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GEOMETRIC ANALYSIS AND SYMBOL CALCULUS: FOURIER TRANSFORM MAGNETIC RESONANCE IMAGING AND WAVELETS

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Und deines Geistes höchster Feuerflug
Hat schon am Gleichnis, hat am Bild genug.
Johann Wolfgang von Goethe (1749 - 1832)

One picture is worth a thousand words, provided one uses another thousand words to justify the picture.
Harold M. Stark (1995)

Die Welt als Phantom und Matrize.
Günther Anders, 1956

Theorien kommen zustande durch ein vom empirischen Material inspiriertes Verstehen, welches am besten im Anschluß an Plato als Zur–Deckung–Kommen von inneren Bildern mit äußeren Objekten und ihrem Verhalten zu deuten ist. Die Möglichkeit des Verstehens zeigt aufs Neue das Vorhandensein regulierender typischer Anordungen, denen sowohl das Innen wie das Außen des Menschen unterworfen sind.
Wolfgang Pauli, 1961

In order to understand the principles of MR imaging, one must successfully navigate through an elaborate structure whose essence is very much like a mathematical subject.
Alfred L. Horowitz (1995)

Timing is everything...
Robert B. Lufkin, The MRI Manual, 1990

Variations on a theme by Kepler.
Victor W. Guillemin and Shlomo Sternberg, 1990

The object of imaging systems is to probe targets and generate an image of the targets probed. Magnetic resonance imaging (MRI) has provided the radiologist with a new and clearer high tech window through which to visualize living human anatomy and detect the morphological alterations caused by a wide range of pathologies, with no radiation risk to the patient.

Magnetic resonance tomography represents one of the most significant advances in medical imaging in this century. The progress of MRI as a clinical tool has been extraordinary, outstripping the rate of development of any other medical imaging technique. Barely ten years have passed since the introduction of clinical systems, and yet MRI is established as a routine radiodiagnostic technique in most hospitals, with a tissue discrimination unrivalled by any other medical imaging technique. The speed of growth is a testimony to the clinical significance of this intricate image processing technique. Specifically the multiplanar capability and high resolution imaging are unmatched by any other current clinical imaging technique. The physical and mathematical principles on which this non-invasive imaging technology is based are completely different from those associated with the other, more familiar imaging modalities. The MRI process is as complex as the computer controlled synergy of the MRI scanner organization the parameter of which must be adjusted by the operator.

The MRI scanner is quantum electrodynamical (QED) device, the components of which implement and detect the wavelet interference and resonance phenomena underlying the symplectic MRI filter bank processing. Notice the fact that QED has approved the semi-classical approach to quantum holography. In this paper, in semi-classical QED treatment of the basic MRI system organization is presented which has its deep roots in the Keppler phase triangulation procedure of physical astronomy and in the symplectically invariant symbol calculus of pseudodifferential operators. It is based on geometric analysis of affine transvections and non-commutative Fourier analysis which allow for modelling the interference patterns of phase coherent wavelets by distributional harmonic analysis on the Heisenberg nilpotent Lie group G of quantum mechanics. Goemetric quantization yields the tomographic slices by the planar coadjoint orbit stratification $\mathcal{O}_\nu$, $\nu \neq 0$, of the unitary dual $\hat{G}$ of the Heisenberg group G.

The resonance of affine wavelets is treated by harmonic analysis on the affine solvable Lie group $\mathbf{GA(R)}$ which is intrinsically operating on non-trivial blocks of G. It allows to acquire the coordinates of high resolution tomographic scans within the tomographic slices by the Lauterbur spatial encoding technique of quantum holography.

1. General introduction. The Radon transform $\mathcal{R}$ of geometric analysis ([23]) maps a function f in the Schwartz space $S(\mathbf{R}^n)$ into the set of its integrals over hyperplanes ($n \geq 2$):

$$f \rightsquigarrow \mathcal{R} f(\theta, s) = \int_{\langle x|\theta\rangle = s} f(x)\,dx \quad (\theta \in \mathbf{S}_{n-1},\ s \in \mathbf{R})$$

Because $\mathcal{R}$ solves the reconstruction problem of X-ray computed tomography (CT) by means of line integrals, it opened up a window for geometric analysis on medical imaging and visualization. A first-generation X-ray CT scanner was developed for clinical use in 1972 and 1973 by Godfrey Hounsfield. Several of the long-standing problems with standard dynamic X-ray CT were overcome with the introduction of spiral X-ray CT scanning in 1990. However, the limited intrinsic spatial resolution and, of course, the invasiveness of the ionizing radiation remained. As a result of experience with ionizing radiation, the absorption rate of a spinal examination with a modern X-ray CT examination needs about thirty times the absorption rate highly sensitive X-ray film. Because the genotoxic and oncogenic effects of the new fourth-generation of X-ray CT scanners that utilizes an X-ray transmitter rotating about the patient, and a circular stationary receiver aperture for the implementation of a fan beam data collection strategy, are not negligible, intracranial X-ray CT examinations should be restriced to the radiological evaluation of emergency patients presenting with acute head trauma.

Apart of the invasivenees of X-ray CT, there are also technological and mathematical disadvantages with this modality. The X-ray CT scanners have no multiplanar capabilities, and the cylindrical symmetry of their emitter-detector geometries embodies an axial singularity within each of the tomographic slices. In contrast to the MRI modality which relies on spherical symmetry due to coil constructions designed by expansions into spherical harmonics ([49]), the reconstruction of X-ray CT images from the data which are sampled by the scanner leads to the ill-posed problem of inverting the Radon transform $\mathcal{R}$. Nevertheless, it is remaining historical merit of the X-ray CT modality to having introduced the computer into the field of radiodiagnostic medicine. X-ray CT scans are still helpful with differential diagnoses although MRI has invaded CT's space.

X-ray CT is preferred imaging modality for studying tissues that contain little water and therefore hydrogen nuclei, such as the lungs or the calcifications of tissues. Because X-ray techniques do not image the airways directly, recently a new MRI method has been developed at Princeton University which uses spin-polarized helium gas rather than protons as detectable material. The patient inhales the hyperpolarized gas to fill the lungs, which are imaged with a conventional QED based MRI detector assembly. In a normal state, the magnetic moments associated with spins have almost no net polarization, so the diagnostic gas must be optically pumped by a diode-laser array to induce spin alignment. In 1995, the first hyperpolarized-helium MRI scan of the lungs of a human subject was acquired. In this pulmonary evaluation, the tomographic slices of a thickness of approximately 2 cm had a two-dimensional in-plane resolution of about $1.25\,\text{mm} \times 2.5\,\text{mm}$. In comparison, radioactive xenon single-photon emission computed tomography (SPECT) scans only achieve spatial relation on the order of 10 mm. This is one of the reasons why MRI is the superior medical imaging modality. It actually offers high resolution while using materials that are nontoxic, non-radioactive, and non-invasive.

The introduction and vertiginous development of new imaging techniques have revolutionized modern radiology and radiodiagnostic imaging during the last 15 years. Among these techniques, X-ray CT and MRI are without doubt the ones that have had the most important impact on patient evaluation protocols. Before the development of X-ray CT and MRI, it was impossible to visualize normal brain maturation in vivo. Only autopsy studies had been performed, and they suffered from skewing of sample population. Both X-ray CT and MRI show gross morphologic changes in the maturing brain. However, only the high contrast resolution of MRI permits highly sensitive assessment of gray and white matter changes and allows for unraveling some of the mysteries of the human fetal brain ([54]), and the developing pediatric brain ([2], [6]). This explains why MRI is the initial imaging study of choice to evaluate most neurological disorders ([28]).

The capability of the X-ray CT and MRI modality to differentiate between soft tissues, and the pathological information about diseases that is given by MRI has led to earlier treatment, thus increasing the likelihood of recovery. However, MRI is radically different non-invasive radiodiagnostic imaging technique based on QED, with superior soft tissue contrast resolution over X-ray CT, and with a tissue discrimination unrivaled by any other medical imaging technique. The QED organization of a MRI scanner resembles much more to a pumped laser than to a X-ray CT scanner. The extraordinarily large innate contrast which, for two soft tissues, can be on the order of several hundred percent, is the single most distinguishing feature of MRI when compared to X-ray based modalities like CT ([20]). The spin-lattice relaxation T_1 varies for example from 0.275 s in intestinal tissue to 0.595 s in cerebral parenchyma, a difference of greater than 100 percent. In X-ray CT neuroimaging, however, the difference in electron density, hence in X-ray scattering, between gray and white matter is less than 1 percent. Because MRI also has a strong perspective to image-guided therapy, it forms the most significant imaging advance in medical technology since the introduction of X-rays by Wilhelm Conrad Röntgen in 1895. For X-ray examination one hundred years ago, the exposure to radiation took from 10 minutes to 120 minutes so that some patients and early practitioners of diagnostic radiography suffered severe radiation burns. From 1899 tests of X-rays began for remedying cancer, tuberculosis, and various inflammations with uneven success.

In less than 20 years, MRI with its high sensitivity to pathological processes, excellent anatomical detail, multiplanar imaging capability, and lack of ionizing radiation risk has a major impact on routine clinical diagnoses ([4]). Frequently, MRI is the definitive examination, providing invaluable information to help the surgeon not only to understand the underlying pathoanatomy but also to make the critical decision regarding surgical intervention without planar restriction. Cranial MRI should be performed whenever X-ray CT does not adequately explains the patient's neurological status, specifically in all patients presenting persistent impairment or delayed improvement of consciousness in order to assess the full extent of intraaxial damage.

Essential to diagnosis by any medical imaging modality is detectability and locability. The medical discipline of radiodiagnostics only exists today because of the discovery of bands of radiation in the electromagnetic spectrum which can penetrate human tissue, and thus can be exploited by computerized Fourier analysis for detection, localization, and visualization purposes. The MRI modality employs radiation in the radiofrequency band and therefore represents the final such window in the electromagnetic spectrum for medical imaging and visualization. Since its first introduction, it has fired the imagination of MRI scientists and clinicians because of its high soft tissue contrast with multiparameter dependence of wavelets originating from tissue voxels, i.e., detectability, and flexibility of tomographic slice sectioning together with high spatial resolution, i.e., locability. MRI has revolutionized both the understanding and the radiodiagnostics of pathology of the central nervous system, the musculoskeletal system, and the structure of the vascular system. The musculoskeletal system is that part of the human body outside the central nervous system that is most commonly studied with MRI.

It is now general agreement that MRI has advanced to a degree that if access to it were readily available it would be the preferred initial investigation technique for almost all types of intracranial pathology and for any conditions of the spinal column, joints, and muscles in the body ([8]). Its ability to accurately depict the anatomy of the central nervous system ([3]), the morphology of the musculoskeletal system ([5]), the extent of breast cancer in images of infiltrating ductal carcinoma ([4]), and in most of the potential applications now equals or exceeds that of X-ray CT. Many clinical studies have shown evidence of a clear superiority of MRI over X-ray CT in the detection of alterations in soft tissue composition caused by disease. Post-therapy assessment is ideally done on MRI because it does not involve the use of ionizing radiation. Due to the abundance of information and multi parameter dependence, MRI is a lot harder to read than X-ray CT. MRI is not , however, always more specific.

Despite its widespread use in all modern medical centers as a pre-eminent diagnostic imaging modality, MRI appears as a remarkably recent technologic development which is in a constant state of flux and therefore is as far from maturation as it ever was. Nuclear magnetic resonance (NMR) was initially important in solid state physics but soon also became an essential analytical and structural technique in chemistry. It swept across the disciplines to biochemistry and physiology and on to medical imaging. Similarly to the clinical application of X-ray CT to intracranial scans, MRI also started with tomographic scans of the human brain and then expanded stepwise its clinical role from neurodiagnostic imaging to other parts of the human body, and finally achieved vascular system MRI, and functional MRI (fMRI).

From a clinical point of view, body imaging is a very extensive topic ([56], [45]). However, there are fundamental principles which apply to all clinical MRI and which are applicable regardless of the specific MRI applications. In all medical advanced imaging centers, MRI is now firmly established as the most versatile and precise approach for tomographic imaging much of the human body that has displaced plain radiographs of roentgenography, X-ray CT, diagnostic arthrography, myelography, and even agiography, as the clinical imaging study of choice for a growing number of diseases.

Cerebral MRI has established itself as a standard modality of radiodiagnostic investigation and research of the parenchymal morphology of the human brain. As a result of the recent incursion of MRI protocols into the field of functional brain imaging, brain function can be mapped with MRI sensitized to regional blood-oxygen-level-dependent (BOLD) changes originating from the cortical parenchyma due to neural activation ([32]). Actually fMRI is the only tomographic imaging modality presently available for accomplishing brain activity visualization during mental processes and the performance of cognitive tasks in vivo, and non-invasively without the use of exogenous contrast agents ([28]). Because there is nothing as compelling to the understanding of a neurological disease as being able to directly visualize the neuropathological state of the human brain, the fMRI techniques have significantly expanded the potential clinical role of which includes the localization of epileptic foci, presurgical treatment planning and intraoperative mapping of the diseased brain function, elucidation of function of the normal brain function. These results pave the way to applications to cognitive neuroscience ([10]) as well as the diagnosis and evaluation of pathophysiological conditions ([30]).

The purpose of imaging systems is to probe targets and produce an image of the targets probed. Tomography, from the Greek $\tau\acute{o}\mu o\varsigma$, the slice, and $\gamma\rho\alpha\varphi\acute{\iota}\varsigma$, the crayon, refers to imaging cross-sectional targets of the internal anatomy of the human body from either transmission or scattering data probes. The cross-sections of the main scan plane orientations are central, coronal, and sagittal tomographic slices.

- In contrast to the X-ray CT modality of radiodiagnostic imaging, NMR preserves the differential phase as well as the frequency in the semi-classical QED description of response data phase holograms. MRI is not a transmission imaging technique but a phase coherent radar type tomographic imaging modality of computer controlled actively backscattering cross-sectional targets. The most notable distinguishing feature of MRI when compared to X-ray CT is the absence of a universal gray scale.

Crystallography, from the Greek $\kappa\varrho\upsilon\sigma\tau\alpha\lambda\lambda o\varsigma$, the ice, may be viewed as a mathematical extension of the theory of array antennae. Therefore the problem of probing crystal targets and determining molecular structures by the techniques of X-ray crystallography introduced by Max T. F. von Laue in 1912 had a strong impact on the mathematics. It is problem of determining the positions of the atoms in the crystal when only the amplitudes of the diffraction maxima are available from diffraction experiment. However, the associated phases which are also needed if one is to determine the equal-atom crystal and molecular structures from the experimental observations, are lot in the diffraction experiment. By exploiting a priori structural information of the sample, it can be established that the lost phase information is contained in the measured diffraction intensities and can be recovered provided that the molecular structure is not too lager ([22]). For proteins and virus', however, these methods of solving the phase problem appear to be unsuccessful. Hence phase coherent imaging techniques such as electron holography ([53]) reveal to be more appropriate for determining the position of single atoms in crystal structures. In medical imaging, some of the MRI findings have resulted in deepening and fortifying the knowledge of the underlying pathomechanism of disease processes such a multiple sclerosis (MS), degenerative disc disease of the spine, and disc herniation which may be accompanied by degenerative processes.

In MS the principal constituent of the white matter of the central nervous system, the myelin, is attacked, thus impairing neural function. MS involves the breakdown of previously normally developed myelin. The oligodendroglia cells are destroyed and the axons spared. The normally myelinated regions of the brain are replaced by plaques, which consist of reactive astrocytes, lymphocytes, and macrophages ([55]).

The first MRI scans of MS lesions have been displayed in 1981. Because MRI is considerably more sensitive than X-ray CT, the MRI technique has now completely superseded X-ray CT as a tool for imaging MS plaques, whether for the purposes of radiodiagnostic medicine or for monitoring the course of the disease ([55]). MRI is the only modality that allows for direct visualization of spinal cord demyelinating plaques ([5]). It has provided a unique opportunity to study the evolving pathologic process of MS in vivo ([42]), as well as problems of epileptic processes as they relate to anatomically and functional abnormalities of the human brain such as focal seizures ([30]).

Holography from the Greek ὅλος, the whole, is the signal record of both the amplitude and phase information content of coherent wavelets. The quadrature phased-array multicoil receiver technique of MRI, for long spinal cord imagery say, is based on the holographic principles of phase preservation as used in phase coherent data storage of radar array processing in synthetic aperture radar (SAR) imagery ([35]). The coils of the array are electrically isolated from one another through the use of low-input impedance preamplifiers overlapping electromagnetic fields, and are each connected to a separate MRI receiver, preamplifier, and digitizer. The separate scans are then combined by preserving the phase to form one composite scan that admits maximal coverage, signal-to-noise ratio, and contrast resolution. In SAR imaging, the aperture is synthesized by recording the radio signal received at two or more array antenna elements and later processing the image data phase coherently within a digital processor or an optical processor which acts as a massively parallel multichannel cross-correlator ([33], [34]). The image data present some difficulties for interpretation by human observers or automatic target recognition systems. Among these difficulties is the large dynamic range (five orders of magnitude) of the sensor signal which requires some type of nonlinear data compression.

Synthesizing an aperture by allowing the earth's rotation to sweep a phased-array of fixed antenna elements through space also was at the basis of the development of synthetic aperture radio telescopes by Martin Ryle in 1952. It follows that be ideas of tomography are closely related to the coherent data processing methods of synthetic antenna developed in radar imaging and radio astronomy. The transmission systems, antennae, and receiver coils of MRI scanners should be considered from this point of view. In particular, this aspect hold for the quadrature phased-array multicoil receiver as used in long spinal cord imagery.

In X-ray CT, the Radon transform $\mathcal{R}$ provides a decomposition into plane waves whereas the MRI modality is based on response data phase holograms generated by computer controlled wavelet interference. The echo wavelet elicited is generated by the relaxation-weighted spin isochromat densities (literally, 'spins of the same colour') themselves in response to a computer controlled external perturbation. According to the semi-classical approach to quantum holography which has been approved by QED, the spin precession under the strong external magnetic flux density generated by a superconductive magnet serves as the natural reference for the arrays of phase-locked Bloch vectors. It generates a natural symplectic structure on the planar tomographic slices which is basis for the FTMRI modality. The ensuing phase shift of period 4 has become routine in the implementation of various multiple spin echo train by pulse transmission gates with a quartz controlled stable local oscillator ([41]) similar to the STALO technique of SAR transmission.

A factor that sets clinical imaging experts apart from specialists in other medical disciplines who consider MRI scanners as black boxes ([11]), and forms the strongest explanation for the imaging expert's claim to an indispensable role in medicine is an understanding of the sciences of imaging and visualization. Both are based on mathematical methods which include image segmentation ([25]). From the mathematical point of view, the FTMRI modality depends on

- Non-commutative affine geometry
- Non-commutative Fourier analysis

Tomographic imaging means, in the context of radiodiagnostic medicine, displaying internal anatomy in cross-sections by non-commutative geometry of mappings. Non-commutative Fourier analysis allows for translation of the geometry of affine mappings into the frequency domain of phase coherent wavelets to make the geometry accessible to computer controlled wavelet backscattering, interference and phase conjugation. Interference and phase conjugation, however, are resonance phenomena both of which require parallel synchronization. In this sense, computer controlled synchronized timing is everything in FTMRI.

In the case of the FTMRI modality, which is the most common among the MRI technique, the appropriate choice of mappings are the affine transvections. A transvection is an endomorphism of a real vector space which leaves pointwise fixed every element of a homogeneous hyperplane ([14]). Every transvection is an affine bijection of determinant 1 whose set of fixed points is exactly the given hyperplane. Transvections generate the unimodular groups and play with respect to the symplectic group an analog role as do symmetries with respect to the orthogonal group. Their tensor powers are called tranvectants.

In FTMRI, there are basically three standard types of affine transvections: central, coronal, and sagittal transvections which are implemented together with their affine bijections by the Heisenberg nilpotent Lie group G. It is a surprising fact that despite the complexity of the MRI system organization, the structure of this specific nilpotent Lie group is rich enough to allow for a study of the non-invasive diagnostic FTMRI modality. The key to this semi-classical QED approach to FTMRI is an embedding of Fourier analysis into the symplectically invariant symbol calculus of pseudodifferential operators in conjunction with the geometric quantization by the planar coadjoint orbit stratification of the unitary dual $\hat{G}$.

2. Fundamental principles. The purpose of tomographic imaging systems is to probe cross-sectional targets and generate an image of the tomographic slices from the data probes.Taking into account that the majority of textbooks on the clinical application of MRI systems to diagnostics, agiography and functional imaging include an introductory chapter concerned with the basic principles of MRI, the question arises, why a mathematical treatment of the MRI modality should be developed. The answer is that these introductions to the fundamental principles of MRI are in most of the cases insufficient, even when the medical sections of the textbooks are of an excellent quality. In the same vein is the presentation of the basic principles of MRI in lectures. For example, a prominent staff member of one of the American top universities and widely acclaimed authority in the area of fMRI explained to his audience the flatness of the tomographic slices excited by the MRI scanner as a cause of the homogeneity of the external magnetic flux density. A couple of minutes before, however, he emphasized the fact that a linear magnetic field gradient switched along the patient's axis parallel to the bore of the strong magnet is necessary for the tomographic slice selection. Indeed, switching of linear magnetic field gradients is basic for slice positioning by affine transvections, and the Lauterbur spatial encoding technique of quantum holography which applies frequency modulating linear ramp actions inside the excited tomographic slice. A magnetic field gradient, however, introduces a computer controlled inhomogeneity into the external magnetic flux density so that actually its homogeneity can not be responsible for the flatness of the excited tomographic slice. The inhomogeneity which is generated by the linear slice-selection gradient during the exciting electromagnetic pulse has to be compensated by a subsequent linear ramp of reversed slope of equal duration to maintain zero net dephasing for all spins that were excited.

It is obvious that the reason for the flatness of the tomographic slices excited by the MRI scanner need a deeper argument. The flatness of the tomographic slices actually depends upon the square integrability of the infinite dimensional irreducible unitary linear representations U of the Heisenberg nilpotent Lie group G modulo its one-dimensional center C. Equivalently, it depends upon the fact, that these irreducible unitary linear representations U of G and the coadjoint orbits in side $\mathrm{Lie}(G)^*$ associated to their unitary isomorphy classes are determined by the unitary central characters χ_U which are defined on the line C. On the other hand, the square integrability mod C explains the double-slit experiment of photonics and therefore is at the heart of quantum mechanics. It follows from the coadjoint orbit stratification of the unitarydual $\hat{G}$ that the flatness of the tomographic slices excited by the MRI scanner and the tomographic slice selection procedure are basic consequences of fundamental principles of QED ([15]).

3. Kernel distributions. The energy required to flip a proton through 180 degrees in a magnetic field of flux density of 1.0 T is 2.82×10^{-26} J $= 0.176$ μeV. By the Einstein relation this is the energy of a 42.573 MHz photon. In MRI the basic problem is to radiate electromagnetic energy at frequency of 42.573 MHz into the spin assembly which is ordered by the external magnetic field to achieve by resonance a well organized spin distribution. In terms of neural network theory, the distributed spin assemblies must be capable to performing corticomorphic processing similar to SAR imagery.

Distribution theory, usually formulated as a local extension theory in terms of open sets of the Euclidean vector space $\mathbf{R}^n$ or a C^∞-differential manifold, can be thought of as the completion of differential calculus just as Lebesgue integration theory can be thought of as the completion of integral calculus. Distribution theory when applied to unitary linear representations of Lie groups G provides the basis for non-commutative Fourier analysis and symbol calculus. Specifically geometric quantization gives rise to distributional harmonic analysis on the Heisenberg group G is suitable to describe relaxation-weighted spin isochromat densities excited in the planar coadjoint orbits $\mathcal{O}_\nu$, $\nu \neq 0$, of the stratification of the unitary dual $\hat{G}$ of G in terms of the symplectically invariant symbol calculus. The symbol calculus allows to embed the von Neumann approach to quantum mechanics which is based on the category of Hilbert spaces into the Dirac approach which is based on complex locally convex topological vector of tempered distributions. For the preparation of the symbol calculus, some generalities of the calculus of Schwartz kernels are needed ([50]). The full QED meaning of the kernel distributions for the semi-classical approach to FTMRI can be appreciated in the context of the planar coadjoint stratification of the unitary dual $\hat{G}$ of the Heisenberg group G which provides two-dimensional Fourier analysis with an extra symplectic structure. This symplectic structure has been traced back by André Weil to Carl Ludwig Siegel's papers on the theory of quadratic forms ([58]).

For simplicity, let G denote a unimodular Lie group and dg a choice of Haar measure on G. Then dg forms the element 1_G of the complex vector space $\mathcal{D}'(G)$ of distributions on G. The topological antidual of the complex vector space $\mathcal{D}(G)$ of infinitely differentiable, compactly supported, complex-valued functions on G under its natural anti-involution of complex conjugation

$$\psi \rightsquigarrow \overline{\psi}$$

is the vector space $\mathcal{D}'(G)$ of those antilinear forms on $\mathcal{D}(G)$ which are continuous with respect to the canonical inductive limit topology of $\mathcal{D}(G)$. As a realization of the Wigner invariance theorem, the involutory anti-automorphism of $\mathcal{D}'(G)$ which is contragredient to the natural anti-involution of $\mathcal{D}(G)$ plays a crucial role in FTMRI in modeling the refocusing phenomenon of phase conjugation. Note that the concept of phase conjugation is present already in Aristotelian physics. Note also that the complex vector spaces $\mathcal{D}(G)$ and $\mathcal{D}'(G)$ form their own anti-spaces. Because nuclearity is preserved by countable inductive limits and Hausdorff projective limits, the complex vector spaces $\mathcal{D}(G)$ and $\mathcal{D}'(G)$ nuclear locally convex topological vector spaces under the canonical inductive limit topology and the weak dual topology, respectively.

Let U denote a linear representation of G acting continuously on the Hilbert subspace $\mathcal{H}$ of $\mathcal{D}'(G)$ by Hilbert space automorphisms of $\mathcal{H}$. Then $\mathcal{H}$ forms a closed vector subspace of $\mathcal{D}'(G)$ under its weak dual topology in the sense that the linear injection

$$\mathcal{H} \hookrightarrow \mathcal{D}'(G)$$

forms a continuous mapping. The norm topology of $\mathcal{H}$ is finer than the topology induced on $\mathcal{H}$ by the weak dual topology of $\mathcal{D}'(G)$, the mapping

$$G \times \mathcal{H} \ni (g, \psi) \rightsquigarrow U(g)\,\psi \in \mathcal{H}$$

is simultaneously continuous, and the vector subspace $\mathcal{H}^{+\infty}$ of smooth vectors for the unitary linear representation U is everywhere dense in $\mathcal{H}$ with respect to the norm topology. Thus, for every element of the complex vector space $\mathcal{H}^{+\infty}$, the trajectory through $\psi \in \mathcal{H}^{+\infty}$ which is defined by

$$\tilde{\psi} : g \rightsquigarrow U(g)\,\psi$$

forms an infinitely differentiable mapping of G in $\mathcal{H}$ ([51]). The mapping

$$\psi \rightsquigarrow \tilde{\psi}$$

defines a continuous linear embedding of $\mathcal{H}$ into the topological vector space

$$\mathcal{C}^0(G;\, \mathcal{H}) \cong \mathcal{C}^0(G)\hat{\otimes}\mathcal{H}$$

of continuous functions of G with values in the complex Hilbert space $\mathcal{H}$ under the locally convex vector space topology of compact convergence.

For the case of the Heisenberg group G with center C, and $\mathcal{H} = L^2(\mathbf{R})$, the standard complex Hilbert space over the bi-infinite time scale $\mathbf{R}$, the continuous linear embedding

$$\mathcal{H} \hookrightarrow \mathcal{C}^0(G/C; \mathcal{H})$$

performed at resonance frequency $\nu \neq 0$ by the linear Schroödinger representation U^ν of G in $\mathcal{H}$ with projective kernel C is at the basis of the symplectically invariant symbol calculus approach to FTMRI. According to the semi-classical QED approach, the canonical projection

$$G \to G/C$$

allows by passing to the quotient mod C for implementation of the phase dispersion and its compensation by phase conjugation in the laboratory coordinate frame attached to the cross-section G/C to C in G.

The continuous injection $\mathcal{H} \hookrightarrow \mathcal{C}^0(G; \mathcal{H})$ identifies $\mathcal{H}$ with a closed vector subspace $\tilde{\mathcal{H}}^0$ of $\mathcal{C}^0(G; \mathcal{H})$, and $\mathcal{H}^{+\infty}$ with the closed vector subspace

$$\tilde{\mathcal{H}}^{+\infty} = \{\tilde{\psi} | U(g')\tilde{\psi}(g) = \tilde{\psi}(g'.g),\ g', g \in G\}$$

of $\mathcal{C}^\infty(G; \mathcal{H})$. Notice that G defines an infinitely differentiable action on the vector space $\tilde{\mathcal{H}}^{+\infty}$ of smooth trajectories via left translations by the reflected group elements $g'^{-1} \in G$. Of course, each smooth trajectory $\tilde{\psi} \in \tilde{\mathcal{H}}^{+\infty}$ through $\psi \in \mathcal{H}^{+\infty}$ defines a distribution on G with values in $\mathcal{H}$ by the standard prescription

$$\tilde{\psi}(f) = \int_G \tilde{\psi}(g)\, f(g)\, dg\,,$$

where $f \in \mathcal{D}(G)$ denotes an arbitrary test function on G. It is called the trajectory distribution associated to $\psi \in \mathcal{H}^{+\infty}$.

Let μ denote a compactly supported scalar measure on G and $\check{\mu}$ the image of under the involutory homeomorphism $g \rightsquigarrow g^{-1}$ of G onto itself. Then, as observed above,

$$U(g')\,\tilde{\psi} = \varepsilon_{g'} \star \tilde{\psi} \qquad (g' \in G)$$

for $\tilde{\psi} \in \tilde{\mathcal{H}}^{+\infty}$ and the Dirac measure $\varepsilon_{g'}$ located at $g' \in G$. The weak integral taken in $\mathcal{H}$

$$U(\mu) = \int_G U(g)\, d\mu(g)$$

extends U by averaging over G. If μ is absolutely continuous with respect to dg and admits a density f which is integrable over G with respect to the measure dg and vanishes in a neighborhood of the point at infinity, then $U(f) = U(f.dg)$ is given by the prescription ([58])

$$U(f) = \int_G U(g)\, f(g)\, dg\,.$$

The extension $U(\mu)$ satisfies the distributional convolution identity

$$U(\mu)\, \tilde{\psi} = \check{\mu} \star \tilde{\psi}$$

for all trajectory distributions $\tilde{\psi} \in \tilde{\mathcal{H}}^{+\infty}$. The elements of the complex vector space $\tilde{\mathcal{H}}^{-\infty}$ of distributions $K \in \mathcal{D}'(G; \mathcal{H})$ on G with values in $\mathcal{H}$ satisfying the convolution identity

$$U(\mu)\, K = \check{\mu} \star K$$

for all compactly supported scalar measure μ on G are called kernel distributions of $\mathcal{H}$ for the given unitary linear representation U of G in $\mathcal{H}$. Equivalently,

$$U(\mu)\, K(f) = K(\mu \star f)$$

holds for all test functions $f \in \mathcal{D}(G)$. If the complex vector space $\tilde{\mathcal{H}}^{-\infty}$ is endowed with the locally convex vector space topology induced by $\mathcal{D}'(G; \mathcal{H})$, the continuous inclusions

$$\tilde{\mathcal{H}}^{+\infty} \hookrightarrow \tilde{\mathcal{H}}^{0} \hookrightarrow \tilde{\mathcal{H}}^{-\infty}$$

hold. In this sequence, the vector space of trajectory distributions $\tilde{\mathcal{H}}^{+\infty}$ is isomorphic to the space $\mathcal{H}^{+\infty}$ of smooth vectors for U, and the vector space $\tilde{\mathcal{H}}^{0}$ is isomorphic to the representation space $\mathcal{H}$ of the unitary linear representation U of G.

The important step is to realize that the unitary representation U of G in $\mathcal{H}$ gives rise to a continuous representation $\tilde{U}$ of G acting in the trajectory space

$$\tilde{\mathcal{H}}^{0} \cong \mathcal{H}$$

according to the prescription

$$\tilde{U}(g') : \psi \rightsquigarrow (U(g')\,\psi)^{\sim} \quad (g' \in G)\,.$$

The polarized symbol map $\tilde{U}$ extends to an infinitely differentiable linear representation $g' \rightsquigarrow \tilde{U}(g')$ of G in $\tilde{\mathcal{H}}^{-\infty}$ by right translations. Thus

$$\tilde{U}(g')\, K = K \star \check{\varepsilon}_{g'}$$

for $K \in \tilde{\mathcal{H}}^{-\infty}$. In particular, it follows

$$\tilde{U}(\mu)\, \tilde{\psi} = \tilde{\psi} \star \check{\mu}$$

by convolving the trajectory $\tilde{\psi} \in \tilde{\mathcal{H}}^{+\infty}$ from the right the right with the reflected compactly supported scalar measure μ on G. Moreover, the filter cascade identity

$$\tilde{U}(S \star T)\, \tilde{\psi} = \tilde{U}(S) \circ \tilde{U}(T)\, \tilde{\psi}$$

holds for all compactly supported complex distributions $S, T \in \mathcal{D}'(G)$.

The kernel distribution $K \in \tilde{\mathcal{H}}^{-\infty}$ of $\mathcal{H}$ for the representation U of G is a called generating kernel distribution ([51]) provided the image $K(\mathcal{D}(G))$ is an everywhere dense vector subspace of $\mathcal{H}$. Let U be a unitary left regular representation of G on the Hilbert subspace $\mathcal{H}$ of $\mathcal{D}'(G)$ in the usual sense that the unitary group action $U(g)\, \psi$ of G on $\psi \in \mathcal{H}$ is performed via left translations $\varepsilon_g \star \psi$ by elements $g \in G$. Then

$$U(\mu)\, \psi = \mu \star \psi$$

hold for each compactly supported scalar measure μ on G and each element $\psi \in \mathcal{H}$. It follows that the Schwartz kernel H of $\mathcal{H}$ in $\mathcal{D}'(G)$ is defined by convolution from the left with a uniquely defined positive definite distribution $H^\bullet$ on G. Therefore the canonical reproducing kernel of the representation U of G is defined by $\star H^\bullet$. The existence of the mapping

$$H : \psi \rightsquigarrow \psi \star H^\bullet$$

is a consequence of the Schwartz kernel theorem ([50], [51]). The mapping $K \rightsquigarrow K^\bullet$ of $\tilde{\mathcal{H}}^{-\infty}$ onto the vector space

$$\mathcal{H}^{-\infty} = \{T \in \mathcal{D}'(G) | f \star T \in \mathcal{H},\ f \in \mathcal{D}(G)\}$$

forms a linear bijection which uniquely extends the inverse bijection

$$\tilde{\psi} \rightsquigarrow \psi$$

of $\tilde{\mathcal{H}}^0$ onto $\mathcal{H}$, and identifies the representation $\tilde{U}$ with the representation $g \rightsquigarrow U(g)$ of left translations $\varepsilon_g \star$. In particular,

$$(U(S)\,K)^{\bullet} = U(S)\,K^{\bullet}$$

hold for all kernel distributions $K \in \mathcal{H}^{-\infty}$ and all compactly supported complex distributions $S \in \mathcal{D}'(G)$. Moreover,

$$U(g)\,T = \varepsilon_g \star T\,,$$

and

$$U(S)\,T = S \star T$$

hold for all elements $g \in G$ and distributions $T \in \mathcal{H}^{-\infty}$ on G.

As a consequence of the preceding reasonings, it follows

- There exists a bijective correspondence between the unitary regular representations of G and the positive definite distributions on G.

The contragredient representation $\check{U}$ of U is defined by the inverse of the transposed operators according to the rule

$$\check{U} : g \rightsquigarrow {}^{t}U(g^{-1})\,.$$

Therefore G acts continuously by $\check{U}$ on the topological antidual $\overline{\mathcal{H}}'$ of $\mathcal{H}$ under its weak dual topology. The coefficients of $\check{U}$ are the complex conjugates of the continuous coefficient functions of U and therefore encode important properties of the unitary linear representation U of G.

4. The synchronization. The Heisenberg group G as a maximal nilpotent subgroup

$$G \hookrightarrow \mathbf{SL}(3, \mathbf{R})$$

forms a connected and simply connected three-dimensional nilpotent Lie group of level 2 generated by transvections ([49]). It is unimodular and represents a non-split central group extension

$$\mathbf{R} \triangleleft G \longrightarrow \mathbf{R} \oplus \mathbf{R}$$

where the invariant subgroup is isomorphic to the center C of G, and defines a smooth action on the cross-section G/C. The one-parameter action $\mathbf{R} \triangleleft G \longrightarrow \mathbf{R} \oplus \mathbf{R}$ defines a smooth dynamical system the infinitesimal generator of which is operating on the complex vector space $\tilde{\mathcal{H}}^{-\infty}$ associated to $\mathcal{H} = L^2(\mathbf{R})$, the standard complex Hilbert space over the bi-infinite time scale $\mathbf{R}$. This infinitesimal generator is basic for the encryption and decryption by the FTMRI modality.

Notice also that the mapping

$$z \rightsquigarrow \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$$

defines a faithful representation of the additive group $\mathbf{R}$ in the plane $\mathbf{R} \oplus \mathbf{R}$ realized by transvections. Note also that the mapping

$$\exp_G : \mathrm{Lie}(G) \longrightarrow G$$

of the Heisenberg Lie algebra $\mathrm{Lie}(G)$, and its inverse

$$\log_{\mathrm{Lie}(G)} : G \rightarrow \mathrm{Lie}(G)$$

are diffeomorphisms: As a connected and simply connected nilpotent Lie group, G is of exponential type.

The central transvections of G associated to a homogeneous line L inside the plane $\mathbf{R} \oplus \mathbf{R}$ of fixed points admit a coordinatization with respect to the laboratory coordinate frame attached to the cross-section G/C. In terms of elementary matrices it reads

$$C = \left\{ \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| z \in \mathbf{R} \right\}.$$

The faithful block representation of C central transvections in the plane $\mathbf{R} \oplus \mathbf{R}$ defined by deletion of the middle row and column of the elementary matrices belonging to the line C

$$\begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{bmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$$

allows for a linear ramp action of the disconnected affine Lie group

$$\mathbf{GA}(\mathbf{R}) = \left\{ \begin{bmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \middle| \alpha \neq 0,\ \beta \in \mathbf{R} \right\}$$

on the homogeneous plane spanned by the lines L and C. The affine bijections represented by the elements of $\mathbf{GA(R)}$ are transvections for $\alpha = 1$, dilations of ratio $\alpha \neq 0$ for $\beta = 0$ including the phase conjugating symmetries for $\alpha = -1$ and $\beta = 0$, and linear gradients for $\alpha \neq 0$ and $\beta \in \mathbf{R}$ which generate the superposition of a frequency modulating linear ramp of slope α. Therefore, $\mathbf{GA(R)}$ is called the frequency modulation group of the Lauterbur spatial encoding technique of quantum holography ([31]).

Because a dilation never commutes with a transvection unless one or both is the identity map, the commutator subgroup of $\mathbf{GA(R)}$ is given by the set of transvections in $\mathbf{R} \oplus \mathbf{R}$

$$[\mathbf{GA(R)}, \mathbf{GA(R)}] = \left\{ \begin{bmatrix} 1 & \beta \\ 0 & 1 \end{bmatrix} \middle| \beta \in \mathbf{R} \right\},$$

and the center of $\mathbf{GA(R)}$ is trivial. The linear ramps action of $\mathbf{GA(R)}$ on the homogeneous plane spanned by the lines L and C can be faithfully pulled back to the affine Larmor frequency scale on the dual of the transversal line C of the non-split central group extension $\mathbf{R} \triangleleft G \longrightarrow \mathbf{R} \oplus \mathbf{R}$. The affine Larmor frequency scale itself allows for slice selection by resonance with the frequency modulating linear ramp.

Let (x, y) denote the coordinates with respect to the laboratory coordinate frame of the homogeneous plane $\mathbf{R} \oplus \mathbf{R} \cong G/C$ of fixed points associated to the central transvections. Then the unipotent matrices

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbf{R} \right\}$$

present a coordinatization of the closed subgroup G of the unimodular group $\mathbf{SL}(3, \mathbf{R})$. Indeed, G is a closed subgroup of the standard Borel subgroup of the linear group $\mathbf{GL}(3, \mathbf{R})$ consisting of the upper triangular matrices

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{pmatrix}$$

with real entries such that

$$\det \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & 0 & x_{33} \end{pmatrix} = x_{11}.x_{22}.x_{33} \neq 0 .$$

The Borel subgroup leaves a flag consisting of a homogeneous line L and a plane containing L globally invariant. The condition

$$x_{11} = x_{22} = x_{33} = 1$$

ensures the aforementioned embedding

$$G \hookrightarrow \mathbf{SL}(3, \mathbf{R})$$

and achieves that the line L is kept pointwise fixed under the action of the Borel subgroup. With respect to this coordinatization, the multiplication law of G reads

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+x' & z+z'+xy' \\ 0 & 1 & y+y' \\ 0 & 0 & 1 \end{pmatrix}$$

so that a choice of left and right Haar measure on G in given by Lebesgue measure

$$dx \oplus dy \oplus dz$$

of the space $\mathbf{R}^3$. Then the center C of G carries the Lebesgue measure dz as a choice of Haar measure.

The preceding polarized presentation of G gives rise to the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

of the central bilinear form. On the other hand, the Heisenberg Lie algebra of nilpotent matrices

$$\mathrm{Lie}(G) = \left\{ \begin{pmatrix} 0 & a & c \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \middle| a, b, c \in \mathbf{R} \right\}$$

as a maximal nilpotent Lie subalgebra

$$\mathrm{Lie}(G) \hookrightarrow \mathrm{Lie}(\mathbf{SL}(3, \mathbf{R}))$$

admits the one-dimensional center

$$\log_{\mathrm{Lie}(G)} C = \left\{ \begin{pmatrix} 0 & 0 & c \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| c \in \mathbf{R} \right\}.$$

The Lie bracket of Lie(G) gives rise to the symplectic matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

of the central transvection. The symplectic matrix J of period 4 can be unitarized by inserting the factor $\frac{1}{2}$ so that the symplectic presentation of G reads

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & x' & z' \\ 0 & 1 & y' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+x' & z+z'+\frac{1}{2}\det\begin{pmatrix} x & y \\ x' & y' \end{pmatrix} \\ 0 & 1 & y+y' \\ 0 & 0 & 1 \end{pmatrix}.$$

Under the symplectic presentation of G, the restriction of the homomorphism $\log_{\mathrm{Lie}(G)} : G \longrightarrow \mathrm{Lie}(G)$ to the center C of G identifies with the identity map of the central factor in Lie(G)

$$\log_{\mathrm{Lie}(G)} | C = \mathrm{id}_{\mathbf{R}}.$$

In terms of the MRI technology, $\frac{1}{2}J$ is called the swapping matrix. It forms the infinitesimal generator at the angle $\frac{\pi}{2}$ of rotations in a symplectic affine plane and acts by doubling the number of variables. It turns out that the swapping matrix allows to detect the spectral information content of phase coherent wavelets on the bi-infinite time scale $\mathbf{R}$.

For the purposes of MRI it is appropriate to use the row vectors of upper triangular matrices for the coordinatization of the non-split central group extension $\mathbf{R} \triangleleft G \longrightarrow \mathbf{R} \oplus \mathbf{R}$ and the associated smooth dynamical system because the row vectors of the matrices represent the coordinates with respect to a basis of the dual of the underlying three-dimensional real vectors space whose coordinates represent the column vectors of the transposed matrices. Apart from the faithful block representation

$$z \rightsquigarrow \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$$

of the line C by central transvections, the coordinatization gives also rise to the faithful block representation

$$y \rightsquigarrow \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$$

of G by coronal transvections. Swapping then yields the faithful block representation of G by sagittal transvections

$$x \rightsquigarrow \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

The frequency modulation group $\mathbf{GA}(\mathbf{R})$ acts from the left on the overlapping blocks of G. By transition to the frequency domain, it is this intrinsic linear ramp action of the group $\mathbf{GA}(\mathbf{R})$ on the blocks inside G which mathematically describes the resonance phenomena on which the MRI modality is based. Therefore the non-invasive diagnostic FTMRI modality can be studied by means of Fourier analysis on this additional Lie group which act on the blocks of G. The synergy of this block organization inside G is realized by the MRI scanner organization.

The canonical commutation relations of quantum mechanics which underlie the Heisenberg uncertainty principle, pushed down by the diffeomorphism $\exp_G$ from Lie(G) to the Lie group G of central, sagittal, and coronal transvections, provide by passing to the quotient mod C the cross-section G/C to the center C in G with the structure of a symplectic affine plane. Thus the symplectic cross-section to C in G carries the structure of a plane consistently endowed with both the structure of an affine plane an a symplectic plane. By passing to the quotient mod C and complexification, the vertical projection G/C of G carries the structure of a complex plane

$$W \cong (\mathbf{R} \oplus \mathbf{R},\ J) \cong \mathbf{C}.$$

Indeed, symplectic vector spaces are symplectomorphic if and only if they have the same dimension, or equivalently, if and only if they are isomorphic as vector spaces. Any transvection having G/C as its plane of fixed points forms actually a symplectic transvection. The tensor powers of the symplectic transvections inside the tensor algebra of W give rise to expansion of holomorphic functions on W in terms of transvectants.

The Keppler phase triangulation procedure (Albert Einstein, 1951: ein wahrhalf genialer Einfall...) which allowed Johann Keppler to establish his three fundamental laws of physical astronomy is based on the ingenious idea of semi-classically interpreting (x, y) as differential phase-local frequency coordinates with respect to a coordinate frame rotating with frequency $\nu \neq 0$. In the semi-classical QED description of response data phase holograms, the differential phases and local frequencies represent the spectral information content of phase coherent wavelet on the bi-infinite time scale $\mathbf{R}$. Notice that the problem of measuring phase angles is far from being as evident as the classical expositions like to make them believe.

The implementation of the parallel synchronization of the Keppler phase triangulation procedure in the rotating coordinate frame implies a transition from the cross-section G/C to the planar coadjoint orbit stratification of the unitary dual $\hat{G}$ of the Heisenberg group G. Due to the coadjoint action of G on the dual $\mathrm{Lie}(G)^*$ of the real vector space $\mathrm{Lie}(G)$ which is given by ([46])

$$\mathrm{CoAd}_G\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & -y \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix},$$

the basic identification reads

$$\mathrm{Lie}(G)^*/\mathrm{CoAd}_G(G) \cong \hat{G}\,.$$

The elements of the symplectic quotient $\mathrm{Lie}(G)^*/\mathrm{CoAd}_G(G)$ are identified as follows:

(i) single points in the singular plane $\nu = 0$,

and

(ii) symplectic affine planes $\{\mathcal{O}_\nu | \nu \neq 0\}$,

respectively. Hence the spectrum of G provides a continuous open mapping onto the central factor $\mathbf{R} \triangleleft G \longrightarrow \mathbf{R} \oplus \mathbf{R}$, taking a copy of $\mathbf{R} \oplus \mathbf{R}$ onto 0, and a single point onto each nonzero resonance frequency $\nu \in \mathbf{R}$. in terms of the C^*-algebra of G, considered as a bundle of C^*-algebras, the fibres are labelled by the vertical coordinate, the frequency scale $\nu \in \mathbf{R}$. The fibres are

(i) the singular C^*-algebra $\mathcal{C}_0(\mathbf{R}\oplus\mathbf{R})$ of continuous complex valued functions on the plane $\mathbf{R}\oplus\mathbf{R}$ vanishing at infinity, when $\nu = 0$, and

(ii) the standard C^*-algebra $\mathcal{LC}(\mathcal{H})$ of compact operators on the complex Hilbert space $\mathcal{H} = L^2(\mathbf{R})$, when $\nu \neq 0$,

with respect to their natural anti-involutions.

In FTMRI, the basic observation of geometric analysis is that the planar coadjoint orbits

$$\mathcal{O}_\nu \in \mathrm{Lie}(G)^*/\mathrm{CoAd}_G(G)$$

form the central tomographic slices (ii) having a rotating coordinate frame attached to it.

- The FTMRI modality allows for visualization of the planar coadjoint orbit stratification $\mathcal{O}_\nu$, $\nu \neq 0$, of the unitary dual $\hat{G}$. It admits the resonance frequency $\nu \neq 0$ as the vertical coordinate which represent the frequency of excitation.

The other basic observation is that the central tomographic slices with their rotating coordinate frames attached can be selected by resonance with the unitary central character $\chi_\nu = \chi_{U^\nu}$ of frequency $\nu \neq 0$ associated to the central transvections of C. Explicitly,

$$\chi_\nu : C \ni \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \rightsquigarrow e^{2\pi\nu i z} \in \mathbf{S}_1 .$$

Specifically the tomographic slices of scout-view scans are selected as single whole planes by standard clinical MRI protocols in a plane orthogonal to the desired image plane by resonance with χ_ν to allow more accurate tomographic slice positioning. The basic resonance processing of single whole planes is a fundamental consequence of the Stone-von Neumann theorem of quantum mechanics ([46]) in conjunction with the linear ramp action of the frequency modulation group $\mathbf{GA}(\mathbf{R})$ on the homogeneous plane spanned by the lines L and C. The linear ramp action of the Lauterbur spatial encoding technique of quantum holography generates the affine Larmor frequency scale on the dual of C. Due to the gyromagnetic ratio γ of protons given by the Larmor frequency ν at 1.0 T

$$\frac{\gamma}{2\pi} = 42.573 \text{ MHz/T} ,$$

the angular resonance frequency in radians per second at a standard external magnetic flux density B_0 of 1.5 T is given by

$$2\pi\nu = 401.241 \text{ MHz}.$$

For comparison: The earth's average magnetic field has a flux density of about 0.5 Gauβ (1 T = 10^4 Gauβ) which induces protons in the human body to process at an angular frequency of about 2.1 kHz. The external magnetic flux densities B_0 in clinical MRI use vary from approximately 0.5 T to 4.0 T and are strong when compared to the earth's magnetic flux density. Therefore it is important to remember that metal objects taken into MRI scan rooms can become lethal projectiles.

The coadjoint orbit model of the unitary dual of the affine solvable Lie group

$$\mathbf{GA}(\mathbf{R}) \hookrightarrow \mathbf{GL}(2, \mathbf{R})$$

of affine bijection associated to the central, coronal, and sagittal transvections reads

$$\mathbf{R} \cup \mathcal{O}_+ \cup \mathcal{O}_-$$

where $\mathcal{O}_\pm$ denotes the open upper/lower complex half-plane. Its coadjoint orbit symmetries

$$(\mathcal{O}_+, \mathcal{O}_-)$$

indicates how to compensate the phase dispersion induced by the computer controlled wavelet distortion of central tomographic slice selection by the phase conjugation symmetry

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathbf{GA}(\mathbf{R}).$$

For

$$\psi \in L^2(\mathbf{R}_+^\times)$$

and

$$w = \beta + i\alpha \in \mathcal{O}_\pm$$

define the function

$$g(w) = \int_{\mathbf{R}_+^\times} e^{2\pi i w t}\, t^n\, \psi(t)\, dt$$

by an application of the Fourier cotransform at spin level

$$n \in \frac{1}{2}\mathbf{N}^\times.$$

The integral converges, as follows from the Cauchy-Schwarz-Bunjakowski inequality and the Laplace transform:

$$|g(w)| \leq \left(\int_{\mathbf{R}_+^\times} t^{2n} e^{-2\alpha t} dt\right) . \left(\int_{\mathbf{R}_+^\times} |\psi(t)|^2 dt\right)$$

$$= \Gamma(2n+1)^{\frac{1}{2}} (2\alpha)^{-(n+\frac{1}{2})} \|\psi\|_2 .$$

The function g is holomorphic/anti-holomorphic on the open upper/lower complex half-plane $\mathcal{O}_\pm$. An application of the Plancherel theorem yields the identity

$$\int_{\mathbf{R}} |g(\beta + i\alpha)|^2 d\beta = 2\pi \int_{\mathbf{R}_+^\times} e^{-2\alpha t} t^{2n} |\psi(t)|^2 dt ,$$

so that g has finite norm $\|g\|_2$ on the Poincaré half-plane $\mathcal{O}_+$ carrying the standard measure

$$\frac{d\alpha . d\beta}{\alpha^2}$$

which is invariant under the canonical linear fractional action of the unimodular group $\mathbf{SL}_2(\mathbf{R})$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} . w = \frac{aw+b}{cw+d}, \quad \text{ded} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 .$$

It follows the isomorphy

$$\mathcal{O}_+ \cong \mathbf{SL}_2(\mathbf{R})/\mathbf{SO}(2, \mathbf{R})$$

and

$$\|g\|_2^2 = \int\int_{\mathcal{O}_\pm} |g(w)|^2 \alpha^{(2n+1)} \frac{d\alpha . d\beta}{\alpha^2} ,$$

or explicitly

$$\|g\|_2^2 = 2\pi \int_{\mathbf{R}_+^\times} \alpha^{(2n-1)} \int_{\mathbf{R}_+^\times} e^{-2\alpha t} t^{2n} |\psi(t)|^2 \, d\alpha dt = 2\pi 2^{-2n} \Gamma(2n) \|\psi\|_2^2 .$$

The affine wavelet transform $\mathcal{W}_n$ of spin level $n \in \frac{1}{2} N^\times$ associated to the matrix

$$\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \in \mathbf{GA}(\mathbf{R})$$

in the orbit of the natural action of the linear group $\mathbf{GL}(2, \mathbf{R})$ by the holomorphic/antiholomorphic discrete series representations of $\mathbf{SL}_2(\mathbf{R})$ on the open upper/lower complex half-plane $\mathcal{O}_\pm$ with stabilizer at $i \in \mathcal{O}_+$ and $-i \in \mathcal{O}_-$, respectively, reads

$$\mathcal{W}_n : \psi \rightsquigarrow (w \rightsquigarrow \beta^{2n+1} \int_{\mathbf{R}_+^\times} e^{(2\pi i \alpha\beta - \beta^2)t}\, t^n\, \psi(t)\, dt$$

The attenuation factor of the affine wavelet transform $\mathcal{W}_n$ indicates the scattering effect of phase dispersion in response to an external perturbation. For echo wavelet detection, the phase dispersion must be corrected by phase conjugation because the external distortion destroys phase coherence and degrades the quantum holograms encoded by the MRI scanner organization. The distortion correction is performed by the inverse wavelet transform produced by an application of the symmetry $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ in the frequency modulation group $\mathbf{GA}(\mathbf{R})$. As a realization of the Wigner invariance theorem, the symmetry inverts the wavelet transform by recalling in inverse order shape'. The correction processing illuminates the importance of the affine wavelet transform $\mathcal{W}_n$ for the Lauterbur spatial encoding technique of quantum holography.

Gradient echo techniques are an essential part of the modern MRI examination and are offered as standard software by every major vendor. They have proved useful in a wide variety of applications, including spinal imagery, cardiac imaging, magnetic resonance angiography (MRA), three-dimensional imaging, ultra-fast echo planar imaging (EPI), and dynamic contrast techniques. An important feature of both gradient recalled echo (GRE) imaging and spoiled GRE imaging (SPGR) is that the gradient slope reversal refocusses only those spins that have been dephased by the linear ramp action itself. Specifically, phase shifts resulting from magnetic field inhomogeneities, static tissue susceptibility gradients, or chemical shifts are not compensated by the GRE and SPGR techniques. Therefore it is important to notice that the principle of rephrasing by computer controlled gradient slope reversed of GRE and SPGR imaging does not correspond completely to the refocusing technique of spin echo (SE) imaging. In SE protocols, the local frequency fan narrowing performed by $\tilde{\psi} \in \tilde{\mathcal{H}}^{+\infty}$ is implemented by the symplectic form of planar coadjoint orbits $\mathcal{O}_\nu$, $\nu \neq 0$, of the stratification of $\hat{G}$. The natural action of $\mathbf{SU}(2, \mathbf{C}) \hookrightarrow \mathbf{SO}(4, \mathbf{R})$

on the compact unit sphere $\mathbf{S}_2$ of $\mathbf{R}^3$ with stabilizer at the north-pole

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

provides the Hopf projector $\mathbf{S}_3 \longrightarrow \mathbf{S}_2$. It provides by structure transport from the tangent plane of the fuzzy sphere $\mathbf{S}_2$ at the north-pole the complexified coadjoint orbit $\mathcal{O}_\nu \in \mathrm{Lie}(G)^*/\mathrm{CoAd}_G(G)$ with the parametrization

$$\begin{pmatrix} 0 & x+iy \\ x-iy & 0 \end{pmatrix}$$

of two contragredient rotating coordinate frames of frequency $\nu \neq 0$. the complex parametrization is consistent with the structure of $\mathcal{O}_\nu \in \mathrm{Lie}(G)^*/\mathrm{CoAd}_G(G)$ as a symplectic affine plane. Parallel synchronization of the Keppler phase triangulation procedure by computer controlled radiofrequency pulses performing phase conjugation with respect to the rotating coordinates frame attached to the planar coadjoint orbit $\mathcal{O}_\nu$, $\nu \neq 0$, of the stratification of $\hat{G}$ to compensate the dephasing effect of transverse magnetization decay within the planar coadjoint orbit $\mathcal{O}_\nu \in \mathrm{Lie}(G)^*/\mathrm{CoAd}_G(G)$, and the switching of a linear gradient pulse which performs phase conjugation with respect to the affine Larmor frequency scale along the dual of the transversal line C to correct the phase dispersion induced by the computer controlled timing diagram of a specific exciting pulse sequence at resonance frequency $\nu \neq 0$. The parallel synchronization parameters which are central to the understanding of soft tissue contrast way, the waiting time intervals: the time to echo T_E and the time to repeat T_R. In this way, the computer controlled score of the standard symmetric SE modulation functions is a linear ramp. The strength and duration of the pulse determines the flip angle vector. Because the implementation of choreographic scores which perform parallel synchronizations of Keppler configurations is at the heart of MRI protocols, computer controlled synchronized timing is everything in FTMRI.

Soon after the initial discovery of the phenomenon of NMR, Erwin Louis Hahn performed as a former radar technician an experiment that, in conceptual originality and elegance, transcended all that had preceded it, the symmetric SE experiment. The originality and elegance provide a clear superiority of MRI over X-ray CT. The symmetric SEM imaging technique, which represents the most generic MRI technique, has long been the workhorse of clinical MRI because of its generally high signal intensity and soft tissue contrast compared with other imaging techniques. In joint imaging, for instance, the symmetric SE technique is most commonly used so that even today, the symmetric SE protocol remains the gold standard for diagnostic MRI due to its inherent contrast properties, its robust signal strength, its flexibility for contrast adaption, and its relative insensitivity to artifacts that plague other techniques such as GRE imaging.

The symmetric SE imaging technique is characterized by a $\frac{\pi}{2}$ pulse that flips the Bloch vector into the selected planar coadjoint orbit of G followed by a contragredient π pulse performing phase conjugation, and generates a refocused echo wavelet by local frequency fan narrowing at the time T_E. This computer controlled synchronized pulsing is repeated at time intervals of length T_R to record the response data phase holograms by switching the linear frequency gradient pulse and by stepwise increasing the slope $\alpha > 0$ of the frequency modulating linear ramps $\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} \in \mathbf{GA(R)}$ in the score.

- The echo time T_E is the time the MRI scanner waits after the applied pulse to receive the echo wavelet elicited at resonance frequency $\nu \neq 0$.
- The repetition time T_R represents the time that elapses from the $\frac{\pi}{2}$ pulse of one repetition to the same pulse in the next repetition.

Advanced methods of sampling the acquisition data, such as the fast SE imaging technique (FSE) or turbo SE acquisition, significantly decrease the scan times by acquiring trains of multiple spin echoes instead of one in each repetition interval of length T_R. The number of phase encoding steps that are simultaneously acquired is termed the echo train length (ETL). Typically, an ETL ≤ 16 can be used in clinical neurodiagnostic imaging ([28]). STALO controlled transmission gates for sequences of coherent pulses and substantial computer memory, however, are required to handle a long ETL. For high resoluting PSE studies even and ETL scan parameter of 64 may be chosen. After each echo, the phase encoding is rewound to preserve phase coherence and allow formation of the subsequent spin echoes. In the sense of parallel data acquisition, computer controlled synchronized timing is everything in FTMRI.

Recall that an irreducible unitary linear representation U of G with projective kernel C is square integrable mod C if and only if one of the non-trivial coefficient functions of U, hence all the coefficient functions of U, belong to the complex Hilbert space $L^2(G/C)$. By Godement's theorem ([52]), an equivalent condition is the isomorphy of U to a subrepresentation of the regular representation acting on $L^2(G/C)$. Equivalently, U is a discrete summand of the linear representation

$$U^\chi = \mathrm{ind}_C^G(\chi_U)$$

of G which is unitarily induced by the unitary central character

$$\chi_U = U|C$$

of G, hence the term discrete series representation of G ([43]). Two isomorphic irreducible unitary linear representations of G are either simultaneously square integrable mod C or not ([46]). The strong geometric condition of flatness of the symplectic affine planar coadjoint orbits $\mathcal{O}_\nu \in \mathrm{Lie}(G)*/\mathrm{CoAd}_G(G)$ admitting the vertical coordinate $\nu \neq 0$ implies the square integrability mod C of the associated isomorphy classes in $\hat{G}$ in irreducible unitary linear representation U^ν of G ([7]). It follows the fundamental identity of central tomographic slice selection at resonance frequency ν which represents the frequency of excitation:

$$U^\nu|C = \chi_\nu \quad (\nu \neq 0)$$

and each irreducible unitary linear representation V^ν of G satisfying the resonance condition of the unitary central character

$$V^\nu|C = \chi_\nu \quad (\nu \neq 0)$$

is unitarily isomorphic to U^ν, hence attached to the same planar coadjoint orbit $\mathcal{O}_\nu \in \mathrm{Lie}(G)*/\mathrm{CoAd}_G(G)$, and therefore a discrete summand of the unitary linear representation $U^{\chi_\nu} = \mathrm{ind}_C^G(\chi_\nu)$ of G. According to the Stone-von Neumann theorem of quantum mechanics, the isomorphism of U^ν onto V^ν is given by the unitary action of the metaplectic group $\mathbf{Mp}(2, \mathbf{R})$ on $\mathcal{H} = L^2(\mathbf{R})$ ([27]). Specifically

$$\sigma(J) = \overline{\mathcal{F}}_{\mathbf{R}}$$

is the Fourier cotransform of the bi-infinite time scale $\mathbf{R}$.

- For $\nu \neq 0$, the unitary action σ of the metaplectic group $\mathbf{Mp}(2, \mathbf{R})$ links the symplectic structure J of the planar coadjoint orbit $\mathcal{O}_\nu \in \mathrm{Lie}(G)*/\mathrm{CoAd}_G(G)$ to the spectral information content of phase coherent wavelets on the bi-infinite time scale $\mathbf{R}$.

The semidirect product of $\mathbf{Mp}(2, \mathbf{R})$ and G is the Jacobi group $G^J = \mathbf{Mp}(2, \mathbf{R}) \propto G$. Its Lie algebra $\mathrm{Lie}(G^J)$ reflects the canonical commutation relations of quantum mechanics

$$[X, Y] = X.Y - Y.X = 1$$

in the real Weyl algebra of non-commutative polynomials in two variables $\{X, Y\}$ of degree ≤ 2.

The Schwartz kernel of $\mathcal{H}$ in $\mathcal{H}$ is the canonical isomorphism

$$j : \overline{\mathcal{H}}' \longrightarrow \mathcal{H}$$

admitting the inverse kernel of $\overline{\mathcal{H}}'$ in $\overline{\mathcal{H}}'$ which is given by

$$j^{-1} : \mathcal{H} \longrightarrow \overline{\mathcal{H}}'.$$

The identity

$$\check{U}^\nu | C = \overline{\chi}_\nu \quad (\nu \neq 0)$$

for the unitary central character of the contragredient representation $\check{U}^\nu$ acting on $\overline{\mathcal{H}}'$ shows that $\check{U}^\nu$ is also square integrable mod C for $\nu \neq 0$, that it is isomorphic to $U^{-\nu}$, that its isomorphy class as an element of $\hat{G}$ is associated to the conjugate planar coadjoint orbit $\mathcal{O}_{-\nu} \in \mathrm{Lie}(G) * /\mathrm{CoAd}_G(G)$, and that it forms a discrete summand of the unitary linear representation $U^{\chi_{-\nu}} = \mathrm{ind}_C^G(\chi_{-\nu})$ of G. The covariant representation

$$\check{U}^\nu \hat{\otimes} U^\nu$$

of G acts unitarily on the complex Hilbert space of Hilbert-Schmidt integral operators

$$\mathcal{L}_2(\mathcal{H}) \hookrightarrow \mathcal{LC}(\mathcal{H})$$

of $\mathcal{H}$ which is endowed with its covariant tracial scalar product. It weakly contains the one-dimensional identity representation of G having as its associated coadjoint orbit the single point set

$$\{0\} \in \mathrm{Lie}(G)^* / \mathrm{CoAd}_G(G)$$

which allows to map $\mathcal{O}_\nu$ onto $\mathcal{O}_{-\nu}$ by central reflection:

$$\mathcal{O}_\nu \longleftrightarrow \mathcal{O}_{-\nu} \quad (\nu \neq 0).$$

This central reflection is basic for the transactional interpretation of quantum mechanics ([15]). The action of the linear Schroödinger representation U^ν of G at the matrix $\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in G$ on the phase coherent wavelet ψ in the complex Schwartz space $S(\mathbf{R}) \hookrightarrow L^2(\mathbf{R})$ on the bi-infinite time scale $\mathbf{R}$ reads

$$U^\nu(\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}) : \psi \rightsquigarrow (t \rightsquigarrow e^{2\pi i\nu(z+it)}\psi(t+x))$$

The action of the linear Schroödinger representation Uv implements the iteration of the smooth one-parameter action of $\mathbf{R}$ on $S(\mathbf{R})$ by translation of the bi-infinite time scale $\mathbf{R}$, and the dual action of $\mathbf{R}$ on $S(\mathbf{R})$, which correspond to the faithful block representations of G by sagittal and coronal transvections, respectively. The infinitesimal generator of the associated smooth dynamical system $\mathbf{R} \triangleleft G \longrightarrow \mathbf{R} \oplus \mathbf{R}$ which is operating on the complex vector space $\tilde{\mathcal{H}}^{-\infty}$ of kernel distributions of $\mathcal{H} = L^2(\mathbf{R})$ is given by the formula

$$\omega_\nu = \tilde{U}^\nu(-\dot{\varepsilon}_0)\,.$$

Using the Newtonian notation,

$$\dot{\varepsilon}_0 = \frac{d\varepsilon_0}{dt}$$

denote the temporal derivative of the Dirac measure ε_0 at the origin of the bi-infinite time scale $\mathbf{R}$ associated to the smooth dynamical system $\mathbf{R} \triangleleft G \longrightarrow \mathbf{R} \oplus \mathbf{R}$. In the semi-classical QED description of response data phase holograms excited by $\tilde{\psi}$, the parameter x denotes the differential phase coordinate, and the parameter y denotes the local frequency coordinate with respect to a coordinate frame rotating with frequency $\nu \neq 0$. Finally, z the vertical coordinate of the central tomographic slice $\mathcal{O}_\nu \in \mathrm{Lie}(G)^*/\mathrm{CoAd}_G(G)$, is dual to the overall resonance frequency $\nu \neq 0$ which represents the frequency of excitation. All three coordinates are implemented by computer controlled wavelet perturbations. Their embodiment requires distortion corrections by phase conjugation within the score.

To complement the semi-classical QED description, it should be noted that an alternate realization of the linear Schroödinger representation U^ν of G is given by the Bargmann-Fock model of lowering and raising operators of quantum mechanics operating on holomorphic functions on W ([46]). The standard L^2 Sobolev inequality establishes that the complex vector space of smooth vectors for U^ν acting on $\mathcal{H} = L^2(\mathbf{R})$ is formed by the Schwartz space

$$\mathcal{H}^{+\infty} = S(\mathbf{R})$$

of complex-valued smooth functions on the bi-infinite time scale $\mathbf{R}$, rapidly decreasing at infinity such that all their derivatives are also rapidly decreasing at infinity. It is well known that $\mathcal{H}^{+\infty}$ is a complex nuclear locally convex topological vector space of complex distributions on $\mathbf{R}$ in the sense that the canonical injection

$$\mathcal{D}(\mathbf{R}) \hookrightarrow \mathcal{H}^{+\infty}$$

is continuous and admits an everywhere dense image ([50]). By extension, it provides $\mathcal{H}^{+\infty}$ with its natural anti-involution. The complex vector space $\tilde{\mathcal{H}}^{-\infty}$ which is isomorphic to $\mathcal{H}^{-\infty}$ under the locally convex vector space topology induced by $\mathcal{D}'(G; \mathcal{H})$ forms the Sobolev space consisting of all tempered distributions $T \in S'(\mathbf{R} \oplus \mathbf{R})$ such that their symplectic convolution products satisfy

$$f \star_\nu T \in L^2(\mathbf{R} \oplus \mathbf{R})$$

for all functions $f \in S(\mathbf{R} \oplus \mathbf{R})$. It contains the irreducible $S(\mathbf{R} \oplus \mathbf{R})$ - module $S(\mathbf{R} \oplus \mathbf{R})$ as well as the irreducible $L^2(\mathbf{R} \oplus \mathbf{R})$ - module $L^2(\mathbf{R} \oplus \mathbf{R})$in the sense of the symplectic convolution product $\star_\nu$ of $\tilde{\mathcal{H}}^{-\infty}$. Thus the continuous inclusions

$$S(\mathbf{R} \oplus \mathbf{R}) \hookrightarrow L^2(\mathbf{R} \oplus \mathbf{R}) \hookrightarrow \mathcal{H}^{-\infty} \hookrightarrow S'(\mathbf{R} \oplus \mathbf{R})$$

hold. Due to the swapping symplectic matrix $\frac{1}{2}J$ associated to the Lie bracket of Lie(G), the symplectic convolution product induced by the linear Schrodinger representation U^ν on the planar coadjoint orbit $\mathcal{O}_\nu \in \mathrm{Lie}(G)^*/\mathrm{CoAd}_G(G)$ takes the explicit form

$$f \star_\nu g(x, y; \nu) =$$

$$\frac{1}{2} \int_{\mathbf{R}\oplus\mathbf{R}} f(x', y'; \nu).g(x - x'), y - y'; \nu)\, e^{\pi i \nu \det \begin{pmatrix} x & y \\ x' & y' \end{pmatrix}} dx'\, dy'.$$

The normalization factor in front of the integral comes from the formal degree of the linear Schrödinger representation U^ν of G. Taking into account the iteration of mooth actions implemented by U^ν, the symplectic convolution product $f \star_\nu g$ reveals to be jointly continuous for

$$f, g \in S(\mathbf{R} \oplus \mathbf{R}) \cong S(\mathbf{R}) \hat{\otimes} S(\mathbf{R}) \cong S(\mathbf{R}; S(\mathbf{R}))$$

and $\nu \neq 0$. Hence the irreducible $S(\mathbf{R} \oplus \mathbf{R})$ - module $S(\mathbf{R} \oplus \mathbf{R})$ is a complex Fréchet algebra under symplectic convolution. The symplectic Fourier transform is defined by the involutory isomorphism

$$S(\mathbf{R} \oplus \mathbf{R}) \ni f \rightsquigarrow \hat{f} \in S(\mathbf{R} \oplus \mathbf{R}),$$

where

$$\hat{f} = f \star_\nu (1_\nu \otimes 1_\nu)$$

The symplectic convolution with the constant distribution $1_\nu \otimes 1_\nu$ on the symplectic affine plane $\mathcal{O}_\nu \in \mathrm{Lie}(G)^* / \mathrm{CoAd}_G(G)$ explicitly reads

$$\hat{f}(x, y; \nu) = \frac{1}{2} \int_{\mathbf{R} \oplus \mathbf{R}} f(x', y'; \nu)\, e^{\pi i \nu . \det \begin{pmatrix} x & y \\ x' & y' \end{pmatrix}} dx'\, dy'$$

$$((x, y) \in \mathbf{R} \oplus \mathbf{R}).$$

It follows from the Stone-von Neumann theorem of quantum mechanics ([46]) or from a smooth version of the Takesaki-Takai duality theorem ([18]) applied to the dynamical system $\mathbf{R} \triangleleft G \longrightarrow \mathbf{R} \oplus \mathbf{R}$, that phase averaging of the proton-weighted spin isochromat density

$$f \in L^2(\mathbf{R} \oplus \mathbf{R}) \hookrightarrow S'(\mathbf{R} \oplus \mathbf{R})$$

by the mod C square integrable linear Schrodinger representation U^ν of G leads to the Hilbert-Schmidt integral operator

$$U^\nu(f) \in \mathcal{L}_2(\mathcal{H}),$$

realized by an integral operator of kernel in $L^2(\mathbf{R} \oplus \mathbf{R})$. The integrated form $U^\nu(f)$ of U^ν extends the evaluation at the Dirac measure $\varepsilon_{\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}}$ by an average over the cross-section G/C to the center C with respect to the measure

$$f(x, y; \nu).dx \otimes dy.$$

- The square integrability mod C of the linear Schrodinger representation U^ν of G for $\nu \neq 0$ allows to embed by means of the symbol calculus the von Neumann approach to quantum mechanics which is based on the category of Hilbert spaces into the Dirac approach which is based on complex locally convex topological vector spaces of tempered distributions.

Let $\langle\, .\, ,\, .\rangle$ denote the bracket which defines the topological vector space antiduality

$$(S(\mathbf{R}), S'(\mathbf{R}))\,.$$

It implements the involutory anti-automorphism of the complex vector space $S'(\mathbf{R})$ the tempered distributions on the bi-infinite time scale $\mathbf{R}$ which is contragredient to the natural anti-involution of $S(\mathbf{R})$. Moreover, it provides $S'(\mathbf{R})$ with its weak dual topology under which $S'(\mathbf{R})$ forms a complex nuclear locally convex topological vector space ([50]).

- The complex vector spaces $S(\mathbf{R})$ and $S'(\mathbf{R})$ form their own antispaces, and the sesquilinear form $\langle\, .\, ,\, .\rangle$ is consistent with the internal scanal product of the standard complex Hilbert space $L^2(\mathbf{R}) \hookrightarrow S'(\mathbf{R})$.

The holographic transform attached to the symplectic affine plane $\mathcal{O}_\nu \in \mathrm{Lie}(G)^*/\mathrm{CoAd}_G(G)$ at resonance frequency $\nu \neq 0$ is defined by

$$\mathcal{H}_\nu : S(\mathbf{R}) \ni \psi \rightsquigarrow \left((x, y) \rightsquigarrow \langle U^\nu(\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix})\psi, 1_\nu\rangle . 1_\nu \otimes 1_\nu\right).$$

According to the semi-classical QED description of response data phase holograms, the tempered distribution $1_\nu \in S'(\mathbf{R})$ represents the proton density-weighted rotation at the frequency $\nu \neq 0$ of the unitary central character χ_ν associated to the planar coadjoint orbit $\mathcal{O}_\nu \in \mathrm{Lie}(G)^*/\mathrm{CoAd}_G(G)$, and $\mathcal{H}_\nu$ performs the temporal cross-correlation of the proton density-weight $1_\nu \in S'(\mathbf{R})$ with the echo wavelet $\psi \in S(\mathbf{R})$ elicited at resonance frequency $\nu \neq 0$. The echo wavelet $\psi \in S(\mathbf{R})$ represents the non-linear spin isochromat's response to the exciting pulse sequence at resonance frequency $\nu \neq 0$. Specifically,

- $1_\nu \otimes 1_\nu$ forms the central reference of the rotating coordinate frame attached to the planar coadjoint orbit $\mathcal{O}_\nu \in \mathrm{Lie}(G)^*/\mathrm{CoAd}_G(G)$ at frequency of excitation $\nu \neq 0$.

An application of the Paley-Wiener-Schwartz theorem allows to smooth out on the Fourier transform side the Poisson bracket by an expansion in terms of tranvectants of the symplectic affine plane $\mathcal{O}_\nu \in \mathrm{Lie}(G)^*/\mathrm{CoAd}_G(G)$.

- For $\nu \neq 0$, the holographic transform $\mathcal{H}_\nu$ attached to the symplectic affine plane $\mathcal{O}_\nu \in \mathrm{Lie}(G)^*/\mathrm{CoAd}_G(G)$ forms a quantum analog of the Liouville density in phase space. The FTMRI modality is a quantum holographic process.

Because the continuous linear mapping $\mathcal{H} - V$ defined by the coefficient of the contragredient representation

$$\check{U}^\nu \cong U^{-\nu}$$

of G corresponding to $1_\nu \in S'(\mathbf{R})$ commutes with the left regular action of G/C on $L^2(G/C)$, and application of Schur's theorem ([46]) to the von Neumann algebra formed by the weakly closed commutant of $U(G)$ in the group of automorphisms of $\mathcal{H}$ yields

$$\mathcal{H}_{-\nu} \circ j \circ \mathcal{H}_\nu = \mathrm{id}_{S'(\mathbf{R}\oplus\mathbf{R})},$$

and similarly

$$\mathcal{H}_\nu \circ j^{-1} \circ \mathcal{H}_{-\nu} = \mathrm{id}_{S'(\mathbf{R}\oplus\mathbf{R})}.$$

Thus the identity

$$\overline{\mathcal{H}}'_\nu = \mathcal{H}_{-\nu}$$

holds for the inverse kernel of $\mathcal{H}_\nu$. In consistency with the Bochner-Plancherel-Schwartz characterization of positive definite distributions, the tempered distribution

$$\mathcal{H}^\bullet_\nu = 1_\nu \otimes 1_\nu$$

on the planar coadjoint orbit $\mathcal{O}_\nu \in \mathrm{Lie}(G)^*/\mathrm{CoAd}_G(G)$ of unitary central character χ_ν provides the uniquely defined canonical reproducing kernel $\star_\nu \mathcal{H}^\bullet_\nu$ of the linear Schrödinger representation U^ν, $\nu \neq 0$, of G. Notice that $\mathcal{H}^\bullet_\nu \in \tilde{\mathcal{H}}^{-\infty}$ represents the orientation class of $S'(\mathbf{R} \oplus \mathbf{R})$. Conversely, the preceding identity and its infinitesimally generating equivalent imply the square integrability mod C of U^ν and the flatness of the planar coadjoint orbit $\mathcal{O}_\nu$, $\nu \neq 0$, of the stratification of $\hat{G}$ associated to the isomorphy class of U^ν in the unitary dual $\hat{G}$ of G ([19]).

- The flatness of the symplectic affine planes $\mathcal{O}_\nu$, $\nu \neq 0$, of the stratification of $\hat{G}$ allows to identify the trace of the holographic transform $\mathcal{H}_\nu$ attached to the planar coadjoint orbit $\mathcal{O}_\nu \in \mathrm{Lie}(G)^*/\mathrm{CoAd}_G(G)$ of the Heisenberg group G at resonance frequency ν as follows:

$$\mathrm{tr}\mathcal{H}_\nu = \mathcal{H}_\nu^\bullet$$

  In terms of the exterior differential 2-form representing the swapping matrix $\frac{1}{2}J$, the infinitesimal generator ω_ν of the smooth dynamical system $\mathbf{R} \triangleleft G \longrightarrow \mathbf{R} \oplus \mathbf{R}$ is the rotational curvature form

$$\omega_\nu = \pi i \nu . dx \wedge dy = -\frac{\pi}{2}\nu . dw \wedge d\overline{w} .$$

  The symplectic character formula holds in the density space $\tilde{\mathcal{H}}^{-\infty}$, where (x, y) denote the differential phase-local frequency coordinates with respect to a coordinate frame rotating with frequency $\nu \neq 0$, and $w = x + iy \in W$. The upsampling Pfaffiant of the canonical symplectic form $\frac{1}{\pi i}\omega_\nu$ of the planar coadjoint orbit $\mathcal{O}_\nu \in \mathrm{Lie}(G)^*/\mathrm{CoAd}_G(G)$ reads

$$\mathrm{Pf}\Big(\frac{1}{\pi i}\omega_\nu\Big) = \nu .$$

The identity

$$U^\nu(\mu)\mathcal{H}_\nu = \mu \star_\nu \mathcal{H}_\nu$$

which holds for all compactly supported scalar measures on the symplectic affine plane $\mathcal{O}_\nu \in \mathrm{Lie}(G)^*/\mathrm{CoAd}_G(G)$ implies

$$\mathcal{H}_\nu(\psi) = \tilde{\psi} \star_\nu \mathcal{H}_\nu^\bullet .$$

Hence

$$\langle \mathcal{H}_\nu(\psi), \mathcal{H}_\nu^\bullet \rangle = \tilde{\psi}$$

for all elements $\psi \in \mathcal{H}^{-\infty}$. It follows that $\mathcal{H}_\nu \in S'(\mathbf{R} \oplus \mathbf{R})$ represents a reproducing kernel of $\mathcal{H} = L^2(\mathbf{R})$ in $\mathcal{H}^{-\infty}$. The uniquely defined canonical reproducing kernel $\star_\nu \mathcal{H}^\bullet$ where $\mathcal{H}^\bullet \in \tilde{\mathcal{H}}^{-\infty}$ acts as the differential phase-local frequency reference of the rotating coordinate frame of frequency $\nu \neq 0$ which is attached to the planar coadjoint orbit $\mathcal{O}_\nu \in \mathrm{Lie}(G)^*/\mathrm{CoAd}_G(G)$. According to the semi-classical QED description of response data phase holograms, it can be thought of as the reference of a Keppler configuration consisting of precession arrays of phase-locked Bloch vectors inside the symplectic affine plane $\mathcal{O}_\nu$, $\nu \neq 0$ of the stratification of $\hat{G}$.

According to Harish-Chandra's philosophy the correct way to think of characters of Lie groups is as distributions. In terms of the symplectic presentation of G, the symplectic character formula displayed supra can be looked at as a generalization of the relation

$$\exp \circ \operatorname{tr} = \det \circ \exp$$

to Schwartz kernels. From the calculus of Schwartz kernels in the locally convex topological vector space $S'(\mathbf{R} \oplus \mathbf{R})$ of tempered distributions follow the symplectic filter bank identities

$$\sqrt{\nu}.\mathcal{H} = \sqrt{\star_\nu \mathcal{H}_\nu^\bullet \mathcal{L}_2(\mathcal{H})} \quad (\nu > 0),$$

and

$$\sqrt{-\nu}.\mathcal{H} = \sqrt{\star_{-\nu} \mathcal{H}_{-\nu}^\bullet \mathcal{L}_2(\mathcal{H})} \quad (\nu < 0).$$

The kernel K_f^ν associated to the Hilbert-Schmidt integral operator $U^\nu(f) \in \mathcal{L}_2(\mathcal{H})$ for $f \in L^2(\mathbf{R} \oplus \mathbf{R})$ extends from $f \in S(\mathbf{R} \oplus \mathbf{R})$ to its antidual $S'(\mathbf{R} \oplus \mathbf{R})$ by the rule

$$K_f^\nu(x, y) = 2e^{-\pi i \nu x y}.f \star_\nu \mathcal{H}_\nu^\bullet(x, y) \quad ((x, y) \in \mathbf{R} \oplus \mathbf{R}).$$

The preceding identity leads to the following result which explains the important role played by the symplectic filter bank processing in FTMRI. Several methods of MRI such as the spin presaturation technique by gradient spoiling for motion artefact reduction in variable-thickness slabs are based on symplectic filter bank processing.

- The generating kernel distribution K_f^ν with respect to the rotating coordinate frame attached to the symplectic affine plane $\mathcal{O}_\nu \in \mathrm{Lie}(G)^*/\mathrm{CoAd}_G(G)$ obtains by symplectic filtering of the proton-weighted spin isochromat density f with the uniquely defined canonical reproducing kernel $\star_\nu \mathcal{H}_\nu^\bullet$ associated to the holographic $\mathcal{H}_\nu$ at resonance frequency $\nu \neq 0$.

The irreducible unitary linear representation V^ν of G satisfying the resonance condition of the unitary central character and therefore attached to the same planar coadjoint orbit $\mathcal{O}_\nu$ of the stratification of $\hat{G}$ defines a holographic transform isomorphic to $\mathcal{H}_\nu$ by the unitary action

σ of the metaplectic group $\mathbf{Mp}(2, \mathbf{R})$ on the Hilbert space $\mathcal{H} = L^2(\mathbf{R})$ and therefore a generating kernel distribution isomorphic to K_f^ν.

5. Relaxation filtering. A standard clinical symmetric SE protocol consists of a computer controlled $\frac{\pi}{2}$ pulse which flips by resonance the Bloch vector into the selected central tomographic slice. Because there are also phased-arrays of contragredient Bloch vectors rotating in the opposite sense to the spin precession at resonance frequency v, the following contragredient π pulse at $\frac{1}{2}T_E$ performs a local frequency fan narrowing by phase conjugation. According to the (T_1, T_2) relaxation decomposition of G, the tissue specific spin relaxation rates

$$\frac{1}{T_k} \quad (k \in \{1, 2\})$$

and the flip angle θ of the Bloch vector provide a non-linear amplitude modulation

$$w_{T_1,T_2,\theta} : t \rightsquigarrow \beta_{T_1,\theta}(t).e^{-\frac{|t|}{T_2}} \sin\theta$$

of the exciting pulse sequence $\psi \in S'(\mathbf{R})$ at resonance frequency $\nu \neq 0$. Due to the Bloch gyroscopic equation which is rotationally invariant, $w_{T_1,T_2,\theta}$ includes the central spin relaxation weight([49])

$$\beta_{T_1,\theta} : t \rightsquigarrow e^{-\frac{|t|}{T_1}} \cos\theta + (1 - e^{-\frac{|t|}{T_1}})$$

for normalized equilibrium magnetization.

The modulation of the proton-weighted spin isochromat density f by the relaxation weight $\tilde{w}_{T_1,T_2,\theta}$ allows for control of the contrast resolution by synchronization of the temporal filter bank design parameters T_E and T_R of the exciting pulse sequence $\psi \in S'(\mathbf{R})$ at resonance frequency $\nu \neq 0$, with the dephasing signal

$$K^\nu_{f \star_\nu \tilde{w}_{T_1,T_2,\theta}} \in S'(\mathbf{R} \oplus \mathbf{R}).$$

The preceding symplectic convolution represents the dephasing effect of transverse magnetization decay within the planar coadjoint orbit $\mathcal{O}_\nu \in \mathrm{Lie}(G)^* / \mathrm{CoAd}_G(G)$.

- Spin relaxation performs a symplectic filtering of the proton-weighted spin isochromat density f with temporal filter bank design parameters T_E and T_R.

By judicious choice of the pulse sequence timing parameters T_E and T_R, the degree of T_1-weighting and T_2-weighting on the image are controlled. Consequently tissues with differing spin densities and/or relaxation times may be distinguished on appropriately weighted images.

In standard clinical MRI protocols, short T_E and short T_R provides T_1-weighting, long T_E and long T_R provides T_2-weighting, short T_E and long T_R provides proton density weighting. T_1-weighted scans are most useful for analyzing details of parenchymal morphology and are employed in conjunction with the gadolinium contrast enhancing agent because enhancing lesions are bright under T_1- weighting. In GRE imaging, the SPGR modality provides T_1-weighting because it eliminates and residual transverse magnetization or T_2 effect due to the phase dispersion that is caused by the application of additional spoiler gradients. T_2-weighted scans are very sensitive to increased water content and can visualize edema to great advantage. T_2-weighted scans are also most sensitive to difference in susceptibility. Proton density-weighted scans are helpful from the standpoint of diagnostic specificity because they give far better tissue differentiation than does X-ray CT.

Beyond these general guidelines, the imaging strategies differ depending upon their goals. In joint imaging, for instance, T_1-weighted scans better display the hyperintense bone marrow that contrasts with the signal void cortex. The intraarticular fluid collection and the hyaline cartilage are clearly delineated on T_2-weighted tomographic slices ([4]). With the introduction of FSE techniques, the information obtained by symmetric SE scans in the relatively long MRI examination time of 9 to 10 minutes has been reduced to 5 minutes. FSE protocols have brought a revolution to body, musculoskeletal, and spinal MRI, but has been slower to find a role in cerebral MRI examinations.

In MRI protocols, the relaxation times satisfy

$$T_2 < T_1 \,,$$

whereas the synchronization parameters of echo and repetition times satisfy

$$T_E < T_R \,.$$

In standard MRI protocols of clinical neurodiagnostic imaging spinal imaging ([60]), short T_E typically means 20 ms, and long TE means 160 ms, whereas short T_R typically means 600 ms and long T_R means 3000 ms. Due to the low signal-to-noise ratio, long T_E, short T_R scans are not used in clinical MRI protocols. In this sense again, computer controlled synchronized timing is everything in FTMRI.

It should be observed that there exists fast MRI methods such as the PRESTO pulse sequence which achieve $T_E > T_R$ by spin echo shifting ([36]). The shifted echo is realized by bringing spins in phase at the desired delayed echo time with respect principal gradient and by dephasing other possible gradients and spin echoes. The result is an increased sensitivity to dynamic susceptibility effects while maintaining a short total imaging time. Because the measurement of changes in microscopic magnetic susceptibility effects forms the basic of studying brain activation with MRI, the PRESTO technique is a fast fMRI method.

6. FTMRI reconstruction. The MRI scan generated in the rotating coordinate frame which is attached to the symplectic affine plane $\mathcal{O}_\nu \in \mathrm{Lie}(G)^*/\mathrm{CoAd}_G(G)$ can be thought of as the result of a cascade of symplectic filtering processes ([29]). In consistency with the Heisenberg uncertainty principle, transition from the rotating coordinate frame to the laboratory coordinate frame attached to G/C is performed by the partial Fourier cotransform

$$\overline{\mathcal{F}}^2_{\mathbf{R}\oplus\mathbf{R}}\, \mathcal{H}^\bullet_\nu(x, y) = 1_\nu(x) \otimes \varepsilon_{-\nu y}\,.$$

Freezing of the local frequency coordinate y with respect to the rotating coordinate frame attached to the symplectic affine plane $\mathcal{O}_\nu \in \mathrm{Lie}(G)^*/\mathrm{CoAd}_G(G)$ at the overall resonance frequency $\nu \neq 0$ is performed by resonance with the affine Larmor frequency scale of the faithful block representation

$$y \rightsquigarrow \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}.$$

For the purpose of multiple line collecting filter bank organization of the response data phase holograms in the laboratory frame, adjusting the spectra of single whole resonance lines ([29]) of image data

$$\{1_\nu(x) \otimes \varepsilon_{-\nu y} | (x, y) \in \mathbf{R} \oplus \mathbf{R}\}$$

of $\mathcal{H}^\bullet_\nu \in \tilde{\mathcal{H}}^{-\infty}$ to the symplectic structure J of the planar coadjoint orbit

$$\mathcal{O}_\nu \in \mathrm{Lie}(G)^*/\mathrm{CoAd}_G(G)$$

is achieved by the affine Larmor frequency scale

$$\{\varepsilon_{\nu x} \otimes 1_\nu(y) | (x, y) \in \mathbf{R} \oplus \mathbf{R}\}$$

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$$\mathcal{O}_\nu \in \mathrm{Lie}(G)^*/\mathrm{CoAd}_G(G)$$

is achieved by the affine Larmor frequency scale

$$\{\varepsilon_{\nu x} \otimes 1_\nu(y) | (x, y) \in \mathbf{R} \oplus \mathbf{R}\}$$

swapped by σ and associated to the faithful block representation

$$x \rightsquigarrow \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

introduced earlier in the context of the Lauterbur spatial encoding technique of quantum holography.

The partial Fourier cotransform $\overline{\mathcal{F}}^2_{\mathbf{R}\oplus\mathbf{R}}$ is incapsulated in the symplectically invariant symbol map $\tilde{U}^\nu(K_f^\nu)$ in the sense of the theory of pseudo - differential operators ([24])

$$\tilde{U}^\nu(K_f^\nu) : (x, y) \rightsquigarrow \tilde{U}^\nu(\begin{pmatrix} 1 & x & 0 \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix})(K_f^\nu)$$

of the kernel ([58])

$$K_f^\nu \in L^2(\mathbf{R} \oplus \mathbf{R})$$

associated to the Hilbert-Schmidt integral operator $U^\nu(f) \in \mathcal{L}_2(\mathcal{H})$ for the proton-weighted spin isochromat density $f \in L^2(\mathbf{R} \oplus \mathbf{R})$. Indeed, the kernel function $\tilde{U}^\nu(K_f^\nu) \in L^2(\mathbf{R} \oplus \mathbf{R})$ takes the form

$$\tilde{U}^\nu(K_f^\nu)(x, y) = e^{-2\pi i \nu x y} . \overline{\mathcal{F}}^2_{\mathbf{R}\oplus\mathbf{R}}(K_f^\nu)(x, y) \ (x, y) \in \mathbf{R} \oplus \mathbf{R}) .$$

It represents the response data phase hologram quantum electrodynamically performed by the MRI scanner organization from the excited proton-weighted spin isochromat density f in the laboratory coordinate frame attached to the cross-section G/C by interference with the echo wavelet $\psi \in S(\mathbf{R})$ elicited at resonance frequency $\nu \neq 0$.

The central aspect of MRI is the encoding of differential phase - local frequency coordinates by response data phase holograms. According to the semi classical QED description, response data phase holograms form symplectic spinors ([27]) which act as learning matrices consisting of matched filter banks. The matched filter bank processing to increase the signal to noise ratio ([37]) forms the link between MRI and coherent optical holography ([21], [47]). The two main advantages offered by the holographic storage technology are high information capacity in small volumes and massive parallelism in data access. Considered as learning matrices ([9]), the phase holograms form the link between MRI and corticomorphic neutral network processing. In the sense of photonics ([48]), phase coherence is everything in MRI.

The FTMRI modality is based on the marriage of the computer to quantum mechanics. Readout of the response data phase holograms $\tilde{U}^{\nu}(K_f^{\nu})$ stored in the laboratory coordinate frame of G/C after transition from the rotating coordinate frame attached to the planar coadjoint orbit $\mathcal{O}_{\nu}$, $\nu \neq 0$, of the stratification of $\hat{G}$ follows by the symplectic Fourier transform

$$\tilde{U}^{\nu}(K_f^{\nu})(x, y) = 2(e^{-2\pi i\nu x'y'} f(x', y'; \nu))^{(2x, 2y)} \quad ((x, y) \in \mathbf{R} \oplus \mathbf{R})$$

of pseudodifferential operators. Indeed, because $\tilde{U}^{\nu}(K_f^{\nu})$ averages the phases of the proton weighted spin isochromat densities in the symplectic affine plane $\mathcal{O}_{\nu} \in \mathrm{Lie}(G)^*/\mathrm{CoAd}_G(G)$, the symplectically invariant symbol calculus provides the reconstruction formula in the laboratory coordinate frame of G/C for the proton density-weighted spin isochromat density $f \in L^2(\mathbf{R} \oplus \mathbf{R})$

$$f(x, y; \nu) = \frac{1}{2} e^{\pi i\nu xy} . \widehat{\tilde{U}^{\nu}(K_f^{\nu})}(\frac{1}{2}x, \frac{1}{2}y) \quad ((x, y) \in \mathbf{R} \oplus \mathbf{R}) .$$

Thus the canonical reproducing kernel

$$\star_{\nu} \mathcal{H}_{\nu}^{\bullet} = \star_{\nu}(1_{\nu} \otimes 1_{\nu})$$

forms the cryptographic key for the decoding of the quantum holographically encoded MRI encryption.

- The symplectic Fourier transform $f \rightsquigarrow \hat{f}$ allows for a decryption of the response data phase holography encoded by the MRI scanner organization as a FTMRI ciphertext.

Due to the amplitude modulation of the exciting pulse sequences $\psi \in S'(\mathbf{R})$ at resonance frequency $\nu \neq 0$ by the spin relaxation weight $w_{T_1, T_2, \theta}$ the excited proton-weighted spin isochromat density $f \in L^2(\mathbf{R} \oplus \mathbf{R})$ sensitively depends upon the dynamical data

$$T_1, \; T_2, \; \theta$$

of the symplectic relaxation filter

$$f \star_{\nu} \tilde{w}_{T_1, T_2, \theta} \in S'(\mathbf{R} \oplus \mathbf{R})$$

and therefore upon the waiting times T_E and T_R of the pulse sequence ψ which is used for exciting the proton-weighted spin isochromat density f in the symplectic affine plane $\mathcal{O}_\nu \in \mathrm{Lie}(G)^*/\mathrm{CoAd}_G(G)$, and training the learning matrix of the neural network which underlies the phase memory of spin precession at resonance frequency $\nu \neq 0$. Due to the contrast processing by the symplectic relaxation filtering, there exists no universal gray scale in MRI, unlike scans performed by the X-ray CT modality. Therefore, side by side correlations of X-ray CT and MRI scans are particularly helpful with differential diagnoses ([20], [2]).

The dependance of the symplectic relaxation filter from the flip angle θ of the Bloch vector in conjunction with compensation of the phase dispersion by gradient inversion is used in fast low-angle shot (FLASH) protocols or partial flip angle imaging. FLASH plays an important role in magnetic resonance mammorgraphy (MRM), for instance, because it allows to identify even a carcinoma of the breast as small as 3 mm. For the standard symmetric SE pulse sequence which remedies the dephasing by phase conjugation, the amplitude modulation is performed by the spin relaxation weight factor. As a result of the symplectically invariant symbol calculus, the relaxation-weighted amplitude modulation arises

$$e^{-m\frac{T_E}{T_2}}\left(1-e^{-\frac{T_R}{T_1}}\right) f,$$

where $m \geq 1$ denotes the echo number. In the sense of the temporal filter bank design parameters T_E, T_R of spin relaxation, timing is everything in FTMRI.

The idea of filtering is also extremely valuable in three-dimensional MRI processing, specifically in MRA which is clinically used to provide images of vascular structures. From the symplectic character formula it follows that the Plancherel measure μ of G is concentrated on the dual of the line C. It is dual to the downsampled measure

$$\frac{1}{\mathrm{Pf}(\mathcal{X}_\nu^\bullet)} dz$$

on C, and therefore μ implements the affine Larmor frequency scale by

$$\mu = \nu . d\nu .$$

It has been less than a decade since the realization that blood flow could be imaged by MRI techniques. In MRA, tomographic slices

$\mathcal{O}_\nu \in \mathrm{Lie}(G)^*/\mathrm{CoAd}_G(G)$ are integrated along C against μ in order to receive three-dimensional slabs by the Cavalieri fibration principle of the integration theory of homogeneous spaces. Image post-processing techniques such as anisotropic filtering by convolution with a second-difference kernel can be designed to enhance small vessel details and to suppress noise and stationary tissue detail ([16]) so that MRA forms the most sensitive techniques for the visualization of flow phenomena which can replace even the recently developed magnetic transfer contrast techniques ([1]). Advances in filter bank strategies rather than acquisition techniques in conjunction with visualization by the maximum-intensity projection (MIP) algorithm may be the key toward optimization of MRA as a screening technique of intracranial aneurysms or vascular malformations and atherosclerotic disease of the carotid arteries ([57]).

7. Fast MRI techniques. Conventional MRI protocols, specifically conventional symmetric SE protocols, are an intrinsically slow techniques ([3]). It typically takes several minutes to acquire each set of scans so that a complete MRI examination which may require the application of several pulse sequences and assessment by scout - view scans in more than one planar coadjoint orbit $\mathcal{O}_\nu \in \mathrm{Lie}(G)^*/\mathrm{CoAd}_G(G)$ can take as along as an hour to complete. Therefore it was the factor speed which has limited more widespread use and dissemination of MRI in the decade since the clinical MRI modality has been routinely available. Fast readout algorithms are based on symmetry reasonings (Quater FTMRI) and FFT techniques of computerized Fourier analysis. To reduce signal-to-noise, conjugate response data synthesis is applied in the digital evaluation processing of the response data phase holograms encoded by the MRI scanner organization. In addition, correction algorithms for spatially dependent phase shifts are implemented ([38]).

An alternative approach refers to the discrete non-split central group extension

$$\mathbf{Z} \triangleleft \Gamma \longrightarrow \mathbf{Z} \oplus \mathbf{Z}$$

which implements the J-invariant Gaussian lattice

$$\Gamma \hookrightarrow W$$

as stroboscopic lattice of the Keppler phase triangulation procedure by integral transvections. Notice that the stroboscopic lattice Γ immobilizes the rotating coordinate frame and implies the symplectic filtering version of the Poisson summation formula

$$e^{\pi ixy}.\sum_{n\in\mathbf{Z}} e^{2\pi i(z+ny)}\,\psi(n+x) = e^{-\pi ixy}.\sum_{n\in\mathbf{Z}} e^{2\pi i(z-nx)}\overline{\mathcal{F}}_{\mathbf{R}}\psi(n+y)\,.$$

On the right hand side, the Poisson summation formula includes the one-dimensional Fourier cotransform $\overline{\mathcal{F}}_{\mathbf{R}}$ acting on $\psi \in S(\mathbf{R})$. This symplectic filtering formula which governs the relationship between Fourier series and Fourier transforms gives rise to and intertwining operator of the translation of the compact nilmanifold G/Γ and the linear Schrodinger representation U^{ν} of G. In photonics this intertwining operator is termed system transfer function. Its inverse is nothing else than the periodization by Γ of the polarized symbol isomorphism

$$\tilde{U}^{\nu} : S'(\mathbf{R}\oplus\mathbf{R}) \longrightarrow S'(\mathbf{R}\oplus\mathbf{R})\,.$$

Notice that $\tilde{U}^{\nu}$ has been used to make the response data phase holograms, recorded with respect to the rotating coordinate frame attached to the symplectic affine plane $\mathcal{O}_{\nu} \in \mathrm{Lie}(G)^{*}/\mathrm{CoAd}_{G}(G)$, stationary. Thus the Keppler phase triangulation procedure in the rotating coordinate frame finds its natural formulation in the setting of the compact Heisenberg nilmanifold G/Γ. In additions, an application of the theory of currents on W ([50]) to the symplectic character formula projected to G/Γ yields by passing to the quotient mod Γ the Pick area formula ([13]) via the J-invariant Gaussian lattice Γ. The Pick approach to the spatial resolution of FTMRI by the area A under the frequency or readout gradient is given by

$$A = \#\overset{\circ}{A} + \frac{1}{2}(\#\partial A) - 1\,,$$

where $\#\overset{\circ}{A}$ denotes the number of interior sampling points, and $\#\partial A$ the number of ramp sampling points. The proof exploits the connection of the Bohr-Sommerfeld quantization rules with cyclic cohomology ([18]) in the plane W.

The stroboscopic lattice Γ gives rise to the ultra-fast EPI procedure of FTMRI which collects all the response data in a single-shot or snapshot protocol ([39]). The first half of the exciting pulse sequence is essentially identical to a standard clinical symmetric SE protocol in that there is a $\frac{\pi}{2}$ pulse that flips the Bloch vector into the selected planar coadjoint orbit of G followed by a contragredient π pulse performing phase conjugation, and generates a refocused echo wavelet by local frequency fan narrowing at the time T_E. The second half of the exciting pulse sequence is where EPI differs significantly from a standard symmetric SE acquisition. The readout gradient oscillates rapidly from positive to negative amplitude to form a train of gradient echoes. Each echo in the EPI echo train is phase encoded differently by the phase encode blips. Each oscillation of the readout gradient corresponds to one resonance line of image data and each phase encode blip corresponds to a transition from one line to the next one. Thus the most significant different between the symmetric SE acquisition and the EPI acquisition is that the EPI pulse sequence aquires multiple resonance lines of image data during one interval of length T_R.

Although the EPI technique was described fifteen years ago, it has only been recently that EPI methods have been studied and applied in other than only a few research laboratories. The principal reasons for this are that ultra-fast EPI protocols place significant demands on the gradient hardware, the receiver hardware, and the homogeneity of the magnetic flux density. The EPI technique has many similarities to FSE. Both techniques collect multiple resonance lines of image data during each interval of length T_R, both techniques have and associated ETL which forms an approximate measure of the extent to which the data acquisition will be faster than a conventional symmetric SE techniques. There is, however, also a significant different between EPI and FSE protocols in that the EPI echo train does not contain and refocusing pulse. The lack of refocusing pulses in EPI protocols is responsible for the unique capability of EPI for snapshot tomographic imagery. Each line of the response data phase holograms is collected by reversing the readout gradient slope to form another gradient echo. The realization of the ideal Dirac comb kernel associated to the stroboscopic lattice Γ, however, involves the need for rapidly switching high gradient slope systems. Increasing the gradient slew rate in combination with increasing the receiver bandwidth of the image represent significant hardware requirements for performing high quality snapshot EPI protocols. Gradient rise time which is half of the gradient ramp time, and maximum slope must each be improved by at least a factor two over today's standard. EPI protocols that use more than one shot to complete the image acquisition are referred to as multi - shot EPI protocols, and those that use only one shot to acquire the image are referred as single-shot or snapshot tomographic images of the cranium with 0.75 mm in-plane resolution and 3 mm to 5 mm tomographic slice thickness at acquisition rates of 10 - 50 scans per second. At these rates MRI applications enter an entirely new domain such as whole-heart perfusion studies in a single breath hold, and whole-brain functional neuroimaging ([17]). Though not considered to be a safety issue, peripheral nerve stimulation impose limitations on the gradient ramp time. Therefore it is quite understandable that in the last couple of years EPI protocols have arguable become the most talked about of MRI acquisition capabilities. From completing the entire MRI examination in a matter of seconds to expanding the applications realm of FTMRI into radiodiagnostic territories traditionally claimed by other imaging modalities the perceived potential of EPI processing is indeed great ([59]).

The most rapid development of EPI application is in the area of perfusion, diffusion and fMRI of the brain. The significant morbidity associated with neurosurgical procedures that involve the cerebral cortex can result from a disruption of primary sensory and motor cortical areas. Neurosurgeons who attempt to preserve these primary areas during therapeutic operations are handicapped by variation in the normal organization of these areas and by the reorganization that may allow neurological disease, particularly in pediatric neurosurgery. To overcome this difficulty, intraoperative cortical mapping in the awake patient can be used in an attempt to define the extent of a primary area in an individual cerebrum and to allow to the neurosurgeon a decision that balance therapeutic need against potential functional impairment. A routine non-invasive procedure to identify the primary area, such as that afforded by ultra-fast EPI protocols is valuable in clinical management.

In MRI, a highly homogeneous magnetic flux density is required to achieve phase coherence over the desired imaging volume; typically, better than 1 part per million root-mean-square deviation over a 30 cm diameter of spherical volume is accepted. Using a linear magnetic field gradient imposed onto the homogeneous magnetic field density by the linear ramp action of the frequency modulation group $\mathbf{GA}(\mathbf{R})$, and accurate encoding of the in-plane positions of proton-weighted spin isochromat densities is made possible. In addition, the presence of a homogeneous magnetic flux density allows for the excitation of planar tomographic slices of even thickness and accurate flip angle in response to the spin manipulation pulses within the radiofrequency window. A special shielding is required to eliminate interference effects from external radiofrequency sources which distort and degrade the quantum holograms encoded by the MRI scanner organizations. The phase constrained encoding (PACE) technique ([26]) is a techniques of efficiently correcting inhomogeneity of the magnetic flux density induced by geometric distortion. It is particularly appropriate for the correction in open access magnet designs used in interventional MRI applications. Similar to the EPI technique, PACE also relies on the Dirac comb kernel associated to the stroboscopic lattice Γ of the Keppler phase triangulation procedure by integral transvections.

8. Outlook. From the mathematical point of view, the disciplines of optical and digital signal processing overlap significantly. Over the years, however, the notational conventions and methodologies of the two areas have diverged to the point that engineers and scientists in one area often find similar or identical work in the other area nearly unrecognizable. The combination of mathematical physics, electrical engineering, and computerized Fourier analysis applied to FTMRI has had almost as big an impact on the medical discipline of radiodiagnostics as Rontgen's discovery of X-rays one hundred years ago. Because even highly respected clinical investigators of MRI are not familiar with QED and the mathematics underlying this sophisticated phase holographic imaging modality, it is not surprising that the surface of knowledge actually obtainable via MRI techniques has been barely scratched. However, the future of the FTMRI modality which is the most common among the MRI techniques, continues to be bright, with the advent of routine second and subsecond imaging with current delivery of the next generation of MRI scanners. With further improvement in image quality, reduction in scan time and substantial refinements, fMRI of cortical activation will have a major impact on cognitive neuroscience. The endogenous BOLD contrast technique ([44]) to map the functional anatomy of the human brain by means of the phase dispersion due to the spatial change of the local resonance frequency within the tomographic slice, will be useful for acquiring maps of specific eloquent cortical regions such as the visual cortex, the motor strip, and the silent speech area ([17]). BOLD - EPI is a particularly useful data acquisition modality is a particularly useful data acquisition modality for the localization of seizure foci in epileptic disorders.

In the past two years the application of fMRI has exploded. Specifically the use of fMRI for the purposes of interventional medicine is highly promising. The most important interventional applications include image-guided frameless stereotaxy using multiplanar and three-dimensional presentation of MRI data sets, MRI - guided biopsies which provide histological diagnosis of primary breast lesions not visible by X-ray mammorgraphy, MRI - guided endoscopy, and real - time intraoperative imaging during minimally invasive procedures. Nevertheless interventional MRI is still in its infancy. Like MRA, EPI, and MRM it has tremendous clinical potential, but this potential will take time to develop and mature. It is still too early to define its precise clinical range.

9. Summary. MRI has developed into a major scientific field in the area of radiodiagnostic imaging. A fascinating aspect of this powerful clinical imaging modality is that the contrast resolution admits a dynamic implementation. In MRI, the contrast is not fixed in advance such as in other radiodiagnostic imaging modalities but can adapted to the specific needs of a clinical examination. The marked variation of relaxation times over the full spectrum of human tissue is the principal cause of the exceptional image detail seen on MRI scans. Clinicians who initially were pleasantly surprised by the imaging capabilities of MRI are now active participants in the gathering of data in this rapidly expanding field of non-invasive radiodiagnostics. The uniques QED basis of NRM in conjunction with the imaging capabilities of current computer technology has made this imaging modality a target of interest for clinicians, physicists, biologists, engineers, and mathematicians. This basis is relatively new to radiodiagnostic imaging, particularly in comparison to the well-developed, long-standing familiarity with X-ray, ultrasound, and radionuclide imaging techniques. Raymond Damadian, Paul C. Lauterbur, and Peter Mansfield made believers of the previous sceptic ([21]) as MRI scanners were being designed and manufactured at an unprecedented rate ([12], [31], [40]). Even a decade after its first clinical applications appeared in the literature, applications of MRI continuously develop while software and hardware technology evolve. The more the MRI technique is used, the more new applications are discorved, and the higher are requirements of computer performance. Therefore the future of MRI continues to be bright concerning its theoretical and clinical performance, with current delivery of the next generation of MRI scanners providing substantial further improvement in image quality and reduction in scan time.

The main contributions of MRI to medical science have not been discovering new diseases, but rather facilitating and improving the diagnostic accuracy of known disease processes by virtue of its unprecedented soft tissue contrast resolution, tissue characterization, and multiplanar imaging capabilities of pathoanatomy ([45]). In addition, the possibility of non-invasively imaging vascular structures and functional imaging have been the focus of many research programs during the last few years, with dramatic progress ([32]). As a result of the increased contrast resolution, it brought neuroanatomists even further toward understanding the brain's complex architecture in adults and the maturing brain in pediatric subjects ([6]). It made MRI the clinical imaging study of choice for a growing number of diseases. MRI has exerted considerable impact on the practices of neuroradiology, neurology, and neurosurgery. The roads made accessible by clinical MRologists take in the direction of better medical care, the common goal of the medical profession. Geometric analysis has the capability to support this development.

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