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Short Communication

MATRIX TRANSFORMATIONS OF ENTIRE DIRICHLET SERIES[1]

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1. The matrix transformation is one of the methods for summing series and sequences using an infinite matrix. Matrix transformations of power series one complex variable has been studied previously by several authors. Most of papers dealed with Nölund matrices, i.e., triangular matrices of a special form (see e.g., [4, 5]). For the general case of matrices there seem to be very few articles. Recently Borwein and Jakimovski [1] considered matrix transformations of power series in the complex plane C and obtained some results on this direction.

In the present note we study matrix transformations of the class of multiple Dirichlet series with complex frequencies that define entire functions in $\mathbf{C}^n$.

Note that the techniques used in [1] are essentially one-dimensional, of power series and, therefore, do not work for Dirichlet series considered in our article.

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2. We use the following basic notation.

If $z, \varsigma \in \mathbf{C}^n$ then $|z| = (z_1\bar{z}_1 + \cdots + z_n\bar{z}_n)^{1/2}$; $(z, \varsigma) = z_1\varsigma_1 + \cdots + z_n\varsigma_n$.

$\mathcal{O}(\mathbf{C}^n)$ denotes the space of entire functions in $\mathbf{C}^n$ with the compact-open topology, i.e., the topology of uniform convergence on compact subsets of $\mathbf{C}^n$.

Let $(\lambda^k)_{k=1}^{\infty}$, $\lambda^k = (\lambda_1^k, \ldots, \lambda_n^k), k = 1, 2, \ldots$, be a sequence of complex vectors in $\mathbf{C}^n$. We associate to it the following sequence space

$$E_0 = \{c = (c_k);\ |c_k|^{1/|\lambda^k|} \to 0,\ k \to \infty\}.$$

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This is the class of sequences of coefficients of multiple Dirichlet series (with the frequencies $\Lambda = (\lambda^k)$) of the form

$$\sum_{k=1}^{\infty} c_k e^{\langle \lambda^k, z\rangle}, \quad z \in \mathbb{C}^n. \tag{1}$$

Throughout this note we assume that the following condition holds

$$\lim_{k\to\infty} \frac{\log k}{|\lambda^k|} = 0. \tag{2}$$

From [3, Theorem 1] it follows that, with the sequence of coefficients from the space E_0, the Dirichlet series (2.1) converges, or equivalently, converges absolutely, for the topology of the space $\mathcal{O}(\mathbb{C}^n)$ and therefore, represents an entire function in $\mathbb{C}^n$.

We give some properties such as normality, perfectness for the spaces E_0 as well as a description of the Köthe of this space. Here we follow the terminology of [2]. These properties have their own interest and, on the other hand, will be used for further study.

First note that whenever E_0 contains (c_k) it also contains (d_k) with $|d_k| \leq |c_k|$ for $k = 1, 2, \ldots$. So this space is normal.

Denote by E_0^α the Köthe dual of the space E_0, i.e.,

$$E_0^\alpha = \left\{ (u_k);\ \sum_{k=1}^{\infty} c_k u_k \text{ converges absolutely for all } (c_k) \in E_0 \right\}.$$

Also we introduce the following set

$$E_0^c = \left\{ (u_k);\ \sum_{k=1}^{\infty} c_k u_k \text{ converges for all } (c_k) \in E_0 \right\}.$$

We make a characterization of the Köthe dual for the space E_0.

Proposition 1. *If $(d_k) \in E_0^c$, then*

$$\limsup_{k\to\infty} |d_k|^{1/|\lambda^k|} < +\infty. \tag{3}$$

Conversely, if the sequence (d_k) satisfies condition (3) and, in addition, the sequence (λ^k) satisfies condition (2), then $(d_k) \in E_0^\alpha$.

Corollary. *If (2) holds, then $(d_k) \in E_0^c$ if an only if $(d_k) \in E_0^\alpha$, i.e., $E_0^\alpha = E_0^c$. In this case these sequence spaces can be defined as follows*

$$E_0^c = E_0^\alpha = \{(d_k) \ \ \textit{satisfies condition (3)}\}.$$

It is clear that $E_0 \subset E_0^{\alpha\alpha}$. A question arises: when does the inverse inclusion hold? We can prove the following result.

Proposition 2. *Suppose that codition (2) holds. Then the sequence space E_0 is perfect, i.e., $E_0^{\alpha\alpha} = E_0$.*

We now proceed to the main result of our note on matrix transformations of entire Dirichlet series.

3. Denote by $E_0^{\mathcal{U}}$ the class of all matrices $\left(u_{jk}\right)_{j,k=1}^{\infty}$ having the property that whenever the sequence $c = (c_k) \in E_0$ the sequence of functions $\left(f_j(z)\right)_{j=1}^{\infty}$ given by

$$f_j(z) := \sum_{k=1}^{\infty} u_{jk}\, c_k\, e^{\langle \lambda^k, z\rangle}, \ j = 1, 2, \dots,$$

converges uniformly on every compact subset of $\mathbf{C}^n$, each Dirichlet series $\sum_{k=1}^{\infty} u_{jk}\, c_k\, e^{\langle \lambda^k, z\rangle}$ being convergent in $\mathbf{C}^n$, $j = 1, 2, \dots$

Theorem 1. *A matrix $U = (u_{jk})$ belongs to the class $E_0^{\mathcal{U}}$ if and only if the following coditions hold*

$$\exists \lim_{j\to\infty} u_{jk} = u_k, \ k = 1, 2, \dots,$$

and

$$\sup_{j\geq 1, k\geq 1} |u_{jk}|^{1/|\lambda^k|} < +\infty.$$

Proposition 1 is used in the proof of Theorem 1.

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