is clear that φ satisfies (2.3). This completes the proof of the lemma.

Proof of Theorem 1.1. Let T_1 be given and let φ^0 be a solution of (1.3) – (1.4) on $[t_0,\, t_0+T_1]$ and $\varepsilon > 0$. By virtue of Lemma 2.2, the condition (ii) of the theorem implies that there exists $\rho > 0$ and $l(\cdot) \in L^1([t_0, t_0+T_1])$ such that $K + B(0,\rho) \subset D$ and (2.1) is fulfilled, where $K = \{\varphi^0_t : t \in [t_0, t_0+T_1]\}$ is a compact set in C. As it has been shown in the proof of Lemma 2.4, the multifunction $t \to F(t,\varphi^0_t)$ is integrable. Let $r > 0$ be such that

$$r \exp\left(2\int_{t_0}^{t_0+T_1} l(s)ds\right) < \min\{\varepsilon,\rho\}.$$

Since $\dot{\varphi}^0(t) \in ClCoF(t,\varphi^0_t)$, according to Lemma 2.3 $F(t,\varphi^0_t)$ admits an integrable selection f^0 satisfying

$$\max_{t\in[t_0,t_0+T_1]} \left|\int_{t_0}^{t} [f^0(s) - \dot{\varphi}^0(s)]ds\right| < r.$$

Taking account of the fact that $\varphi^0(t_0) = \bar{\varphi}(0)$, from the last inequality we have

$$\left|\varphi^0(t) - \bar{\varphi}(0) - \int_{t_0}^{t} f^0(s)ds\right| < r$$

for all $t \in [t_0, t_0 + T_1]$. Consequently, by Lemma 2.4, the problem (1.1) - (1.2) has a solution φ^ε on $[t_0,\, t_0 + T_1]$ satisfying

$$\max_{s\in[t_0-h,t_0+T_1]} |\varphi^\varepsilon(s) - \varphi^0(s)| = \max_{t\in[t_0,t_0+T_1]} \|\varphi_t^\varepsilon - \varphi_t^0\| \le r \exp\left(2\int_{t_0}^{t_0+T_1} l(s)ds\right) < \varepsilon,$$

completing the proof.

Proof of Theorem 1.2. Set $\rho = \frac{1}{3}\bar{\delta}$ and let $T_0 \in (0, T]$ be such that

$$\max\{|\bar{\varphi}(\theta_1) - \bar{\varphi}(\theta_2)| : \theta_1, \theta_2 \in [-h, 0], |\theta_1 - \theta_2| \le T_0\} < \rho. \tag{2.13}$$

Further, we set

$$\varphi^0(t) = \begin{cases} \bar{\varphi}(0) & \text{if } t \in [t_0,\, t_0 + T_0], \\ \bar{\varphi}(t - t_0) & \text{if } t \in [t_0 - h,\, t_0]. \end{cases}$$

From (2.13) it obviously follows that

$$\|\varphi_t^0 - \bar{\varphi}\| = \max\{|\varphi_t^0(\theta) - \bar{\varphi}(\theta)| : -h \le \theta \le 0\} < \rho, \tag{2.14}$$

for all $t \in [t_0, t_0 + T_0]$. Thus, the compact $K = \{\varphi_t^0 : t \in [t_0, t_0 + T_0]\}$ is contained in $B(\bar{\varphi}, \rho)$ and hence $K + B(0, \rho) \subset B(\bar{\varphi}, \rho) + B(0, \rho) \subset B(\bar{\varphi}, \bar{\delta})$. As in the proof of Theorem 1, since $\varphi_t^0 \in B(\bar{\varphi}, \bar{\delta})$, it follows that there exists an integrable selection f^0 of the multifunction $t \to F(t, \varphi_t^0)$. Set $r = \frac{1}{2}\rho \exp(-2\int_{t_0}^{t_0+T_0} \bar{l}(s)ds)$. Then $r \exp(2\int_{t_0}^{t_0+T_0} \bar{l}(s)ds) = \frac{1}{2}\rho < \rho$. Let $T_1 \in (0, T_0]$ be such that

$$\int_{t_0}^{t_0+T_1} |f^0(s)|ds < \frac{r}{2}.$$

This, in combining with (2.13), yields

$$|\varphi^0(t) - \bar{\varphi}(0) - \int_{t_0}^{t} f^0(s)ds| < r$$

for every $t \in [t_0, t_0 + T_1]$. Moreover, it is clear that

$$r \exp\left(2\int_{t_0}^{t_0+T_1} \bar{l}(s)ds\right) < \rho.$$

Consequently, according to Lemma 2.4, the Cauchy problem (1.1) - (1.2) has a solution on $[t_0,\, t_0 + T_1]$.

3. AN APPLICATION

In this section we apply the theorems 1.1, 1.2 to prove the existence and relaxation of solutions for differential difference inclusions of the form $\dot{x}(t) \in P(t, x(t-\lambda_1(t)), \cdots, x(t-\lambda_n(t)))$. To simplify the notation, we shall consider only the case $n = 2$.

Let $I = [t_0, t_0+T]$ and $P : I \times U \times U \to \mathcal{F}(X)$, where U is an open set in X. Let $\bar{\varphi} \in C = C_X([-h, 0])$ such that $\bar{\varphi}([-h, 0]) \subset U$. Consider the following differential difference inclusion

$$\dot{x}(t) \in P(t, x(t-\lambda(t)), x(t-\mu(t))), \ t \in I \tag{3.1}$$

$$x(s) = \bar{\varphi}(s - t_0), \quad s \in [t_0 - h, t_0], \tag{3.2}$$

where $\lambda, \mu : I \to [-h, 0]$ are given continuous functions. The inclusion (3.1) may be reduced to (1.1) by defining a multifunction $F : I \times C \to \mathcal{F}(X)$ as

$$F(t, \varphi) = P(t, \varphi(-\lambda(t)), \varphi(-\mu(t))) \tag{3.3}$$

and

$$D = \{\varphi \in C : \varphi([-h, 0]) \subset U\}.$$

In what follows $|\cdot|_1$ will denotes any Banach norm in the product $Y = X \times X$. As an immediate consequence of Theorem 1.1, we have

Theorem 3.1. *Suppose that:*

(i) *for each $y = (x_1, x_2) \in U \times U$ the multifunction $P(\cdot, y)$ is measurable and admits at least one integrable selection on I,*

(ii) *for each $y = (x_1, x_2) \in U \times U$ there exists $\delta_y > 0$ and $l_y \in L^1(I)$ such that*

$$H(P(t, y^1), P(t, y^2)) \leq l_y(t)|y^1 - y^2|_1 \ \text{a.e. on } I$$

for any $y^1, y^2 \in B(y, \delta_y)$.

Then, for any $T_1 \in (0, T]$the set of solutions of (3.1) - (3.2) on $I_1 = [t_0, t_0 + T_1]$ is dense for the topology of uniform convergence in the set of solutions of the convexified inclusion

$$\dot{x}(t) \in ClCoP(t, x(t-\lambda(t)), x(t-\mu(t))), \ t \in I_1, \tag{3.4}$$

$$x(s) = \bar{\varphi}(s - t_0), \quad s \in [t_0 - h, t_0].$$

Similarly, as a corollary of Theorem 1.2, we can also formulate a theorem on the existence of solutions for the differential difference inclusion (3.1) - (3.2).

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