VIETNAM JOURNAL OF MATHEMATICS
Volume 23, Number 2, 1995

ON THE EXISTENCE OF BAYESIAN ESTIMATES IN MULTIDIMENSIONAL NONLINEAR STATISTICAL MODELS WITH COMPACT PARAMETER SPACE

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Abstract. *In this note, we investigate the existence of Bayesian estimates for the location parameter $\theta \in \Theta$ and for the variance component σ^2 in the nonlinear statistical models $X = \varphi(\theta) + \varepsilon$, where X is a random vector of observations and Θ is a compact subset of the finite-dimensional normed linear space F.*

INTRODUCTION

An important problem of mathematical statistics is investigating the linear and nonlinear statistical models (see [1], [2]). In [3] we proved the existence of Bayesian estimates for the location parameter and for the variance component in one-dimensional nonlinear statistical models.

In this paper, by the functional analysis method, we shall investigate the existence of Bayesian estimates in multidimensional nonlinear models.

First of all, we give some notations:

$E,\ F$: finite-dimensional normed linear spaces.

$\mathcal{B}(E),\ \mathcal{B}(F)$: σ-algebras of all Borel sets in the spaces $E,\ F$.

$M(n\times q),\ M(p\times r)$: spaces of all $n\times q$-matrices and $p\times r$-matrices.

$\mathcal{B}(n\times q),\ \mathcal{B}(p\times r)$: σ-algebras of all Borel sets in $M(n\times q),\ M(p\times r)$.

$R^n,\ R^p$: n-dimentional and p-dimensional Euclidean spaces with the standard scalar product $\langle\cdot,\cdot\rangle$.

$\overline{K}$: closure of a set K.

1. STATISTICAL MODELS

Let us consider the following statistical models:

$$X = \varphi(\theta) + \varepsilon\,, \tag{1}$$

where:

X is an observed random variable, taking the values in E,

ε is a random error variable, taking the values in E,

θ is an unknown parameter, $\theta \in \Theta$.

Θ is a subset of F, and

φ is a known function, $\varphi : \Theta \to E$.

The model (1) is called a multidimensional linear model if Θ is a linear subspace of F, and φ is a linear function. The model (1) is called a (q, r)–dimensional linear model if $E = M(n \times q)$, $F = M(p \times r)$. If $q = r = 1$ (i.e. $M(n \times q) = R^n$, $M(p \times r) = R^p$), then (1) is called one–dimensional linear statistical model (see [1]).

The model (1) is called a multidimensional nonlinear model if either Θ is a nonlinear subset of F or φ is a nonlinear function (see [2], [3]). If Θ is a compact subset of F, then (1) is called a multidimensional nonlinear model with compact parameter space. If $E = M(n \times q)$, $F = M(p \times r)$ and Θ is a compact subset of $M(p \times r)$, then (1) is called a (q, r)– dimensional nonlinear model with compact parameter space. If $q = r = 1$ (i.e. $M(n \times q) = R^n$, $M(p \times r) = R^p$), then (1) is called an one–dimensional nonlinear model with compact parameter space (see [3]).

As we have known, for a random variable X, there exists a conditional regular distribution $p^{X|\theta}$ (see [4]). We shall use symbol Q_θ to denote $p^{X|\theta}$.

Assume that μ is a σ–finite measure in the space $(E, \mathcal{B}(E))$ and $Q_\theta \ll \mu$ for every $\theta \in \Theta \subset F$. Then by the Radon–Nikodym theorem there exists a function $f_\theta(x)$ such that

$$f_\theta(x) = \frac{Q_\theta(dx)}{\mu(dx)}\,.$$

Definition 1.1. A function $h : (E, \mathcal{B}(E)) \to (F, \mathcal{B}(F))$ is called an estimate of the parameter $\theta \in \Theta \subset F$ if it is a Borel measurable function. A Borel measurable function h is said to be bounded if it satisfies the condition:

$$\sup_{x \in E} \|h(x)\|_F < +\infty\,.$$

Let $B(E, F)$ denote the space of all bounded Borel measurable functions from E to F. Clearly, it forms a class of estimates of the parameter $\theta \in \Theta \subset F$.

A Borel measurable function h from E to F is said to be essentially bounded measurable if there exists a set $B \in \mathcal{B}(E)$, $\mu(B) = 0$ such that:

$$\sup_{x \in E \setminus B} \|h(x)\|_F < +\infty .$$

Let us denote by $L^\infty(\mu, E, F)$ the space of all essentially bounded measurable functions. Clearly, it is a class of estimates of the parameter $\theta \in \Theta \subset F$.

2. ON THE EXISTENCE OF BAYESIAN ESTIMATES FOR THE LOCATION PARAMETER

First, let us consider the compact parameter space $\Theta \subset F$. By $\mathcal{B}(\Theta)$ we denote the σ–algebra of all Borel sets in the space Θ.

A probability measure τ in $(\Theta, \mathcal{B}(\Theta))$ is called a priori distribution of $\theta \in \Theta \subset F$.

Definition 2.1. Suppose H is a function defined by

$$H : E \times \Theta \to F \times \Theta , \quad H(x, \theta) = (h(x), \theta) ,$$

and L is a function

$$L : F \times \Theta \to \overline{R}^+ = [0, +\infty] .$$

Then the composed function defined by

$$L(h(x), \theta) = L \circ H : E \times \Theta \to \overline{R}^+$$

is called a loss function.

Next, let us consider the measurable spaces $(E, \mathcal{B}(E))$, $(F, \mathcal{B}(F))$, $(\Theta, \mathcal{B}(\Theta))$, $(\overline{R}^+, \mathcal{B}(\overline{R}^+))$, where $\mathcal{B}(\overline{R}^+)$ is a σ–algebra of all Borel sets in $\overline{R}^+$.

Now, we define:

$$\mathcal{A} = \{A \times B : A \in \mathcal{B}(E),\ B \in \mathcal{B}(\Theta)\} ,$$

$$\mathcal{C} = \{C \times B : C \in \mathcal{B}(F),\ B \in \mathcal{B}(\Theta)\} .$$

Let $\mathcal{B}(E) \times \mathcal{B}(\Theta)$ denote the σ–algebra generated by $\mathcal{A}$, $\mathcal{B}(F) \times \mathcal{B}(\Theta)$ denote the σ–algebra generated by $\mathcal{C}$. Then, the following propositions are well–known:

Proposition 2.1. *Let L be a $(\mathcal{B}(F)\times\mathcal{B}(\Theta), \mathcal{B}(\overline{R}^+))$-measurable function. Then the loss function $L(h(\cdot),\cdot)$ is a $(\mathcal{B}(E)\times\mathcal{B}(\Theta), \mathcal{B}(\overline{R}^+))$-measurable function.*

Proposition 2.2. *The classes of estimates $B(E,F)$ and $L^\infty(\mu,E,F)$ are Banach spaces with the following norms:*

$$\|h\|_{B(E,F)} = \sup_{x\in E}\|h(x)\|_F\,,$$

$$\|h\|_\infty = \inf_{B:\mu(B)=0}\ \sup_{x\in E\setminus B}\|h(x)\|_F\,.$$

By Propositions 2.1, 2.2, we can define following Bayesian estimate.

Definition 2.2. A functional $\psi : B(E,F) \to \overline{R}^+$ is said to be a Bayesian risk function with a priori distribution τ if

$$\psi(h) = \int\limits_\Theta\int\limits_E L(h(x),\theta)Q_\theta(dx)\tau(d\theta)$$

$$\Big(\psi(h) = \int\limits_\Theta\int\limits_E L(h(x),\theta)f_\theta(x)\mu(dx)\tau(d\theta)\Big)\,.$$

An estimate $\hat{h}\in B(E,F)$ is said to be a Bayesian estimate of the parameter $\theta\in\Theta\subset F$ with a priori distribution if

$$\psi(\hat{h}) = \inf_{h\in B(E,F)}\psi(h)\,.$$

Definition 2.3. A functional $\psi : L^\infty(\mu,E,F) \to \overline{R}^+$ is said to be a Bayesian risk with a priori distribution τ if

$$\psi(h) = \int\limits_\Theta\int\limits_E L(h(x),\theta)Q_\theta(dx)\tau(d\theta)$$

$$\Big(\psi(h) = \int\limits_\Theta\int\limits_E L(h(x),\theta)f_\theta(x)\mu(dx)\tau(d\theta)\Big)\,.$$

An estimate $\hat{h}\in L^\infty(\mu,E,F)$ is said to be a Bayesian estimate of the parameter $\theta\in\Theta\subset F$ with a priori distribution τ if

$$\psi(\hat{h}) = \inf_{h\in L^\infty(\mu,E,F)}\psi(h)\,.$$

Theorem 2.1. *Let K be a class of all estimates of the parameter $\theta\in\Theta\subset F$ satisfying the following conditions:*

(i) $h(E) \subset \Theta\,,\ \forall h \in K$.

(ii) *For any* $\varepsilon > 0$, *there exist a finite partition* $\{E_i\}_{i=1}^m \subset E$ *and points* $x_i \in E_i$, $i = 1, \ldots, m$ *such that*

$$\sup_{x \in E_i} \|h(x) - h(x_i)\|_F < \varepsilon\,,\ \forall h \in K\,,\ \forall i = 1, \ldots, m\,.$$

(iii) *There exists* $C > 0$ *such that:*

$$|L(y,\theta) - L(y',\theta)| \le C\|y - y'\|_F\,,\ \forall y, y' \in F\,,\ \forall \theta \in \Theta\,.$$

Then K is a relatively compact subset of the space $B(E,F)$ and in the class $\overline{K}$ there exists a Bayesian estimate.

Theorem 2.2. Let K be a class of estimates of the parameter $\theta \in \Theta \subset F$, satisfying the following conditions:

(i) $h(E) \subset \Theta\ (\mathrm{mod}\mu)\,,\ \forall h \in K$.

(ii) *For any* $\varepsilon > 0$, *there exist a finite partition* $\{E_i\}_{i=1}^m \subset E$ *and points* $x_i \in E_i$, $i = 1, \ldots, m$ *such that*

(a) $\|h(x_i)\|_F$, *is uniformly bounded for* $h \in K$ *and* $i = 1, \ldots, m$.

(b) *For each* $h \in K$ *there exists* $B \in \mathcal{B}(E)$ *with* $\mu(B) = 0$ *such that*

$$\sup_{x \in E_i \setminus B} \|h(x) - h(x_i)\|_F < \varepsilon\,,\ \ (i = 1, \ldots, m)\,.$$

(iii) *There exists* $C > 0$ *such that*

$$|L(y,\theta) - L(y',\theta)| \le C\|y - y'\|_F\,,\ \forall y, y' \in F\,,\ \forall \theta \in \Theta\,.$$

Then K is a relatively compact subset of the space $L^\infty(\mu, E, F)$ and in the class $\overline{K}$ there exists a Bayesian estimate.

3. ON THE EXISTENCE OF BAYESIAN ESTIMATES FOR THE VARIANCE COMPONENT

In this section we shall investigate the existence of Bayesian estimate for the variance component in the (q,r)–dimensional models.

First, let R^{nq} be the nq–dimensional Euclidean space. Let us consider the following mapping $T : M(n \times q) \to R^{nq}$, defined by

$$T(A) = \vec{A} = (a_{11}, a_{21}, \ldots, a_{n1}, a_{12}, a_{22}, \ldots, a_{n2}, \ldots, a_{1q}, a_{2q}, \ldots, a_{nq}),$$

where $A = (a_{ij}) \in M(n \times q)$ and $\vec{A}$ is a nq-dimensional vector.

Obviously, T is a linear mapping of $M(n \times q)$ onto R^{nq}. Moreover, T is an isometry of $M(n \times q)$ onto R^{nq}.

Now, let us consider the following (q, r)-dimensional nonlinear model:

$$X = \varphi(\theta) + \varepsilon, \tag{1'}$$

where X is an observed random matrix, taking the values in $M(n \times q)$ and ε is a random error matrix, taking the values in $M(n \times q)$.

The covariance matrix $D(\vec{\varepsilon})$ of the nq-dimensional random error vector $\vec{\varepsilon}$ is called the variance component of the random error matrix ε. We will denote by $\mathrm{Var}\,\varepsilon$ the variance component of the random matrix ε. For each k, let $M^{\geq}(k \times k)$ denote the space of all non-negative definite $s \times s$-matrices. We shall assume that, for non-linear model (1'), $\mathrm{Var}\,\varepsilon = \psi(\sigma^2)$, where $\psi : M^{\geq}(s \times s) \to M^{\geq}(nq \times nq)$ is a known nonlinear function and σ^2 is an unknown parameter, $\sigma^2 \in M^{\geq}(s \times s) \subset M(s \times s)$.

In this section we shall estimate unknown parameter $\sigma^2 \in M^{\geq}(s \times s)$. This unknown parameter σ^2 also is said to be a variance component.

Definition 3.1. A Borel function $h : (M(n \times q), \mathcal{B}(n \times q)) \to (M(s \times s), \mathcal{B}(s \times s))$ is called an estimate of the variance component $\sigma^2 \in M^{\geq}(s \times s)$.

As we have known, for a random matrix X there exists a conditional regular distribution $p^{X|\sigma^2}$, where $\sigma^2 \in M^{\geq}(s \times s)$. Denote by Q_{σ^2} the conditional regular distribution $p^{X|\sigma^2}$ and assume that $Q_{\sigma^2} \ll \mu$, for every σ^2. Then, there exists $f_{\sigma^2}(x)$ such that

$$f_{\sigma^2}(x) = \frac{Q_{\sigma^2}(dx)}{\mu(dx)}.$$

Denote by $B(M(n \times q), M(s \times s))$ the space of all bounded Borel measurable functions and by $L^\infty(\mu, M(n \times q), M(s \times s))$ the space of all essentially bounded measurable functions from $M(n \times q)$ to $M(s \times s)$. Then, they are Banach spaces with the norms

$$\|h\|_{B(M(n\times q), M(s\times s))} = \sup_{x \in M(n\times q)} \|h(x)\|_{M(s\times s)},$$

$$\|h\|_\infty = \inf_{B:\mu(B)=0} \sup_{x\in M(n\times q)\setminus B} \|h(x)\|_{M(s\times s)} \,.$$

Theorem 3.1. *Let $K \subset B(M(n\times q), M(s\times s))$ be a class of estimates of the variance component $\sigma^2 \in M^{\geq}(s\times s)$, satisfying the following conditions*

(i) $h(M(n\times q)) \subset M^{\geq}(s\times s)\,,\ \forall h\in K.$

(ii) *For any $\varepsilon > 0$, there exist finite partition $\{E_i\}_{i=1}^m \subset M(n\times q)$ and points $x_i \in E_i$, $i = 1,\ldots,m$ such that*

$$\sup_{x\in E_i} \|h(x) - h(x_i)\|_{M(s\times s)} < \varepsilon\,,\ \ \forall h\in K\,,\ \forall i = 1,\ldots,m\,.$$

(iii) *There exists $C > 0$ such that*

$$|L(y,\sigma^2) - L(y',\sigma^2)| \leq C\|y-y'\|_{M(s\times s)},\ \forall y,y' \in M(s\times s)\,,$$
$$\forall \sigma^2 \in M \geq (s\times s)\,.$$

Then, K is a relatively compact subset of $B(M(n\times q), M(s\times s))$ and in $\overline{K}$ there exists a Bayesian estimate.

Theorem 3.2. Let $K \subset L^\infty(\mu, M(n\times q), M(s\times s))$ be a class of estimate of the variance component $\sigma^2 \in M^{\geq}(s\times s) \subset M(s\times s)$, satisfying the following conditions:

(i) $h(M(n\times q)) \subset M^{\geq}(s\times s)\,,\ \forall h\in K.$

(ii) *For any $\varepsilon > 0$, there exist finite partition $\{E_i\}_{i=1}^m \subset M(n\times q)$ and points $x_i \in E_i$, $i = 1,\ldots,m$ such that*

(a) $\|h(x_i)\|_{M(s\times s)}$, *is uniformly bounded for $h\in K$ and $i = 1,\ldots,m$.*

(b) *For each $h \in K$, there exists $B \in \mathcal{B}(n\times q)$, with $\mu(B) = 0$ such that*

$$\sup_{x\in E_i\setminus B} \|h(x) - h(x_i)\|_{M(s\times s)} < \varepsilon,\ \ \forall i = 1,\ldots,m\,.$$

(iii) *There exist $C > 0$ such that*

$$|L(y,\sigma^2) - L(y',\sigma^2)| \leq C\|y-y'\|_{M(s\times s)},\ \forall y,y' \in M(s\times s)\,,$$
$$\forall \sigma^2 \in M^{\geq}(s\times s)\,.$$

Then K is a relatively compact subset of $L^\infty(\mu, M(n\times q), M(s\times s))$ and in $\overline{K}$ there exists a Bayesian estimate.

4. THE PROOFS

In this section we shall prove the results in the preceding sections.

Proof of Theorem 2.1. Since Θ is a compact subset of F, by the condition (i), we have

$$\sup_{x\in E} \|h(x)\|_F < +\infty .$$

Hence, we obtain that $K \subset B(E, F)$.

Now, let $F^m = F \times \cdots \times F$ (m factors). Clearly, F^m is a finite-dimensional normed linear space with the norm:

$$\|y\|_{F^m} = \max_{1\le i\le m} \|y_i\|_F .$$

Let us consider the function $\Phi : B(E, F) \to F^m$, defined by

$$\Phi(h) = (h(x_1), h(x_2), \ldots, h(x_m)) .$$

By the condition (i), we see that $\Phi(K)$ is a bounded set of F^m. It follows that $\Phi(K)$ is a totally bounded set of F^m. Consequently, there exist balls $B(t_j, \varepsilon)$, $j = 1, 2, \ldots, r$ such that

$$\Phi(K) \subset \bigcup_{j=1}^{r} B(t_j, \varepsilon) .$$

By a similar argument of the Proposition 1.1 in [3], there is a function $h_j \in K$ such that

$$\|h(x_i) - h_j(x_i)\|_F < 2.\varepsilon, \ \forall i = 1, \ldots, m .$$

Moreover, since h, $h_j \in K$, by the condition (ii), we have

$$\sup_{x\in E_i} \|h(x) - h(x_i)\|_F < \varepsilon , \ \forall i = 1, \ldots, m ;$$

$$\sup_{x\in E_i} \|h_j(x) - h_j(x_i)\|_F < \varepsilon , \ \forall i = 1, \ldots, m .$$

It follows that

$$\sup_{x \in E_i} \|h(x) - h_j(x)\|_F < 4\varepsilon\,, \; \forall i = 1, \ldots, m\,.$$

Consequently

$$\sup_{x \in E} \|h(x) - h_j(x)\|_F = \max_{1 \leq i \leq m} \sup_{x \in E_i} \|h(x) - h_j(x)\|_F < 4\varepsilon\,.$$

This means that

$$K \subset \bigcup_{j=1}^{r} B(h_j, 4\varepsilon)\,.$$

This shows that K is a totally bounded set of $B(E, F)$ and it follows that K is a relatively compact subset of $B(E, F)$.

Next, we shall prove that

$$h(E) \subset \Theta\,, \; \forall h \in \overline{K}\,.$$

Indeed, take any $h \in \overline{K}$. Then, there exists a sequence $(h_n) \subset K$ such that

$$\|h_n - h\|_{B(E,F)} \to 0, \quad \text{as} \quad n \to \infty\,.$$

It follows that

$$\|h_n(x) - h(x)\|_F \to 0, \quad \text{as} \quad n \to \infty\,.$$

On the other hand, we have

$$h_n(E) \subset \Theta\,, \; \forall n \in N\,.$$

By a similar argument of the Proposition 1.4 in [3], we obtain that

$$h(E) \subset \Theta\,, \; \forall h \in \overline{K}\,.$$

Finally, let us consider the functional $\psi : B(E, F) \to \overline{R}^+$ defined by

$$\psi(h) = \int_\Theta \int_E L(h(x), \theta) Q_\theta(dx) \tau(d\theta)$$

$$(\psi(h) = \int_{\Theta}\int_{E} L(h(x),\theta) f_\theta(x)\mu(dx)\tau(d\theta)) .$$

By a similar argument of the Proposition 2.1.2 in [3], we obtain that ψ is a continuous function on the compact subset $\overline{K} \subset B(E,F)$. Consequently, there exists an $\hat{h} \in \overline{K}$ such that

$$\psi(\hat{h}) = \inf_{h\in\overline{K}} \psi(h) .$$

By the Definition 2.2., $\hat{h}$ is a Bayesian estimate and the proof of the theorem is completed.

Proof of Theorem 2.2. Since $L^\infty(\mu, E, F)$ is a Banach space, we obtain by a similar argument as in the proof of Theorem 2.1, that K is a relatively compact subset of $L^\infty(\mu, E, F)$.

Let us consider the functional $\psi : L^\infty(\mu, E, F) \to \overline{R}^+$ defined by

$$\psi(h) = \int_{\Theta}\int_{E} L(h(x),\theta) Q_\theta(dx)\tau(d\theta)$$

$$(\psi(h) = \int_{\Theta}\int_{E} L(h(x),\theta) f_\theta(x)\mu(dx)\tau(d\theta)) .$$

Clearly, ψ is a continuous function on the compact subset $\overline{K}$ and hence there exists in $\overline{K}$ a Bayesian estimate.

Proof of Theorem 3.1. By a similar argument of Theorem 2.1, we see that K is a relatively compact subset of $B(M(n \times q), M(s \times s))$.

Next, we want to show that if

$$h(M(n\times q)) \subset M^{\geq}(s\times s) ,\ \forall h \in K$$

then

$$h(M(n\times q)) \subset M^{\geq}(s\times s) ,\ \forall h \in \overline{K} .$$

Indeed, take any $h \in \overline{K}$. Then there exists a sequence $(h_m) \subset K$ such that $h_m \to h$ (as $m \to \infty$) in the norm of $B(M(n \times q), M(s \times s))$. It follows that $h_m(x) \to h(x)$ for each $x \in M(n \times q)$. Since $h_m(x) \in M^{\geq}(s \times s)$, by the definition of the non–negative definite matrix, we have $\langle t, h_m(x)t\rangle \geq 0$ for all $t \in R^s$. This implies $\langle t, h(x)t\rangle \geq 0$ for all $t \in R^s$, and thus $h(x) \in M^{\geq}(s\times s)$. Since x is arbitrary in $M(n \times q)$, we have $h(M(n \times q)) \subset M^{\geq}(s \times s)$, as to be shown.

Finally, consider the functional $\psi : B(M(n\times q), M(s\times s)) \to \overline{R}^{+}$ defined by

$$\psi(h) = \int\limits_{M^{\geq}(s\times s)} \int\limits_{M(n\times q)} L(h(x),\sigma^2)Q_{\sigma^2}(dx)\tau(d\sigma^2)\,.$$

Clearly ψ is a continuous function and hence in $\overline{K}$ there exists a Bayesian estimate.

Proof of Theorem 3.2. We shall prove that, if

$$h(M(n\times q)) \subset M^{\geq}(s\times s)\ (\mathrm{mod}\mu)\,,\ \forall h\in K\,,$$

then,

$$h(M(n\times q)) \subset M^{\geq}(s\times s)\ (\mathrm{mod}\mu)\,,\ \forall h\in \overline{K}\,.$$

Indeed, take any $h\in\overline{K}$, by a similar argument of the Theorem 3.1, there exists a sequence $(h_m)\subset K$ such that, for each m $\langle t,\, h_m(x)t\rangle \geq 0$ $(\mathrm{mod}\,\mu)$ and, as $t\to\infty$, $\langle t,\, h_m(x)t\rangle \to \langle t,\, h(x)t\rangle$ $(\mathrm{mod}\,\mu)$, $\forall t\in R^s$. Let us define

$$A = \{x\in M(n\times q) : \langle t,\, h_m(x)t\rangle \xrightarrow{m\to\infty} \langle t,\, h(x)t\rangle\}\,,$$

$$B_m = \{x\in M(n\times q) : \langle t,\, h_m(x)t\rangle \geq 0\}\,,$$

$$B = \bigcap_{m\in N} B_m\,.$$

Then clearly, for each $x\in A\cap B$, $\langle t,\, h(x)t\rangle \geq 0$, $\forall t\in R^s$, and $\mu\,(M(s\times s)\setminus(A\cap B)) = 0$. Consequently, $h(x)\in M^{\geq}(s\times s)$ $(\mathrm{mod}\,\mu)$, which implies that

$$h(M(n\times q)) \subset M^{\geq}(s\times s)\ (\mathrm{mod}\mu)\,,\ \forall h\in \overline{K}\,.$$

Remark: Let us consider one–dimensional model. Suppose that $E = R^1$, $F = R^1$, $\Theta = [a,\, b]$.

Take any continuous function h whose support is a compact subset $[u,\, v]\subset R^1$. Clearly, h is a uniformly continuous function on $[u,\, v]$. Therefore, for every $\varepsilon > 0$, there exists $\delta > 0$ such that:

$$|h(x) - h(y)| < \varepsilon\,,\ \text{whenever}\ |x-y| < \delta\,.$$

Let us divide the interval $[u, v]$ into the subintervals, whose length $< \delta$. Then we get the following subintervals: $E_1, E_2, \ldots, E_{m-1}$. Next, let

$$E_m = (-\infty, u) \cup (v, +\infty) .$$

Thus, $\{E_i\}_{i=1}^m$ is a finite partition of R^1.

Now, take the points $x_i \in E_i$, $i = 1, \ldots, m-1$, and $x_m \in (v, +\infty)$.

By the above consideration, we have

$$|h(x) - h(x_i)| < \varepsilon , \ \forall x \in E_i , \ \forall i = 1, \ldots, m-1 .$$

Moreover,

$$h(x) = h(x_m) = 0 , \ \forall x \in E_m .$$

Therefore,

$$\sup_{x \in E_i} |h(x) - h(x_i)| < \varepsilon , \ \forall i = 1, \ldots, m .$$

Usually, the collection of all continuous functions on R^1 whose support is compact is denoted by $C_{[u,v]}(R^1)$.

Let $K = \{h\}$ be a subset of $C_{[u,v]}(R^1)$ satisfying the above properties. Then $K \neq \emptyset$ and K satisfies the conditions of the Theorem 2.1. Consequently, K is a relatively compact subset of $B(R^1, R^1)$ and in $\overline{K}$ there exists a Bayesian estimate.

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Received September 23, 1993

Revised May 24, 1995

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