VIETNAM JOURNAL OF MATHEMATICS
Volume 23, Number 2, 1995

ASYMPTOTIC ERROR EXPANSIONS IN DIFFERENCE METHODS FOR EQUATIONS WITH DISCONTINUOUS COEFFICIENTS[1]

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Abstract. *The paper deals with numerical methods for solving third boundary value problems for a stationary diffusion–convection equation with discontinuous coefficients. A difference scheme which satisfies the maximum principle with any grid stepsize is investigated and an asymptotic expansion of higher order is presented.*

1. INTRODUCTION

In a finite difference method when the error admits an asymptotic expansion with respect to the grid stepsize, the Richardson extrapolation to the limit can be used for accelerating the rate of convergence of the method. It reduces the necessary number of algebraic equations to be solved and thereby provides a very efficient algorithm concerning both computing time and storage requirements. Many delicate investigations about such expansions have been done (see [1] - [6] and the references in them). However very few authors pay attention to the problems with discontinuous coefficients. In [3, p. 68 - 98] an asymptotic error expansion for the first boundary value problem for second order self-adjoint ordinary differential equations with discontinuous coefficients is presented. The non self-adjoint equation with continuous coefficients is considered in [7, 8] but without asymptotic error expansion. In this paper we consider the third boundary value problem for unidimensional stationary diffusion–convection equations with discontinuous coefficients. A difference scheme which satisfies the maximum principle for any grid stepsize and the existence of an asymptotic error expansion of higher order are presented.

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[1]This publication is completed with the financial support from the National Basic Research Program in Natural Sciences, Vietnam.

2. DIFFERENTIAL PROBLEM

Let μ be a non-negative integer and $\xi \in]0, 1[$, $\alpha \in]0, 1[$. We define

$$C^{\mu+\alpha}[a,b] = \Big\{\Phi \,|\, \Phi \in C^{\mu}[a, b],\ |\Phi^{(\mu)}(x) - \Phi^{(\mu)}(x')| \le \text{const.}|x - x'|^{\alpha},\ x, x' \in [a, b]\Big\},$$

$$Q_{\xi}^{\mu+\lambda}[0, 1] = \Big\{\Phi \,|\, \Phi \in C^{\mu+\lambda}[0, \xi],\ \Phi \in C^{\mu+\lambda}[\xi, 1],\ \lambda = 0 \text{ or } \alpha\Big\}.$$

The differential problem to be considered is

$$Lu := (Au')' + Bu' - qu = f,\ \ 0 < x < 1,\ x \ne \xi, \tag{2.1}$$

$$\begin{aligned} l_{\xi}u &:= Au'|_{x=\xi+0} - Au'|_{x=\xi-0} - \sigma_{\xi}u(\xi) = g_{\xi}, \\ &u(\xi + 0) = u(\xi - 0) = u(\xi), \end{aligned} \tag{2.2}$$

$$\begin{aligned} l_0 u &:= A(0)u'(0) - \sigma_0 u(0) = g_0, \\ l_1 u &:= A(1)u'(1) + \sigma_1 u(1) = g_1, \end{aligned} \tag{2.3}$$

where $A,\ B,\ q,\ f$ are given functions satisfying

$$A \ge \text{const} > 0,\ q \ge 0,\ A \in Q_{\xi}^{m+1+\lambda}[0, 1],\ B, q, f \in Q_{\xi}^{m+\lambda}[0, 1], \tag{2.4}$$

for a non-negative integer m and the given real numbers $\sigma_0, \sigma_1, \sigma_{\xi}, g_0, g_1, g_{\xi}$ satisfying

$$\sigma_0 \ge 0,\ \ \sigma_1 \ge 0,\ \ \sigma_{\xi} \ge 0,\ \ \sigma_0 + \sigma_1 > 0. \tag{2.5}$$

Applying the method of [1, ch. 1] we can prove the following lemma.

Lemma 1. *The problem (2.1) - (2.5) has unique solution u:*

$$u \in Q_{\xi}^{m+2+\lambda}[0, 1] \cap C[0, 1],\ \ \lambda = 0 \text{ or } \alpha. \tag{2.6}$$

3. DIFFERENCE SCHEME (DISCRETE PROBLEM)

3.1. Grid

Let N_0 and N_1 be two positive integers and let us denote

$$N = N_0 + N_1,\ \ h_0 = \xi/N_0,\ \ h_1 = (1 - \xi)/N_1,\ \ \gamma = h_1/h_0 = \text{const}.$$

$$x_i = ih_0\,,\ \ 0 \le i \le N_0 \ \text{ and } x_i = \xi + (i - N_0)h_1\,,\ N_0 < i \le N\,,$$

In the following we put

$$h_\nu = \begin{cases} h_0 & \text{when } x_i < \xi\,, \\ h_1 & \text{when } x_i > \xi\,. \end{cases}$$

The set $\omega_h = \{x_i,\ i = \overline{0,\ N}\}$ is called a grid on [0, 1], each point x_i is called a grid point. The grid point x_{N_0} falls exactly at $x = \xi$.

A function v defined at each point of ω_h is called a grid function or discrete function. The value of v at x_i is denoted by v_i. We define the difference quotients v_x and $v_{\bar{x}}$ of v at $x = x_i$ as follows:

$$v_{x_i} = \frac{v_{i+1} - v_i}{x_{i+1} - x_i}\,,\quad v_{\bar{x}_i} = \frac{v_i - v_{i-1}}{x_i - x_{i-1}}\,.$$

3.2. Discrete problem (difference scheme)

Let

$$B^+ = 0.5(B + |B|)\,,\quad B^- = 0.5(B - |B|)$$

so that

$$B = B^+ + B^-\,,\quad |B| = B^+ - B^-\,.$$

Define the discrete functions a, b^+, b^-, d, f, $R^{(\nu)}$, $r^{(\nu)}$ and r by putting:

$$\begin{aligned} a_i &= A(x_i - 0.5(x_i - x_{i-1}))\,; \\ b_i^+ &= B^+(x_i)/A(x_i)\,;\ \ b_i^- = B^-(x_i)/A(x_i)\,; \\ d_i &= q(x_i)\,;\ \ f_i = f(x_i)\,,\ i \ne N_0,\ 0,\ N\,; \\ R_i^{(\nu)} &= 0.5h_\nu |B(x_i)|/A(x_i)\,,\ \ \nu = 0, 1\,; \\ r^{(\nu)} &= 1 - R^{(\nu)} + (R^{(\nu)})^2\,,\ \ \nu = 0, 1\,,\ i \ne N_0\,; \\ r_i &= \begin{cases} (r^{(0)})_i & \text{for } i < N_0\,, \\ (r^{(1)})_i & \text{for } i > N_0\,. \end{cases} \end{aligned}$$

Consider also the quantities:

$$\begin{aligned} S_0 &= 0.5h_0B(0)/A(0)\,,\ \ s_0 = 1 + S_0 + (S_0)^2\,; \\ S_1 &= 0.5h_1B(1)/A(1)\,,\ \ s_1 = 1 - S_1 + (S_1)^2\,; \\ S_\xi^- &= 0.5h_0B(\xi - 0)/A(\xi - 0)\,,\ \ s_\xi^- = 1 - S_\xi^- + (S_\xi^-)^2\,; \\ S_\xi^+ &= 0.5h_1B(\xi + 0)/A(\xi + 0)\,,\ \ s_\xi^+ = 1 + S_\xi^+ + (S_\xi^+)^2\,. \end{aligned}$$

Note that for any h_0 we have

$$r_i,\ s_0,\ s_1,\ s_\xi^-,\ s_\xi^+ \ge 3/4\,. \tag{3.1}$$

We shall consider the following discrete problem (difference scheme), associated with the differential problem (2.1)–(2.5)

$$\begin{aligned} L_h v &:= r(av_{\bar{x}})_x + b^+ a^{(+1)} v_x + b^- a v_{\bar{x}} - dv = f\,,\\ a^{(+1)} &= a_{i+1}\,,\ f_i = f(x_i)\,,\ 0 < i < N,\ i \ne N_0\,, \end{aligned} \tag{3.2}$$

$$\begin{aligned} l_{\xi h} v &:= s_\xi^+ a_{N_0+1} v_{xN_0} - s_\xi^- a_{N_0} v_{\bar{x}N_0}\\ &\quad - \{\sigma_\xi + 0.5[h_1 q(\xi+0) + h_0 q(\xi-0)]\} v_{N_0} = f_{N_0}\,,\\ f_{N_0} &= g_\xi + 0.5[h_1 f(\xi+0) + h_0 f(\xi-0)]\,, \end{aligned} \tag{3.3}$$

$$\begin{aligned} l_{0h} v &:= s_0 a_1 v_{x0} - [\sigma_0 + 0.5 h_0 q(0)] v_0 = f_0\,,\\ f_0 &= g_0 + 0.5 h_0 f(0)\,, \end{aligned} \tag{3.4}$$

$$\begin{aligned} l_{1h} v &:= s_1 a_N v_{\bar{x}N} + [\sigma_1 + 0.5 h_1 q(1)] v_N = f_N\,,\\ f_N &= g_1 - 0.5 h_1 f(1)\,. \end{aligned} \tag{3.5}$$

4. THE RESULTS

4.1. Monotony

Since

$$r > 0,\ a > 0,\ b^+ \ge 0,\ b^- \le 0,\ d \ge 0,\ q \ge 0\,,$$

$$s_0 > 0,\ s_1 > 0,\ s_\xi^+ > 0,\ s_\xi^- > 0\,,$$

$$\sigma_0 \ge 0,\ \sigma_1 \ge 0,\ \sigma_0 + \sigma_1 > 0,\ \sigma_\xi \ge 0\,,$$

we can prove the following property.

Theorem 1. *For any grid stepsize h_0, the difference scheme (3.2)–(3.5) satisfies the maximum principle, that is:*

1) If v is not constant and

$$L_h v \ge 0\,,\ l_{\xi h} v \ge 0\,,\ l_{0h} v \ge 0\,,\ l_{1h} v \le 0\,,$$

then v does not attain its positive maximum value in ω_h.

2) If v is not constant and

$$L_h v \le 0\,,\ l_{\xi h} v \le 0\,,\ l_{0h} v \le 0\,,\ l_{1h} v \ge 0\,,$$

then v does not attain its negative minimum value in ω_h.

If the difference scheme satisfies the maximum principle then we say that it is monotone, and if the difference scheme is monotone for and h_0 then we say that it is unconditionally monotone.

In this sense the scheme (3.2) – (3.5) is unconditionally monotone. Using this fact, we can prove the existence and uniqueness of its solution.

4.2. Solution of the discrete problem

The problem for v can be written as follows.

$$A_i v_{i-1} - C_i v_i + B_i v_{i+1} = Y_i\,,\ \ 1 < i < N\,, \tag{4.1}$$

$$v_0 = p_0 v_1 + Y_0\,,\ \ v_N = p_1 v_{N-1} + Y_N\,, \tag{4.2}$$

where

$$A_i = a_i(r_i - h_\nu b_i^-)\,,\ \ B_i = a_{i+1}(r_i + h_\nu b_i^+)\,, \tag{4.3}$$

$$C_i = A_i + B_i + h_i^2 d_i\,,\ \ i \ne N_0\,, \tag{4.4}$$

$$0A_{N_0} = s_\xi^- a_{N_0}\,,\ \ B_{N_0} = s_\xi^+ a_{N_0+1}/\gamma\,,\ \ \gamma = h_1/h_0 = \text{const}\,, \tag{4.5}$$

$$C_{N_0} = A_{N_0} + B_{N_0} + h_0\{\sigma_\xi + 0.5[h_1 q(\xi+0) + h_0 q(\xi-0)]\}\,, \tag{4.6}$$

$$p_0 = s_0 a_1/\{s_0 a_1 + h_0[\sigma_0 + 0.5 h_0 q(0)]\}\,, \tag{4.7}$$

$$p_1 = s_1 a_N/\{s_1 a_N + h_1[\sigma_1 + 0.5 h_1 q(1)]\}\,, \tag{4.8}$$

$$Y_i = h_\nu^2 f_i\,,\ \ i \ne N_0, 0, N\,, \tag{4.9}$$

$$Y_{N_0} = h_0 f_{N_0}\,, \tag{4.10}$$

$$Y_0 = -h_0 f_0/\{s_0 a_1 + h_0[\sigma_0 + 0.5 h_0 q(0)]\}\,, \tag{4.11}$$

$$Y_N = h_1 f_N/\{s_1 a_N + h_1[\sigma_1 + 0.5 h_1 q(1)]\}\,, \tag{4.12}$$

and f_i ($i = 0, 1, 2, \ldots, N$) are defined as in (3.2) – (3.5).

By assumptions (2.4), (2.5), properties (3.1) and (4.3) – (4.8), for any stepsize h_0 we have

$$A_i > 0\,,\ B_i > 0\,,\ C_i \ge A_i + B_i\,,\ \ 0 < i < N\,, \tag{4.13}$$

$$0 < p_0 \le 1\,,\ \ 0 < p_1 \le 1\,,\ p_0 + p_1 < 2\,, \tag{4.14}$$

$$|A_i - B_i| \le M_1 h_0\,,\ \ M_1 = \text{const} > 0\,,\ i \ne N_0\,, \tag{4.15}$$

$$\frac{1}{M_2} < \frac{A_i}{B_i} < M_2\,,\ \ M_2 = \text{const} > 0\,,\ \ i = N_0\,. \tag{4.16}$$

With the aid of (4.13), (4.14), we can write the sweeping formulae for computing the solution of the problem (4.1), (4.2), that is (3.2) – (3.5), which are numerically stable (see [8, p. 42 – 44]).

4.3. Stability

Now for any grid function z defined on ω_h , we define the norm

$$\|z\| = \max\{|z_i|\}\,,\ \ 0 \le i \le N\,.$$

Taking into account the sweeping formulae for solving the problem (4.1), (4.2) (see [8, p. 42 – 44]) and the relations (4.3) – (4.16) we can prove

Lemma 2. *For any positive grid stepsize* h_0,

$$\|v\| \le \text{const}\cdot\|f\|\,,$$

where $f = (f_0, f_1, \dots, f_N)$ *are the right-hand members of the problem* (3.2) – (3.5) *and* v *is the solution of that problem.*

4.4. Asymptotic expansion for the error

First we introduce a notation. Assuming that

$$w \in Q_\xi^{\mu+\lambda}[0,\,1] \cap C[0,\,1]\,. \tag{4.17}$$

We denote by $\rho(h)$ a quantity which depends on w and h ($h > 0$) with the following properties:

$$\rho(h) = \begin{cases} o(h) & \text{if } \lambda = 0\,, \\ O(h^\alpha) & \text{if } \lambda = \alpha\,, \end{cases}$$

where $o(h) \to 0$ when $h \to 0$ and $\left|\frac{O(h^\alpha)}{h^\alpha}\right| \le \text{const}$.

Lemma 3. *For any* $h > 0$*, we have*

a)

$$w(x + h/2) = w(x) + \sum_{i=1}^{\mu} \frac{h^i}{2^i i!} w^{(i)}(x) + h^\mu \rho(h)\,,$$

$$x,\ x + h/2 \in [0,\,\xi] \ \textit{or}\ [\xi,\,1]\,;$$

b)

$$w(x-h/2) = w(x) + \sum_{i=1}^{\mu} \frac{(-1)^i h^i}{2^i i!} w^{(i)}(x) + h^\mu \rho(h)\,,$$
$$x,\ x-h/2 \in [0,\,\xi] \ \text{or}\ [\xi,\,1]\,;$$

c)

$$A\Big(x+\frac{h}{2}\Big)\frac{w(x+h)-w(x)}{h} = \sum_{k=0}^{[(\mu-1)/2]} h^{2k}\psi_k(w) + \frac{h}{2}\sum_{k=0}^{[(\mu-2)/2]} h^{2k}\varphi_k(w) + h^{\mu-1}\rho(h)\,,$$
$$x,\ x+h \in [0,\,\xi] \ \text{or}\ [\xi,\,1]\,;$$

d)

$$a w_{\bar{x}} = \sum_{k=0}^{[(\mu-2)/2]} h^{2k}\varphi_k(w) + h^{\mu-2}\rho(h)\,,$$
$$x,\ x+h,\ x-h \in [0,\,\xi] \ \text{or}\ [\xi,\,1]\,,$$

where

$$\psi_k(w) = \sum_{i+j=k} \frac{(Aw^{(2i+1)})^{(2j)}(x)}{(2i+1)!(2j)!2^{2i+2j}}\ ;$$
$$\varphi_k(w) = \sum_{i+j=k} \frac{(Aw^{(2i+1)})^{(2j+1)}(x)}{(2i+1)!(2j+1)!2^{2i+2j}}\ .$$

Note that $\psi_0(w) = Aw'$, $\varphi_0(w) = (Aw')'$.

Proof. By Taylor's formula we have

$$w(x+h/2) = w(x) + \sum_{i=1}^{\mu-1} \frac{h^i}{2^i i!} w^{(i)}(x) + \frac{h^\mu}{2^\mu \mu!} w^{(\mu)}(x+\theta h)\,,$$
$$0<\theta<1$$

Since by assumption (4.17) $w^{(\mu)} \in Q_\xi^\lambda[0,\,1]$ and $x,\ x+h \in [0,\,\xi]$ or $[\xi,\,1]$, we have

$$w^{(\mu)}(x+\theta h) - w^{(\mu)}(x) = \rho(h)\,,$$

and hence

$$w^{(\mu)}(x+\theta h) = w^{(\mu)}(x) + \{w^{(\mu)}(x+\theta h) - w^{(\mu)}(x)\}$$
$$= w^{(\mu)}(x) + \rho(h).$$

This first implies a) and then b), c), d).

Applying Lemma 3 we can prove

Lemma 4. *If* $w \in Q_\xi^{\mu+2+\lambda}[0,1] \cap C[0,1]$, $\lambda = 0$ *or* α, *then*

$$L_h w = Lw + \sum_{k=1}^{[\mu/2]} h_\nu^{2k} F_k(w) + h_0^\mu \rho(h_0)\,, \quad i \neq N_0, 0, N\,,$$

$$l_{\xi h} w = l_\xi w + \sum_{k=1}^{[\mu/2]} h_1^{2k} G_k(w)|_{x=\xi+0} - \sum_{k=1}^{[\mu/2]} h_0^{2k} G_k(w)|_{x=\xi-0}$$
$$+ 0.5 h_1 \Big[Lw + \sum_{k=1}^{[\mu/2]} h_1^{2k} F_k(w)\Big]\Big|_{x=\xi+0}$$
$$+ 0.5 h_0 \Big[Lw + \sum_{k=1}^{[\mu/2]} h_0^{2k} F_k(w)\Big]\Big|_{x=\xi-0} + h_0^\mu \rho(h_0)\,,$$

$$l_{0h} = l_0 w + \sum_{k=1}^{[\mu/2]} h_0^{2k} G_k(w)|_{x=0}$$
$$+ 0.5 h_0 \Big[Lw + \sum_{k=1}^{[\mu/2]} h_0^{2k} F_k(w)\Big]\Big|_{x=0} + h_0^\mu \rho(h_0)\,,$$

$$l_{1h} = l_1 w + \sum_{k=1}^{[\mu/2]} h_1^{2k} G_k(w)|_{x=1}$$
$$- 0.5 h_1 \Big[Lw + \sum_{k=1}^{[\mu/2]} h_1^{2k} F_k(w)\Big]\Big|_{x=1} + h_0^\mu \rho(h_0)\,,$$

where $F_k(w)$ *and* $G_k(w)$ *depend upon* w *and the derivatives of* w *as follows:*

$$F_k(w) = \varphi_k(w) + \frac{B}{A}\psi_k(w) + \frac{1}{4}\frac{B^2}{A^2}\varphi_{k-1}(w),$$
$$G_k(w) = \psi_k(w) + \frac{1}{4}\frac{B}{A}\varphi_{k-1}(w) + \frac{1}{4}\frac{B^2}{A^2}\psi_{k-1}(w), \tag{4.18}$$
$$F_k(w) \in Q_\xi^{\mu-2k+\lambda}.$$

Now assume that w_k, $k = 0, 1, \dots, [m/2]$, are functions satisfying

$$w_k \in Q_\xi^{m-2k+2+\lambda}[0, 1] \cap C[0, 1]. \tag{4.19}$$

We put

$$S_m = \sum_{k=0}^{[m/2]} h_0^{2k} w_k, \quad z = v - S_m, \quad w_0 = u, \tag{4.20}$$

where $w_0 = u$ is the solution of the differential problem (2.1) – (2.5) and v is that of the discrete problem (3.2) – (3.5). Applying Lemma 4 we have

Lemma 5. *Under the assumption* (4.19), *we have*

$$L_h z = -\sum_{k=1}^{[m/2]} h_0^{2k}[Lw_k + E_k] + h_0^m \rho(h_o), \quad i \neq N_0, 0, N,$$

$$l_{\xi h} z = -0.5 h_1 \sum_{k=1}^{[m/2]} h_0^{2k}[Lw_k + E_k]|_{x=\xi+0}$$
$$-0.5 h_0 \sum_{k=1}^{[m/2]} h_0^{2k}[Lw_k + E_k]|_{x=\xi-0}$$
$$-\sum_{k=1}^{[m/2]} h_0^{2k}\left[l_\xi w_k + H_k^1|_{x=\xi+0} - H_k^0|_{x=\xi-0}\right] + h_0^m \rho(h_0),$$

$$l_{0h} z = -0.5 h_0 \sum_{k=1}^{[m/2]} h_0^{2k}[Lw_k + E_k]|_{x=0}$$
$$-\sum_{k=1}^{[m/2]} h_0^{2k}(l_0 w_k + H_k^0)|_{x=0} + h_0^m \rho(h_0),$$

$$l_{1h}z = 0.5h_1 \sum_{k=1}^{[m/2]} h_0^{2k}[Lw_k + E_k]|_{x=1} - \sum_{k=1}^{[m/2]} h_0^{2k}(l_1 w_k + H_k^1)|_{x=1} + h_0^m \rho(h_0),$$

where

$$E_k = \begin{cases} \sum_{j=0}^{k-1} F_{k-j}(w_j) & \text{if } 0 < x < \xi, \\ \sum_{j=0}^{k-1} \gamma^{2(k-j)} F_{k-j}(w_j) & \text{if } \xi < x < 1, \end{cases}$$

$$H_k^0 = \sum_{j=0}^{k-1} G_{k-j}(w_j), \quad H_k^1 = \sum_{j=0}^{k-1} \gamma^{2(k-j)} G_{k-j}(w_j),$$

and by (4.18)

$$E_k \in Q_\xi^{m-2k+\lambda}[0, 1] \tag{4.21}$$

Lemma 6. *There exist functions w_k satisfying (4.19) so that the grid function z defined by (4.20) verifies the following discrete problem*

$$L_h z = y_i = h_0^m \rho(h_0), \ i \neq 0, N_0, N,$$
$$l_{\xi h} z = y_{N_0} = h_0^m \rho(h_0),$$
$$l_{0h} z = y_0 = h_0^m \rho(h_0),$$
$$l_{1h} z = y_N = h_0^m \rho(h_0).$$

Proof. We choose $w_0 = u$ and determine w_k, $k = 1, 2, \ldots, [m/2]$, satisfying

$$Lw_k = -E_k, \ 0 < x < 1, \ x \neq \xi \tag{4.22}$$

$$l_\xi w_k = -H_k^1|_{x=\xi+0} + H_k^0|_{x=\xi-0}, \ w_k(\xi + 0) = w_k(\xi - 0), \tag{4.23}$$

$$l_0 w_k = -H_k^0|_{x=0}, \tag{4.24}$$

$$l_1 w_k = -H_k^1|_{x=1}. \tag{4.25}$$

By (2.6) $w_0 = u \in Q_\xi^{m+2+\lambda}[0, 1] \cap C[0, 1]$. So (4.21) yields that $E_1 \in Q_\xi^{m-2+\lambda}[0, 1]$. Hence w_1 is determined by Lemma 1 and belongs to $Q_\xi^{m+\lambda}[0, 1] \cap C[0, 1]$. Again, by (4.21) $E_2 \in Q_\xi^{m-4+\lambda}[0, 1]$ and so w_2 is determined by Lemma 1 and lies in $Q_\xi^{m-2+\lambda}[0,1]\cap C[0,1]$, etc. Thus, with $w_0 = u$ the sequence of problems (4.22) – (4.25) determine w_k successively from $k=1$ to $k=[m/2]$, which satisfy (4.19) and are independent upon h_0 (because the problems (4.22) – (4.25) are independent upon h_0).

With these $[m/2]$ functions w_k, the right–hand side members of the equations in Lemma 5 can be reduced so that Lemma 6 is proved.

Taking Lemma 2 into account we can deduce from Lemma 6 that

$$\|z\| \le \|y\| = h_0^m \rho(h_0)\,.$$

So we have

Theorem 2. *Under the assumptions* (2.4) – (2.5) *there exist* $[m/2]$ *functions* w_k, $k=1,2,\dots,[m/2]$, *which are independent upon* h_0 *but dependent on* γ *and satisfy* (4.19) *such that*

$$v_i - u(x_i) = \sum_{k=1}^{[m/2]} h_0^{2k} w_k(x_i) + h_0^m \rho(h_0)\,. \tag{4.26}$$

This is the asymptotic expansion of order $2[m/2]$ for the error $v_i - u(x_i)$ with respect to the grid stepsize parameter h_0.

Remark. Consider the particular case when $B \equiv 0$ and $\sigma_\xi = 0$. Then the equations (2.1), (2.2) coincide with (3.1), (3.3), (3.4) of [3, p. 87] and the corresponding difference equations coincide with those of [3]. The boundary condition (3.2) in [3] is simpler than (2.3). As a result we have obtained (4.26). In comparison with (3.16) of [3, p. 90] we note that, when m is odd, the first parts of (4.26) and of (3.16) of [3] are the same because in this case $[(m-1)/2] = [m/2]$. When m is even, (4.26) have one term ($h_0^m w_{[m/2]}$) more than (3.16) of [3] because in this case $[(m-1)/2] = [m/2]-1$. Besides that, the remainder in (4.26) is smaller than that in (3.16) of [3] by the factor $\rho(h_0)$ which tends to zero when h_0 tends to zero.

So our problem is more general and our result is somewhat better.

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Received November 30, 1993
Revised July 22, 1994

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