Vietnam Journal of Mathematics
Volume 23, Number 2, 1995

A Short Communication

UNIQUENESS OF GLOBAL QUASI–CLASSICAL SOLUTIONS OF THE CAUCHY PROBLEM FOR SOME SYSTEMS OF FIRST–ORDER NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS[1]

TRAN DUC VAN and LE VAN HAP

Let T be a positive number, $\Omega_T = (0, T) \times \boldsymbol{R}^n$, $\nabla_x = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)$, $n \geq 1$; $\|\cdot\|_n$ and $\langle \cdot, \cdot \rangle$ be the norm and scalar product in $\boldsymbol{R}^n$, respectively.

We consider a solution $u = (u_1, \cdots, u_m)$ of the following system of first–order nonlinear partial differential equations (PDEs)

$$\frac{\partial u_j}{\partial t} + H_j(t, x, u, \nabla_x u_j) = 0\,, \ \ (t,\, x) \in \Omega_T\,, \ \ j = 1, \dots, m\,, \tag{1}$$

subject to the Cauchy conditions

$$u_j(0,\, x) = u_j^0(x)\,, \ \ x \in \boldsymbol{R}^n, \ \ j = 1, \dots, m\,, \tag{2}$$

where each H_j is a function of $(t, x, p, q_j) \in (0, T) \times \boldsymbol{R}^n \times \boldsymbol{R}^m \times \boldsymbol{R}^n$. Vectors $p = (p^1, \cdots, p^m)$ and $q_j = (q_j^1, \cdots, q_j^n)$, $j = 1, \dots, m$, are corresponding to $u = (u_1, \cdots, u_m)$ and $\nabla_x u_j = \left(\frac{\partial u_j}{\partial x_1}, \cdots, \frac{\partial u_j}{\partial x_n}\right)$, respectively.

It is well–known that there may not exist a global classical solution for a first–order nonlinear PDE. Therefore, many kinds of generalized solutions have been introduced and various different methods have been

[1]This work was supported in part by the National Basic Research Program in Natural Sciences, Vietnam.

used in the study of first-order nonlinear PDEs. In [4], the first author proposed a notion of global generalized solutions, namely, *global quasi-classical solutions* of Cauchy problem for a single PDE. Using this notion, that author together with N. D. Thai Son, N. Hoang and R. Gorenflo obtained some new uniqueness and existence results and gave, in particular, the answer to an open uniqueness problem of S. N. Kruzhkov for global solutions of evolution PDEs (see, e. g.[3], [4], [5]).

We note that the uniqueness question is always very subtle and difficult.

The purpose of the present note is to announce some uniqueness theorems for global quasi-classical solutions of system (1) - (2).

Denote by $\mathrm{Lip}(\Omega_T)$ the set of all locally Lipschitz continuous functions u defined on Ω_T and set

$$\mathrm{Lip}([0,T)\times\boldsymbol{R}^n) := \mathrm{Lip}(\Omega_T)\cap C([0,T)\times\boldsymbol{R}^n)\,.$$

As in [3], let $V(\Omega_T)$ be the following subclass of $\mathrm{Lip}([0,T)\times\boldsymbol{R}^n)$:

$$V(\Omega_T) := \{u : u\in \mathrm{Lip}([0,T)\times\boldsymbol{R}^n);\ u \text{ is diffferentiable for all } x\in\boldsymbol{R}^n \text{ and for almost all } t\in(0,T)\}\,.$$

It is obvious that

$$C^1(\Omega_T)\subset V(\Omega_T)\subset \mathrm{Lip}([0,T)\times\boldsymbol{R}^n).$$

Set

$$V_m(\Omega_T) := \underbrace{V(\Omega_T)\times\cdots\times V(\Omega_T)}_{m \text{ times}}\,.$$

Definition. A vector function $u\in V_m(\Omega_T)$ is called a global quasi-classical solution of (1) - (2) if u satisfies the condition (2) for all $x\in\boldsymbol{R}^n$ and u satisfies the system (1) for all $x\in\boldsymbol{R}^n$ and for almost all $t\in(0,T)$.

We now formulate some uniqueness results for global quasi-classical solutions of the problem (1) - (2).

Theorem 1. *Suppose that* $H_j(t,x,p,q_j)$, $j=1,\ldots,m$, *satisfy the following condition: There exist nonnegative functions* $k_j(\cdot)\in L_1(0,T)$ *and nonnegative functions* $h_j(\cdot)$ *locally bounded in* $\boldsymbol{R}^n$ *such that*

$$|H_j(t,x,p,q_j)-H_j(t,x,p',q_j')| \\ \leq k_j(t)\left[(1+\|x\|_n)\|q_j-q_j'\|_n+h_j(x)\|p-p'\|_m\right], \tag{3}$$

for all $x \in \mathbb{R}^n$ and for almost all $t \in (0,T)$; p, $p' \in \mathbb{R}^n$; q_j, $q_j' \in \mathbb{R}^m$. If $u = (u_1, \cdots, u_m)$ and $v = (v_1, \cdots, v_m)$ are global quasi-classical solutions of the problem (1)-(2), then $u \equiv v$ in Ω_T, i. e. $u_j \equiv v_j$ in Ω_T for each $j = 1, \ldots, m$.

Theorem 2. *Suppose that $H_j(t,x,p,q_j)$, $j = 1,\ldots,m$, satisfy the following condition: There exist functions $k_j(\cdot) = k_{jK}(\cdot)$ and $h_j(\cdot) = h_{jK}(\cdot)$ as in Theorem 1 for every compact $K \subset \mathbb{R}^n$ such that (3) holds for all $x \in \mathbb{R}^n$ and for almost all $t \in (0,T)$; p, $p' \in \mathbb{R}^m$; q_j, $q_j' \in K$. If $u = (u_1, \cdots, u_m)$ and $v = (v_1, \cdots, v_m)$ are global quasi-classical solutions of Problem (1)-(2) with*

$$\operatorname*{ess.sup}_{(t,x)\in\Omega_T} \max\{\|\nabla_x u_j\|_n,\ \|\nabla_x v_j\|_n\} < \infty,\ j = 1,\ldots,m,$$

then $u \equiv v$ in Ω_T.

Proofs of Theorems 1, 2 will be based on the following theorem.

Theorem 3. *Let $u \in V_m(\Omega_T)$. If there exist nonnegative functions $k_j(\cdot) \in L_1(0,T)$ and nonnegative functions $h_j(\cdot)$ locally bounded in $\mathbb{R}^n$ such that*

$$\left|\frac{\partial u_j}{\partial t}(t,x)\right| \leq k_j(t)\left[(1+\|x\|_n)\|\nabla_x u_j(t,x)\|_n + h_j(x)\|u(t,x)\|_m\right], \\ j = 1,\ldots,m, \tag{4}$$

for all $x \in \mathbb{R}^n$ and for almost all $t \in (0,T)$, then

$$\max_{j=\overline{1,m}} |u_j(t,x)| \leq \exp\{M(x)f(t)\} \cdot \max_{\|y\|_n \leq N(t,x)} \max_{j=\overline{1,m}} |u_j(0,y)|,$$

where

$$\begin{aligned} N(t,x) &:= (1+\|x\|_n)\exp(f(t)) - 1, \\ M(x) &:= \sup\{|h(y)| : \|y\|_n \leq N(T,x)\}, \\ k(t) &:= \max_{j=\overline{1,m}} k_j(t), \\ h(x) &:= \max_{j=\overline{1,m}} h_j(x), \\ f(t) &:= m\int_0^1 k(\tau)d\tau. \end{aligned}$$

We also get from Theorem 3 the following result which describes a criterion of continuous dependence on initial values for quasi-classical solutions of the systems (1) - (2).

Theorem 4. *Suppose that $H_j(t,x,p,q_j)$, $j = 1,\dots,m$, satisfy the Condition (3) in Theorem 1. If $u = u_1,\cdots,u_m)$ and $v = (v_1,\cdots,v_m)$ are global quasi-classical solutions of the Cauchy problem for the equation (1) with initial conditions:*

$$u_j(0,x) = \varphi_j(x)\,,\ v_j(0,x) = \psi_j(x)\,,\ x \in \boldsymbol{R}^n,\ j = 1,\dots,m\,,$$

then we have the following estimate:

$$\max_{j=1,m} |u_j(t,x) - v_j(t,x)| \le \exp\left\{M(x)\int_0^t mk(\tau)d\tau\right\}\times$$
$$\times \sup\{|\varphi_j(y) - \psi_j(y)| : \|y\|_n \le N(t,x)\}\,.$$

Let us give a very simple example that distinguishes our uniqueness theorems.

Taking $n = 1$, $T = 1$, $m = 2$ and let $J \subset [0,\,1]$ be the Cantor set, i.e., the set of all numbers of the form:

$$t = \sum_{i=1}^{\infty} \frac{\varepsilon_i}{3^i}\,,$$

where ε_i is either 0 or 2. We define a continuous function $\varphi(\cdot)$ on $[0,\,1]$ by:

$$\varphi(t) := \min\{|t-\xi| : \xi \in J\}\,,\ t \in [0,1]\,.$$

It is clear that $\varphi(0) = 0$ and that $\varphi(\cdot)$ is Lipschitz continuous. Set $\psi := \frac{d\varphi(t)}{dt}$.

We consider the following problem:

$$\begin{cases} \frac{\partial u_1}{\partial t}(t,x) + \psi(t)\sin\left(\frac{\partial u_1}{\partial x}(t,x) - u_2(t,x)\right) & = 0\,, \\ \frac{\partial u_2}{\partial t}(t,x) + \psi(t)\frac{\partial u_2}{\partial x}(t,x) & = 0\,, \\ u_1(0,x) & = \frac{\pi}{2}x\,, \\ u_2(0,x) & = \pi\,, \end{cases} \tag{5}$$

where $H_1(t,x,p,q_1) = \psi(t)\sin(q_1 - q_2)$ and $H_2(t,x,p,q_2) = \psi(t)q_2$, $p = (p_1, p_2)$, $q = (q_1, q_2) \in \boldsymbol{R}^2$.

The functions H_1, H_2 satisfy the condition of Theorem 1. Therefore the Problem (5) admits at most one global quasi–classical solution. It is clear that this unique solution is $u_1(t,x) = \varphi(t) + \frac{\pi}{2}x$, $u_2(t,x) = \pi$. Let us note that there exists no classical solution for (5) even in a local sense.

REFERENCES

1. J. P. Aubin and H. Frankowska, *Set valued analysis*, Birkhäuser, Basel, 1990.
2. Ch. Castaing and M. Valadier, *Convex analysis and measurable multifunctions*, Lecture Notes in Math., Vol. 580, Springer–Verlag, New York, 1977.
3. T. D. Van and N. D. Thai Son, *On a class of Lipschitz continuous of several variables*, Proc. Amer. Math. Soc., **121** (1994), 865 – 870.
4. T. D. Van and N. D. Thai Son, *Uniqueness of global quasi–classical solutions of the Cauchy problem for nonlinear partial differential equations of first order*, Differentsialnye Uravneniya, **30** (1994), 712 – 719 (Russian), English transl.: Differential Equations, **30** (1994), 659 – 666.
5. T. D. Van, N. Hoang, and R. Gorenflo, *Existence of global quasi–classical solutions of the Cauchy problem for Hamilton – Jacobi equation*, Differentsialnye Uravneniya, **31** (1995) (Russian, to appear).

Received July 25, 1995

Institute of Mathematics,
Box 631, Bo Ho,
Hanoi. Vietnam