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S-SERIES IN THE WONG – ZAKAI APPROXIMATION FOR STOCHASTIC DIFFERENTIAL EQUATION

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Abstract. *We consider numerical solutions for a scalar stochastic differential equation (SDE)* $dX = a(X)ds + b(X)dW(s)$. *Rümelin proposed the general Runge – Kutta schemes for SDEs. These schemes closely relate to the Wong – Zakai ordinary differential equation (ODE) approximating the SDE. Defining order of an RK scheme as those of its convergence to the Wong – Zakai ODE, we derive the order conditions of general RK schemes. The analysis can be carried out by introducing rooted trees and elementary differentials along with the special ODE. The process is similar to Hairer's P–series.*

1. INTRODUCTION

We consider the stochastic initial value problem (SIVP) for the scalar autonomous Ito stochastic differential equation (SDE) given by

$$\begin{cases} dX(s) &= a(X)ds + b(X)dW(s)\,, \quad s \in [0,T]\,, \\ X(0) &= x_0, \end{cases} \tag{1.1}$$

where $W(s)$ represents the standard Wiener process and the initial value x is fixed. SIVP (1.1) is equivalent to the following stochastic integral equation (SIE) for all s in some interval $[O,T]$:

$$X(s) = X(0) + \int_0^s a(X(r))dr + \int_0^s b(X(r))dW(r)\,. \tag{1.2}$$

Here, the second integral, called a stochastic integral, is interpreted in the sense of Ito. The Stratonovich SDE corresponding to the Ito SDE (1.1) is

$$dX = \underline{a}(X)ds + b(X) \circ dW(s)\,, \tag{1.3}$$

where $\underline{a}(X)$ is a shift from $a(X)$ by

$$\underline{a}(X) = a - \frac{1}{2}b'b \tag{1.4}$$

and the stochastic integral should be interpreted in the sense of Stratonovich.

Wong and Zakai [12] showed that if we replace the Wiener process $W(s)$ in (1.1) by its polygonal approximation on the partition $s_{0,N} = 0 < s_{1,N} < \cdots < s_{N,N} = T$

$$W^{(N)} = W(s_{k,N} + (W(s_{k+1,N}) - W(s_{k,N}))\frac{s - s_{k,N}}{s_{k+1,N} - s_{k,N}}, \quad s_{k,N} \le s \le s_{k+1,N}, \tag{1.5}$$

then the solution $X^{(N)}(s)$ of the corresponding initial value problem for the ordinary differential equation (ODE)

$$\begin{cases} dX^{(N)}(s) &= a(X^{(N)})ds + b(X^{(N)})dW^{(N)}(s), \quad s \in [0,T], \\ X^{(N)}(0) &= x_0, \end{cases} \tag{1.6}$$

converges to the solution of the following Stratonovich SDE

$$dX = a(X)ds + b(X) \circ dW(s), \tag{1.7}$$

w.p.1 as the partition becomes infinitely fine.

Rümelin [9] proposed general explicit Runge–Kutta schemes, investigated their order of convergence and suggested a relationship between Runge–Kutta (RK) schemes and the Wong–Zakai (WZ) approximation. However Clark and Cameron [3] showed that the WZ approximation (1.6) can converge to the solution $X(T)$ of (1.7) in the mean–square sense, no faster than $O((\Delta s)^3)$. Here Δs means the maximum length of the interval in the partition for (1.5). Thus general RK schemes have a barrier with respect to the order of convergence.

However, when we observe how the solution of an PK scheme converges to the WZ approximation, we can exploit another viewpoint of order of convergence. Since the Stratonovich SDE possesses its own significance especially in engineering, such viewpoint may attract attention in the case of the Stratonovich. Furthermore, as seen later in the present paper, analysis of order conditions of the RK schemes using the rooted tree has many interesting points as a typical one for discrete approximations of SDE, and give an insight of algebraic structure for them.

In this paper we will define the order of convergence for general RK schemes to the WZ ODE and call it RK–order. Then we will derive the conditions of RK-order using rooted trees proposed by Butcher [2] or Hairer [6]. This notion is similarly derived as in the P-series proposed by Hairer [5]. Note that the mode of convergence is similar to that of the pathwise approximation [8] for the Stratonovich SDE (1.7).

2. GENERAL RUNGE-KUTTA SCHEMES

We will briefly review the general Runge-Kutta schemes proposed by Rümelin. For notational simplicity, we will denote by $\overline{X}_n$ the numerical solution for the exact solution $X(s_{n,N})$ and take equidistant time step, $s_{n,N} - s_{n-1,N} = \Delta s$, $n = 0, 1, \ldots, N$, $N = T/\Delta s$. Then the general m-stage explicit Runge-Kutta scheme (RK scheme) has the following form:

$$\overline{X}_n = \overline{X}_{n-1} + \sum_{i=1}^{m} p_i A_i \Delta s + \sum_{i=1}^{m} q_i B_i \Delta W_n \,, \tag{2.1}$$

where $\overline{X_0} = x_0$ and $\Delta W_n = W(s_{n,N}) - W(s_{n-1,N})$ and intermediate values are given by

$$\begin{aligned}
A_1 &= a(\overline{X}_{n-1}) \,, \\
B_1 &= b(\overline{X}_{n-1}) \,, \\
A_2 &= a(\overline{X}_{n-1} + \beta_{21} A_1 \Delta s + \gamma_{21} B_1 \Delta W_n) \,, \\
B_2 &= b(\overline{X}_{n-1} + \beta_{21} A_1 \Delta s + \gamma_{21} B_1 \Delta W_n) \,, \\
&\vdots \\
A_m &= a(\overline{X}_{n-1} + \sum_{j=1}^{m-1} \beta_{mj} A_j \Delta s + \sum_{j=1}^{m-1} \gamma_{mj} B_j \Delta W_n) \,, \\
B_m &= b(\overline{X}_{n-1} + \sum_{j=1}^{m-1} \beta_{mj} A_j \Delta s + \sum_{j=1}^{m-1} \gamma_{mj} B_j \Delta W_n) \,,
\end{aligned}$$

and

$$\sum_{i=1}^{m} p_i = \sum_{i=1}^{m} q_i = 1 \,. \tag{2.2}$$

We shall remark some features of the RK scheme (2.1).

The numerical solution $\overline{X}_n$ defined by the m–stage RK scheme converges in mean–square sense to the solution of the following Ito equation.

$$dX = [a + \lambda b'b](X)dS + b(X)dW(s)\,.$$

Here the correction factor λ is equal to 0 for $m = 1$, whereas the identity

$$\lambda = \sum_{i=2}^{m} q_i \sum_{j=1}^{i-1} \gamma_{ij} \tag{2.3}$$

holds for $m \geq 2$. However an RK scheme of high order necessarily yields the equation $\lambda = 1/2$ [9, 10]. Thus an RK scheme with order greater than 1 converges to the solution of Stratonovich SDE (1.7).

Second, Rümelin [9] showed that any RK scheme cannot attain order 3 in mean–square sense for scalar Stratonovich SDE (1.7). This order–barrier is because the Ito's Taylor expansion of the solution of (1.7) contains terms like as $\int W(s)ds$ and $\int sdW(s)$ that depend on the whole path of $W(s)$, not only on samples of $W(t)$.

To detour this order–barrier and to proceed the analysis of the structure of RK schemes for SDE, we introduce a new definition of order. Considering the difference between the solution $X^{(N)}(s)$ of (1.6) and the numerical solution $\overline{X}_n$ of (2.1), we say $\overline{X}_n$ is of RK–order p if the estimation

$$|\overline{X}_n - X^{(N)}(n\Delta s)| = o((\Delta s)^p) \qquad \text{as } \Delta s \downarrow 0$$

holds for any positive integer n and the Wiener process $W(s)$. Here the exponent p can be an integer multiple of $1/2$. For the analysis of this order concept, rooted trees are much helpful like as those in partitioned ODE by Hairer [5].

3. S–TREES

Let us consider the WZ approximation (1.6) for SDE. For notational simplicity, hereafter we omit the superscript (N). When we treat with the Taylor series expansion of the exact solution of (1.6), we have to compute higher derivatives of the solution $X(s)$ with respect to sample path $x(s)$. Here we define differential operators associated with the function a or b:

$$L_a = a\frac{d}{dx} \quad \text{and} \quad L_b = b\frac{d}{dx}\,.$$

We demonstrate this in the following way:

$$\begin{aligned}
x(0) &= x_0 \\
L_a x(0) &= a(x_0) \\
L_b x(0) &= b(x_0) \\
L_a^2 x(0) &= L_a a(x_0) = \frac{da}{dx} a(x_0) \\
L_a L_b x(0) &= L_a b(x_0) = \frac{db}{dx} a(x_0) \\
L_b L_a x(0) &= L_b a(x_0) = \frac{da}{dx} b(x_0) \\
L_b L_b x(0) &= L_b b(x_0) = \frac{db}{dx} b(x_0) \\
L_a^3 x(0) &= L_a^2 a(x_0) = \frac{d}{dx}\left(\frac{da}{dx}\right) a(x_0) \\
&= \frac{d^2 a}{dx^2} a^2(x_0) + \left(\frac{da}{dx}\right)^2 a(x_0)
\end{aligned} \tag{3.1}$$

For a graphical representation of these formulas we need two different kinds of vertices, that is vertices of the bullet " • " and of the small circle " ∘ " which will correspond to a and b, respectively. Formulas (3.1) can then be represented as shown in Figure 1.

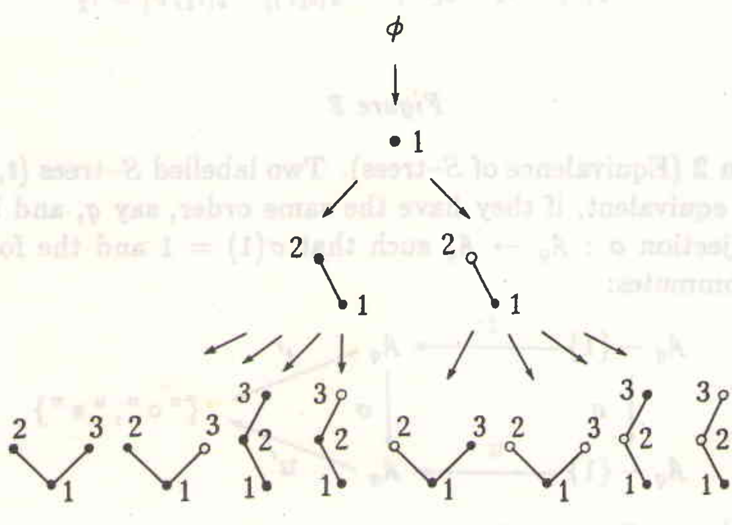


Figure 1

Such interpretation of the higher derivatives by rooted trees can be exploited in the following axiomatic way.

Definition 1 (Labelled S–tree). Let $\mathcal{A}$ be the ordered chain of indices such as $\mathcal{A} = \{1, 2, 3, \dots\}$ and denote by $\mathcal{A}_q$ the subset consisting of the first q indices. Let q be the order of $\mathcal{A}_q$. A labelled S–tree (t, t') of order q is a mapping

$$t : \mathcal{A}_q - \{1\} \to \mathcal{A}_q$$

together with mapping

$$t' : \mathcal{A}_q \to \{\text{“}\bullet\text{”}, \text{“}\circ\text{”}\}.$$

We denote by $\boldsymbol{LTS}^a_q$ the set of those labelled S–trees of order q, whose root is a bullet (i. e. $t'(1) = \text{“}\bullet\text{”}$). Similarly, $\boldsymbol{LTS}^b_q$ is the set of q–th order labelled S–trees with a circle root. Let $\boldsymbol{LTS}_q = \boldsymbol{LTS}^a_q \cup \boldsymbol{LTS}^b_q$.

The symbol S intends to stand for “stochastic”. In Figure 2 we present some elements of labelled S–trees for illustration.

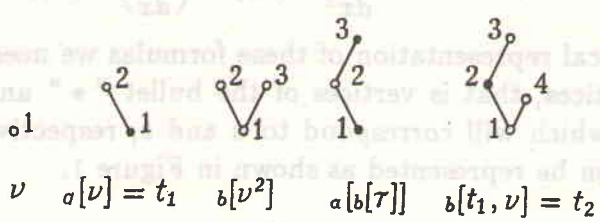
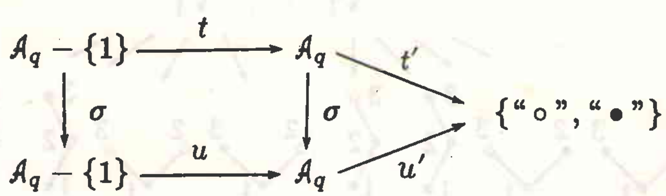
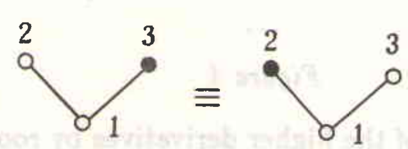


Figure 2

Definition 2 (Equivalence of S–trees). Two labelled S–trees (t, t') and (u, u') are equivalent, if they have the same order, say q, and if there exists a bijection $\sigma : \mathcal{A}_q \to \mathcal{A}_q$ such that $\sigma(1) = 1$ and the following diagram commutes:



For example,



Definition 3. An equivalence class of labelled S-trees of order q with respect to the equivalence relation in the above is called an S-tree of order q. The set of all S-trees of order q with a bullet root is denoted by $\boldsymbol{TS}_q^a$, that with a circle root by $\boldsymbol{TS}_q^b$. For an S-tree t we denote by $\rho(t)$ the order of t and by $\alpha(t)$ the number of elements in the equivalence class t. Let $\boldsymbol{TS}_q = \boldsymbol{TS}_q^a \cup \boldsymbol{TS}_q^b$.

Definition 4 (Composition of S-tree). Let $t_1, \dots, t_m$ be S-trees. We then denote by

$$t = {}_a[t_1, \dots, t_m]$$

the unique S-tree t composed by $t_1, \dots, t_m$ in such a way that the new root is a bullet and the S-trees $t_1, \dots, t_m$ remain as they were if the root and the adjacent branches are chopped off. Similarly, we denote by ${}_b[t_1, \dots, t_m]$ the S-tree whose new root is a circle. We further denote by τ and ν S-trees of order one, i. e. nothing but only the bullet and the circle, respectively.

These notations for simple S-trees are shown in Figure 2.

We use the notation

$$h(t)(x) = \begin{cases} a(x) & \text{if the root of } t \text{ is a bullet,} \\ b(x) & \text{if the root of } t \text{ is a circle.} \end{cases}$$

Definition 5 (Elementary differentials). The elementary differentials, corresponding to (1.6) are defined recursively by

$$F(\tau)(x) = a(x),$$
$$F(\nu)(x) = b(x)$$

and

$$F(t)(x) = \frac{d^m h(t)(x)}{dx^m}(F(t_1)(x), \dots, F(t_m)(x))$$

for $t = {}_a[t_1, \dots, t_m]$ or $t = {}_b[t_1, \dots, t_m]$.

We give some examples of elementary differentials. The elementary differentials corresponding to the S-trees of Figure 2 are.

$$F(t_1)(x) = \frac{da}{dx} \cdot b(x),$$
$$F(t_2)(x) = \frac{d^2 b}{dx^2} \cdot \left(\frac{da}{dx} \cdot b(x),\, b(x)\right).$$

Theorem 1. *For ordered q–tuple $(i_1,\ldots,i_q)$, $i_j \in \{a, b\}$ the following identity holds*

$$\sum_{i_j\in\{a,b\}} L_{i_1}\ldots L_{i_q}x = \sum_{\substack{t\in \boldsymbol{LTS}_q\\ \rho(t)=q}} F(t)(x) = \sum_{\substack{t\in \boldsymbol{TS}_q\\ \rho(t)=q}} \alpha(t)F(t)(x)\,.$$

Proof. The second equality is an immediate consequence of the definition of $\alpha(t)$. The first can be proved by induction on q. We only have to observe that for $(t, t') \in \boldsymbol{LTS}$ with $\rho(t) = q$ we obtain

$$L_aF(t)(X(s)) + L_bF(t)(X(s)) = \sum F(u)(X(s))\,,$$

where the sum is taken over all $(u, u') \in \boldsymbol{LTS}$ of order $q+1$ with $u|_{\{2,\ldots,q\}} = t$ and $u'|_{\{1,\ldots,q\}} = t'$. □

4. S–SERIES OF THE WONG – ZAKAI APPROXIMATION

It is convenient to introduce a new S–tree of order 0, namely ϕ. The corresponding elementary differential is $F(\phi) = x$.

We further set

$$\begin{aligned} \boldsymbol{TS} &= \{\phi\} \cup \boldsymbol{TS}_1 \cup \boldsymbol{TS}_2 \cup \cdots, \\ \boldsymbol{LTS} &= \{\phi\} \cup \boldsymbol{LTS}_1 \cup \boldsymbol{LTS}_2 \cup \cdots. \end{aligned}$$

The sets $\boldsymbol{TS}^a$, $\boldsymbol{LTS}^a$, $\boldsymbol{TS}^b$ and $\boldsymbol{LTS}^b$ are defined similarly.

Definition 6 (S–series). Let c be a certain real–valued function upon $\boldsymbol{TS}$. Then an S–series of the WZ approximation (1.6) is defined by the following formal power series of the variable ξ:

$$S(c,x,\xi) = \sum_{t\in \boldsymbol{LTS}} \frac{\xi^{\rho(t)}}{\rho(t)!}\left\{\frac{\Delta W}{\Delta s}\right\}^{n(t)} c(t)F(t)(x)\,,$$

where $n(t)$ is the number of small circle vertices in a tree t and $\Delta W = W(s_0 + \Delta s) - W(s_0)$. The variable ξ can be substituted by Δs.

For instance, the Taylor series expansion for the solution $X(s)$ of (1.6) can be expressed as

$$X(s_0 + \Delta s) = S(\mathbf{1}, X(s_0), \Delta s)\,,$$

where the function $\mathbf{1}$ means

$$\mathbf{1}(t) = 1 \quad \text{for all } S\text{-trees } t\,.$$

Let $S(\xi)$ stand for $S(c, x, \xi)$ for a certain pair (x, ξ) in case of no fear of confusion.

Remark (S-series representation).
Let $S : (-\xi_0, \xi_0) \to \mathbf{R}$ be arbitrarily often differentiable and $\xi_0 > 0$. S can be represented as an S-series at x with respect to a and b if and only if there exists a map $c : \boldsymbol{TS} \to \mathbf{R}$ such that for all $i \geq 0$ the identities

$$S^{(i)} = \sum_{\substack{t \in \boldsymbol{TS} \\ \rho(t)=i}} c(t)\alpha(t)F(t)(x)\Big(\frac{\Delta W}{\Delta s}\Big)^{n(t)},$$

$$S(\Delta s) = S(c, x, \Delta s)$$

hold.

Theorem 2. *Let $c : \boldsymbol{TS} \to \mathbf{R}$ be a function such that $c(\phi) = 1$. Then the following identity holds*

$$a(S(c, x, \Delta s))\Delta s + b(S(c, x, \Delta s))\Delta W = S(\tilde{c}, x, \Delta s)\,.$$

Here the new function $\tilde{c}$ satisfies the following conditions

$$\tilde{c}(\phi) = 0$$
$$\tilde{c}(\tau) = \tilde{c}(\nu) = 1$$
$$\tilde{c}(t) = \rho(t)c(t_1)\cdots c(t_m)$$

$$\textit{if} \;\; t =_a [t_1, \ldots, t_m] \;\; \textit{or} \;\; t =_b [t_1, \ldots, t_m]\,.$$

That is, in each case, we have the equations as follows

(1) $a(S(c, x, \Delta s))\Delta s = S(\tilde{c}, x, \Delta s)$, *where*

$$\tilde{c}(\phi) = 0,$$
$$\tilde{c}(\tau) = \tilde{c}(\nu) = 1,$$
$$\tilde{c}(t) = \rho(t)c(t_1)\ldots c(t_m)$$

$$\textit{if} \;\; t =_a [t_1, \cdots, t_m]\,,$$

(2) $b(S(c,x,\Delta s))\Delta W = S(\tilde{c},x,\Delta s)$, *where*

$$\begin{aligned}\tilde{c}(\phi) &= 0,\\ \tilde{c}(\tau) &= \tilde{c}(\nu) = 1,\\ \tilde{c}(t) &= \rho(t)c(t_1)\cdots c(t_m)\\ &\textit{if } t =_b [t_1,\cdots,t_m].\end{aligned}$$

Proof of Theorem 2. We will prove only the first case. The conditions on c imply that $S(\xi) = x + O(\xi)$. Thus the function

$$\xi a(S(\xi)) = \Phi(\xi)$$

is defined in a neighbourhood of 0. By the Leibniz-rule we have

$$\Phi^{(i)}(0) = i\,(a \circ S)^{(i-1)}(0)\,. \tag{4.1}$$

Consider now the set

$$\begin{aligned}\boldsymbol{U} = \big\{(u,u') \in \boldsymbol{LTS}^a \,|\, &\mathrm{card}(u^{-1}(i)) \le 1\\ &\text{for } i \ge 2 \text{ and } u'(i) = u'(1) \text{ if } i \notin u^{-1}(1)\big\}\,.\end{aligned}$$

$\boldsymbol{U}$ consists of those monotonically labeled S-trees, whose only possible ramifications are at the root, with the bullet as their root. For $(u,u') \in \boldsymbol{U}$, the corresponding equivalence class can be expressed as $u = {}_a[u_1,\cdots,u_k]$. Using we define

$$G(u)(\xi) = \frac{d^k a(S(\xi))}{dx^k}\left(g^{(i_1)}(\xi),\cdots,g^{(i_k)}(\xi)\right).$$

We obtain by induction on i

$$(a \circ S)^{(i-1)}(\xi) = \sum_{\substack{(u,u')\in U\\ \rho(u)=i}} G(u)(\xi)\,.$$

Putting $\xi = 0$ and using the fact that $S(\xi)$ is an S-series with function $c(t)$ we get

$$
\begin{aligned}
(a \circ S)^{(i-1)}(0) &= \sum_{\substack{(u,u') \in U \\ \rho(u)=i}} \frac{d^k a(S(0))}{dx^k} \left(g^{(i_1)}(0), \cdots, g^{(i_k)}(0)\right) \\
&= \sum_{\substack{(u,u') \in U \\ \rho(u)=i}} \sum_{\substack{t_1 \in LTS \\ \rho(t_1)=i_1}} \cdots \sum_{\substack{t_k \in LTS \\ \rho(t_k)=i_k}} c(t_1) \cdots c(t_k) \times \\
&\quad \times \frac{d^k a(x)}{dx^k} \left(F(t_1)(x), \cdots, F(t_k)(x)\right) \left(\frac{\Delta W}{\Delta s}\right)^{n(t_1)+\cdots+n(t_k)} \\
&= \sum_{\substack{t \in LTS^a \\ \rho(t)=i}} \frac{\tilde{c}(t)}{\rho(t)} F(t)(x) \left(\frac{\Delta W}{\Delta s}\right)^{n(t)}
\end{aligned}
$$

Inserting the above formula into (4.1) we obtain

$$
\begin{aligned}
\Phi^{(i)}(0) &= \sum_{\substack{t \in LTS^a \\ \rho(t)=i}} \tilde{c}(t) F(t)(x) \left(\frac{\Delta W}{\Delta s}\right)^{n(t)} \\
&= \sum_{\substack{t \in TS^a \\ \rho(t)=i}} \alpha(t) \tilde{c}(t) F(t)(x) \left(\frac{\Delta W}{\Delta s}\right)^{n(t)},
\end{aligned}
$$

which is the desired result. □

5. RUNGE–KUTTA SCHEMES FOR STOCHASTIC ODE

The s–stage Runge–Kutta scheme for the WZ approximation (1.6) are generally rewritten by the following formulas:

$$
\begin{aligned}
A_i &= a(Y_i)\Delta s, \\
B_i &= b(Y_i)\Delta W_n, \\
Y_i &= \overline{X}_{n-1} + \sum_{j=1}^{s} \beta_{ij} A_j + \sum_{j=1}^{s} \gamma_{ij} B_j, \\
\overline{X}_n &= \overline{X}_{n-1} + \sum_{i=1}^{s} p_i A_i + \sum_{i=1}^{s} q_i B_i.
\end{aligned}
\tag{5.1}
$$

For the derivation of the order conditions we assume that $A_i + B_i$ can be represented as an S–series at x_0:

$$A_i + B_i \sim S(\Phi_i, x_0, \Delta s). \tag{5.2}$$

This is trivial if the schemes are explicit and can be verified by the implicit function theorem in the general case.

An immediate consequence of (5.2) is that

$$Y_i \sim S(\Psi_i, x_0, \Delta s),$$

where

$$\Psi_i(t) = \begin{cases} 1 & \text{if } \rho(t) = 0, \\ \sum_{j=1}^{s} \beta_{ij}\Phi_j(t) & \text{if } t \in \boldsymbol{TS}^a,\ \rho(t) \ge 1, \\ \sum_{j=1}^{s} \gamma_{ij}\Phi_j(t) & \text{if } t \in \boldsymbol{TS}^b,\ \rho(t) \ge 1. \end{cases}$$

By Theorem 2 the right–hand side of scheme (5.1) can be represented as an S–series, too. Comparing each coefficients we obtain $\Phi_i(t) = \Psi_i'(t)$, which yields the following recurrence relations for $\Phi_i(t)$:

$$\begin{aligned} &\Phi_i(\phi) = 0, \\ &\Phi_i(\tau) = \Phi_i(\nu) = 1, \\ &\Phi_i(t) = \rho(t) \sum_{j_1,\dots,j_m} \beta_{ij_1}\cdots\beta_{ij_k}\gamma_{ij_{k+1}}\cdots\gamma_{ij_m}\Phi_{j_1}(t_1)\cdots\Phi_{j_m}(t_m), \end{aligned} \tag{5.3}$$

where

$$t = {}_z[t_1, \dots, t_m] \quad z \in \{a, b\}$$

and

$$\begin{aligned} &t_j \in \boldsymbol{TS}^a \ \text{ for } 1 \le j \le k, \\ &t_j \in \boldsymbol{TS}^b \ \text{ for } k+1 \le j \le m. \end{aligned}$$

For the numerical solution we now have

$$\overline{X}_1 \sim S(\Phi_i, x_0, \Delta s),$$

where

$$\Psi(t) = \begin{cases} 1 & \text{if } \rho(t) = 0, \\ \sum_{i=1}^{s} p_i\Phi_i(t) & \text{if } t \in \boldsymbol{TS}^a,\ \rho(t) \ge 1, \\ \sum_{i=1}^{s} q_i\Phi_i(t) & \text{if } t \in \boldsymbol{TS}^b,\ \rho(t) \ge 1. \end{cases}$$

Since the exact solution of (1.6) can again be expressed in an S–series, the local truncation error of scheme (5.1) has the form

$$X(s_0+\Delta s)-\overline{X}_1=\sum_{t\in \boldsymbol{TS}}(1-\Phi(t))\alpha(t)F(t)(x_0)\frac{\Delta s^{\rho(t)-n(t)}\Delta W^{n(t)}}{\rho(t)!}.$$

Henceforth we get the following theorem.

Theorem 3. *Scheme (5.1) is accurate of RK–order p if*

$$\Phi(t)=1 \ \ for\ all\ t\in \boldsymbol{TS}\ with\ \rho(t)-\frac{n(t)}{2}\le p, \tag{5.5}$$

where $\Phi(t)$ *is defined (5.4) via (5.3).*

As we mentioned in Sect. 1, the solution for WZ ODE converges to that of the Stratonovich SDE (1.7) as N becomes to infinity. Therefore the numerical solution generated by an RK scheme with RK–order greater than or equal 1 converges to the Stratonovich solution in strong sense (as for the strong convergence, cf. [4] p. 323).

6. AN EXAMPLE

We will derive 2–stage first order (in PK sense) explicit RK scheme. Due to Theorem 3 order conditions up to first order are given in the following table:

t	$\Phi(t)$
•	p_1+p_2
∘	q_1+q_2
	$2q_2\gamma_{21}$

The explicitness of the formula implies $\gamma_{ij}=0\ (i\le j)$, which have been taken into account. For these conditions a solution of the formula parameters follows:

$$p_1=1,\ p_2=0,\ q_1=q_2=\frac{1}{2},\ \gamma_{21}=1.$$

This means a first order explicit RK scheme as follows

$$\begin{aligned} A_1 &= a(\overline{X}_{n-1})\Delta s\,, \\ B_1 &= b(\overline{X}_{n-1})\Delta W_n\,, \\ B_2 &= b(\overline{X}_{n-1} + B_1)\Delta W_n\,, \\ \overline{X}_n &= \overline{X}_{n-1} + A_1 + \frac{1}{2}[B_1 + B_2]. \end{aligned}$$

Taking advantge of Theorem 3 it is possible to construct an RK scheme with any order of accuracy.

7. CONCLUDING REMARKS AND FUTURE ASPECTS

The WZ approximation (1.6) is known to be pathwise convergent to the Stratonovich SDE (1.7). Thus, an explicit way of the solution for (1.6) can be interpreted to provide a concrete means of simulation for qualitative and quantitative analyses of SDE solutions. That is the significance of the RK scheme (2.1) for the WZ approximation in numerics of SDE.

Moreover, the rooted tree analysis presented in this paper may give an essential extension of Hairer's one [6] to the SDE case. Although the WZ approximation has a restriction of the order of convergence in the mean–square sense, our analysis makes a step towards the rooted tree analysis of RK–type schemes for full SDEs. Actually we are planning to further consider an extension to WZ approximation for multi–dimensional SDE or to SDE with multi-dimensional noises.

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