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A MEAN VALUE THEOREM FOR SEMIDIFFERENTIABLE FUNCTIONS

NGUYEN DONG YEN

Abstract. *We obtain a mean value theorem for semidifferentiable functions where a simple topological property of the set of discontinuous points replaces the usual assumption on (semi) continuity of the subdifferentials.*

Consider a function $f : R^n \to R$ and two (different) points $a, b \in R^n$. We want to establish a mean value theorem for *upper C–semidifferentiable* functions [3], which is analogous to the theorem of Lebourgh [5] (see also [1, p. 41]) for locally Lipschitz functions. There are various mean value theorems for nonsmooth functions (see [1] [2] [4] [5] [6] [10] [12] and references therein). Mean value theorems can be established also for set–valued maps [9]. Basic aspects of the topic were outlined in [6]. In the mean value theorem below, instead of a requirement on (semi) continuity of the subdifferentials of f on (a, b), we will use a simple condition on the set of discontinuous points of f on this interval. Our result has a close relation to a theorem by Penot [6, Proposition 2.5]. A discussion will be given at the end of the paper.

The Euclidean norm and the inner product of R^n are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. By C we denote the set of sublinear functionals defined of R^n. Let us recall the notion of upper C–semindifferentiable functions [3] in the following.

Definition 1. A functional $g \in C$ is said to be the upper C–semiderivative of f at $\overline{x} \in R^n$ if $\overline{g}$ is the functional with maximal epigraph among the set of all $g \in C$ satisfy the following condition:

$$\limsup_{x \to \overline{x}} \frac{f(x) - f(\overline{x}) - g(x - \overline{x})}{\|x - \overline{x}\|} \leq 0 .$$

If such $\overline{g}$ dose exist, f is said to be upper C–semidifferentiable at $\overline{x}$.

It has been proved in [11, Theorem 2.2] that if $\overline{g}$ is the upper C–semiderivative of f at $\overline{x}$, then

$$\bar{g}(v) = d^+_{DH} f(\bar{x}; v) := \limsup_{t\downarrow 0,\, v'\to v} t^{-1}(f(\bar{x}+tv) - f(\bar{x})) \quad \forall v \in R^n\,. \quad (1)$$

The value $d^+_{DH} f(\bar{x}; v)$ is known as the upper Dini–Hadamard directional derivative of f at $\bar{x}$ in direction v. The upper Dini directional derivative of f at $\bar{x}$ is in direction v is the number

$$d^+_D f(\bar{x}; v) := \limsup_{t\downarrow 0} t^{-1}(f(\bar{x}+tv) - f(\bar{x}))\,.$$

Therefore, d^+_{DH} is a form of regularization of d^+_D. Symbols $d^-_{DH} f(\bar{x}; v)$ and $d^-_D f(\bar{x}; v)$ will denote, respectively, the lower Dini–Hadamard and the lower Dini directional derivatives. Their definitions are quite similar to the above, provided that lims sup are replaced with lim inf. If $d^+_{DH} f(\bar{x}; v) = d^+_D f(\bar{x}; v)$ then we say that f is (upper) *Dini regular* at $\bar{x}$ is direction v.

By using formula (1) we can show that if $\bar{g}$ is the upper C–semiderivative of f at $\bar{x}$, then $\bar{g}$ is the lowest upper convex approximation of f at $\bar{x}$ in the sense of Pshenichnyi [7].

Our result can be stated as follows:

Theorem 1. (Mean value theorem). *Let the following conditions be satisfied:*

(C_1) *f is upper C–semidifferentiable at every $x \in (a, b) := \{a+t(b-a) : t \in (0, 1)\}$;*

(C_2) *f is (upper) Dini regular at every $x \in (a, b)$ in directions $b-a$ and $a-b$;*

(C_3) *f is continuous at a and b;*

(C_4) *f is continuous on (a, b), or it is discontinuous on this interval but there is at least one isolated discontinuous point $\bar{x} \in (a, b)$.*

Then there exists $c \in (a, b)$ and

$$\xi \in \partial f(c) := \{\xi \in R^n : \langle \xi, v\rangle \le d^+_{DH} f(c; v) \text{ for every } v \in R^n\}$$

such that

$$f(b) - f(a) = \langle \xi, b-a\rangle\,.$$

(We say that $x \in (a, b)$ is an isolated discontinuous point of f if f is discontinuous at $\bar{x}$ and there is an open subinterval U of (a, b) containing $\bar{x}$ such that f is continuous at every point from $U \setminus \{\bar{x}\}$).

Before giving a proof we briefly discuss the assumptions (C_1) - (C_4).

1) By the classical mean value theorem we know that if f is Frechet differentiable at every $x \in (a, b)$, then the conclusion (2) is valid under (C_3). Since (C_1), (C_2) and (C_4) hold automatically in this case, Theorem 1 implies the classical theorem.

2) Conditions (C_4) states something about the character of discontinuity of f on (a, b). It holds obviously if the number of discontinuous point of f on (a, b) is finite. The following example gives an upper C-semidifferentiable function on a whole interval but discontinuous on it.

Example 1. Let $n = 1$, $\overline{x} = 0$, $f(x) = \sin(1/x)$ if $x \neq 0$, $f(0) = 1$. It is easy to see that f is upper C-semidifferentiable on whole R, but discontinuous at $\overline{x}$. ($\overline{g}(\dot{}) = 0$ is the upper C-semiderivative of f at $\overline{x}$).

3) If (C_4) is violated then f behaves too badly: *Every* discontinuous point of it on (a, b) is cluster point of sequence of discontinuous points. Such semidifferentiable functions, if exist, can be seen very rarely. We exclude them from our consideration for simplicity of the presentation.

4) We shall rely on (C_2) in the proof below. But is seems to us that the following properties may hold:

(i) If f is upper C-semidifferentiable on an open domain $\Omega \subset R^n$, then for every $x \in \Omega$, $v \in R^n$, we have $d^+_{DH} f(x; v) = d^+_D f(x; v)$.

(ii) $(C_1) \Rightarrow (C_2)$.

Proof of Theorem 1. Consider the function $\varphi : [0, 1] \rightarrow R$ defined by $\varphi(t) = f(a+t(b-a))+t(f(a)-f(b))$. We have $\varphi(0) = \varphi(1) = f(a)$ and φ is continuous at 0 and 1 (see (C_3)). By using (C_2) we obtain formulas for the Dini directional derivatives of φ at a given point $\tau \in (0, 1)$:

$$
\begin{aligned}
d^+_D(\tau; 1) &= \limsup_{t\downarrow 0} t^{-1}(\varphi(\tau + t) - \varphi(\tau)) \\
&\ \limsup_{t\downarrow 0} t^{-1}(f(a + \tau(b - a) + t(b - a)) \\
&\qquad - f(a + (\tau(b - a))) + (f(a) - f(b)) \\
&= d^+_{DH} f(a + \tau(b - a); b - a) + (f(a) - f(b))\,; \\
d^+_D(\tau; -1) &= \limsup_{t\downarrow 0} t^{-1}(\varphi(\tau - t) - \varphi(\tau)) \\
&= \limsup_{t\downarrow 0} t^{-1}(f(a + \tau(b - a) - t(b - a)) \\
&\qquad - f(a + (\tau(b - a))) - (f(a) - f(b))
\end{aligned}
$$

$$= d^+_{DH} f(a + \tau(b-a); a-b) - (f(a) - f(b)) .$$

Lemma 1. *Under assumptions* (C_1) - (C_4) *there exists* $\tau \in (0, 1)$ *such that*

$$d^+_D\varphi(\tau; 1) \geq 0 \text{ and } d^+_D\varphi(\tau; -1) \geq 0 . \tag{3}$$

Assuming that this lemma has been obtained we continue our proof. By (C_1) for *every* $x \in (a, b)$ we have $d^+_{DH} f(x; \cdot) \in C$. Let $\tau \in (0, 1)$ be the value provided by Lemma 1 and let $c := a + \tau(b-a)$. According to (3),

$$d^+_{DH} f(c; b-a) \geq f(b) - f(a) \text{ and } d^+_{DH} f(c; a-b) \geq f(a) - f(b) .$$

Using the homogeneous property of $d^+_{DH} f(c; \cdot)$ we have

$$d^+_{DH} f(c; t(b-a)) \geq t(f(b) - f(a)), \quad \forall t \in R .$$

This means that

$$\alpha(t(b-a)) := t(f(b) - f(a)), \quad \forall t \in R \tag{4}$$

is a linear functional on the linear subspace generated by vector $b - a$, who is majorized by the sublinear functional $d^+_{DH} f(c; \cdot)$. According to the Hahn - Banach theorem [8] there is a linear functional ξ defined on R^n which is an extention of α and which satisfies $d^+_{DH} f(c; v) \geq \langle \xi, v \rangle$ for every $v \in R^n$. This implies $\xi \in \partial f(c)$. Using (4) we have $\langle \xi, b-a \rangle = a(b-a) = f(b) - f(a)$. Thus (2) is proved.

The following property of upper C-semidifferentiable functions can be observed from the definition.

Lemma 2. *If* f *is upper* C*-semidifferentiable at* $\overline{x}$*, then* f *is upper semicontinuous at* $\overline{x}$*, that is*

$$\limsup_{x \to \overline{x}} f(x) \leq f(\overline{x}) .$$

As a consequence, it follows from (C_1) that φ is upper semicontinuous at every $\overline{t} \in (0, 1)$.

To prove Lemma 1 we consider two cases:

1. f is continuous on (a, b). Combining this with (C_3) we see that φ is continuous on $[0, 1]$.

a) If $\varphi(t) = \varphi(0) = \varphi(1)$ for all $t \in (0, 1)$, then (3) is trivial.

b) Assume that there exists $t \in (0, 1)$ such that $\varphi(t) > \varphi(0) = \varphi(1)$. By the continuity of φ on $[0, 1]$ there is $\tau \in [0, 1]$ such that $\varphi(\tau) = \max\{\varphi(t) : t \in [0, 1]\}$. Since $\varphi(t) > \varphi(0) = \varphi(1)$ then $\tau \in (0, 1)$. As τ is a local maximum of φ then

$$d_D^+\varphi(\tau; 1) \leq 0 \text{ and } d_D^+\varphi(\tau; -1) \leq 0\,. \tag{5}$$

Let $c := a + \tau(b-a)$. Since $0 = \frac{1}{2}(a-b) + \frac{1}{2}(b-a)$, the convexity of $d_{DH}^+ f(c;\,)$ implies

$$\begin{aligned} 0 = d_{DH}^+ f(c; 0) &\leq \frac{1}{2} d_{DH}^+ f(c; a-b) + \frac{1}{2} d_{DH}^+ f(c; b-a) \\ &= \frac{1}{2}\left[d_D^+\varphi(\tau; -1) + (f(a) - f(b))\right] \\ &\quad + \frac{1}{2}\left[d_D^+\varphi(\tau; 1) - (f(a) - f(b))\right]. \end{aligned}$$

Invoking (5) one has $d_D^+\varphi(\tau; 1) = d_D^+\varphi(\tau; -1) = 0$, which implies (3).

c) Assume that there is $t \in (0, 1)$ such that $\varphi(t) < \varphi(0) = \varphi(1)$. By the continuity of φ we find a point $\tau \in (0, 1)$ satisfying $\varphi(\tau) = \min\{\varphi(t) : t \in [0, 1]\}$. Then (3) is obvious.

2. f is discontinuous on (a, b)*, but there exists at least one isolated discontinuous point* $\overline{x} \in (a, b)$. Let $\overline{x} \in (a, b)$ be an isolated discontinuous point of f. Then there exist an interval $(\alpha, \beta) \subset (0, 1)$ containing the point $\overline{t} \in (0, 1)$ defined by the inequality $\overline{x} = a + \overline{t}(b-a)$, such that φ is discontinuous at $\overline{t}$ but continuous on $\alpha, \overline{t})$ and $(\overline{t}, \beta)$.

a) Assume that φ is not monotone on $(\overline{t}, \beta)$. Then there exists a local minimum or local maximum τ of φ on $(\overline{t}, \beta)$. The arguments used in the first case show that (3) holds. If φ is not monotone on $(\alpha, \overline{t})$ we argue similarly.

b) Now let φ be monotone on $(\overline{t}, \beta)$ and on $(\alpha, \overline{t})$. In this case we set

$$\mu = \lim_{t \to \overline{t}-0} \varphi(t) \text{ and } \lambda = \lim_{t \to \overline{t}+0} \varphi(t)\,.$$

By the upper semicontinuity of φ at $\bar{t}$ we have $\mu \le \varphi(\bar{t})$ and $\lambda \le \varphi(\bar{t})$. Moreover, $\mu \ne -\infty$ and $\lambda \ne -\infty$. Indeed, otherwise we would have $d_D^+\varphi(\bar{t};-1) = -\infty$ or $d_D^+\varphi(\bar{t};1) = -\infty$, what is impossible because of (C_1) and (C_2). If $\mu < \varphi(\bar{t})$ or $\lambda < \varphi(\bar{t})$ we can still have that $d_D^+\varphi(\bar{t};-1) = -\infty$ or, respectively, $d_D^+\varphi(\bar{t};1) = -\infty$. Therefore, $\mu = \lambda = \varphi(\bar{t})$. But, then φ is *continuous* at $\bar{t}$, a contradiction to our assumption. We conclude that φ cannot be monotone on both intervals $(\alpha, \bar{t})$ and $(\bar{t}, \beta)$. The Proof of Lemma 1 is complete.

Let us have a discussion about the relation of Theorem 1 to a result in [6], where also no condition on (semi) continuity of the subdifferentials of f is required. As in [6] we set

$$\underline{\partial} f(x) = \{\xi \in R^n : \langle \xi, v\rangle \le d_{DH}^- f(x;v) \ \ \forall v \in R^n\}$$

and denote by *co* D the convex hull of a set $D \subset R^n$. Proposition 2.5 in [6] can be stated as follows.

Theorem 2. *Let f be lower semicontinuous on $[a, b]$ and let the following conditions be fulfilled:*

(A_1) $d_{DH}^- f(x;\dot{\ }) \in C$ *for every* $x \in [a, b]$;

(A_2) $d_{DH}^- f(x;v) = d_D^- f(x;v)$ *for every* $x \in [a, b]$ *and* $v \in R^n$.

Then either there exists $c \in (a, b)$ and $\xi \in \underline{\partial} f(c)$ such that

$$f(b) - f(a) = \langle \xi, b-a\rangle, \tag{6}$$

or there exist $\xi \in co(\underline{\partial} f(a) \cap \underline{\partial} f(b))$ satisfying (6).

(The just cited mean value theorem was obtained in [6] for functions defined on an topological vector space).

Apart from differences in the formulation of Theorems 1 and 2, it is worth noting that they are applicable for two different classes of functions. Indeed, assume that f satisfies (C_1) - (C_4) and that one considers $-f$ instead of f. Since $d_{DH}^-(-f)(x;\dot{\ }) = -d_{DH}^+ f(x;\dot{\ })$, $d_{DH}^-(-f)(x;\dot{\ })$ is a *concave* functional for every $x \in (a, b)$. Then theorem 2 cannot be applied. Conversely, assuming that f satisfies (A_1) and (A_2) one has that $d_{DH}^+(-f)(x;\dot{\ })$ is a *concave* functional for every $x \in [a, b]$. Thus Theorem 1 cannot be applied.

Loosely speaking, Theorem 1 is applicable when the upper Dini–Hadamard directional derivatives $d_{DH}^+ f(x;\dot{\ })$, $x \in (a, b)$, are sublinear; while Theorem 2 works when the lower Dini–Hadamard directional derivatives $d_{DH}^- f(x;\dot{\ })$, $x \in [a, b]$, are sublinear.

The following example with a simple continuous function is to illustrate the situation.

Example 2. Let $n = 1$, $f(x) = x\sin(1/x)$ for $x \neq 0$, $f(0) = 0$. let $[a, b]$ be any segment containing 0. Since $d^-_{DH}f(0;v) = d^-_{DH}(-f)(0;v) = -|v|$ for every $v \in R$, Theorem 2 is not applicable for both f and $-f$. For such function f and segment $[a, b]$ Theorem 1 is applicable.

Professor S. Komlosi has shown to us that the following result can be obtained by modifying slightly the proof of Prosposition 2.5 in [6]:

Theorem 3. *Suppose that f is finite, upper semicontinuous on $[a, b]$ and such that for each $x \in [a, b]$, $d^+_{DH}f(x;\cdot) = d^+_{D}f(x;\cdot)$ is a functional from C. Then either there exist $c \in (a, b)$ and $\xi \in \partial f(c)$ such that (6) holds, or there exist $\xi \in co(\partial f(a) \cup \partial f(b))$ fulfilling (6).*

Note that under the property (C_4) in Theorem 1, the second case in the conclusion of Theorem 3 can be excluded. Recall that (C_4) holds automatically if f has only a finite number of discontinuous points on (a, b).

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REFERENCES

1. F. H. Clarke, *Optimization and nonsmooth analysis*, Wiley, New York, 1983.
2. V. F. Demyanov and A. M. Rubinov, *Quasidifferential calculus*, Optimization Software, New York, 1986.
3. F. Giannessi, *Semidifferentiable functions and necessary optimality conditions*, J. Opt. Theory Appl., **60** (1989), 191–241.
4. J. B. Hiriart–Urruty, *Mean value theorem in nonsmooth analysis*, Numer. Funct. Anal. Optim., **2** (1980), 1–30.
5. G. Lebourg, *Valeur moyene pour un gradient generalise*, C. R. Acad. Sc. Paris, **281** (1975), 785–797.
6. J. P. Penot, *On the mean value theorem*, Optimization, **19** (1988), 147–156.
7. B. N. Pschenichnyi, *Convex analysis and extremal problem*, Nauka, Moscow, 1980, (in Russian).
8. W. Rudin, *Functional analysis*, McGraw–Hill, New York, 1983.
9. P. H. Sach, *Differentiability of set-valued maps in Banach spaces*, Math. Nachr., **139** (1988), 215–235.

10. M. Studniarski, *Mean value theorems and sufficient optimality conditions for nonsmooth functions*, J. Math. Anal. Appl., **111** (1985), 313 – 326.

11. N. D. Yen, *On G–semidifferentiable functions in Euclidean spaces*, J. Opt. Theory Appl., **85** (1995), 377 – 392.

12. D. Zagrodny, *Approximate mean value theorem for upper subderivatives*, Non–linear Analysis, **12** (1988), 1413 – 1428.

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Institute of Mathematics
P. O. Box 631, BoHo,
Hanoi, Vietnam.