VIETNAM JOURNAL OF MATHEMATICS
Volume 24, Number 1, 1996

Short Communication

ON STRUCTURED SINGULAR VALUES AND ROBUST STABILITY OF POSITIVE SYSTEMS UNDER AFFINE PERTURBATIONS

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1. The notion of the structured singular value (or μ-values) introduced in [2] is an important linear algebra tool to study robust stability of linear systems under perturbations of block-diagonal structure. Structured block-diagonal perturbations are of great importance in control and form the object of the so-called μ-analysis which studies the properties of function μ, its algebraic characterizations and its computation.

The aim of the present paper is to develop a μ-analysis of n-dimensional positive linear discrete-time systems and examine, in this connection, their robust stability under arbitrary affine parameter perturbations. As one of the main results, it is shown that real and complex stability radii of positive systems coincide for arbitrary perturbation structures, in particular for block-diagonal disturbances as considered in μ-analysis. Estimates and computable formulae are derived for these stability radii. The results are derived for arbitrary perturbation norms induced by monotonic vector norms [5].

2. We first introduce some notations. Let n, ℓ, q be positive integers. A matrix $P = [p_{ij}] \in \mathbf{R}^{\ell\times q}$ is said to be nonnegative ($P \geq 0$) if all its entries p_{ij} are nonnegative; it is said to be *positive* ($P > 0$) if all its entries are positive. For P, $Q \in \mathbf{R}^{\ell\times q}$, $P \geq Q$ means that $P - Q \geq 0$. The set of all nonnegative $\ell \times q$-matrices is denoted by $\mathbf{R}_+^{\ell\times q}$. For any $P = [p_{ij}] \in \mathbf{C}^{\ell\times q}$ we define $|P| \in \mathbf{R}_+^{\ell\times q}$ by $|P| = [|p_{ij}|]$. The following definition extends the definition of Doyle in [2]. Suppose $M \in \mathbf{C}^{q\times\ell}$, I_q denotes the identity $q\times q$-matrix, $\emptyset \neq \mathcal{D} \subset \mathbf{C}^{\ell\times q}$ and span $\mathcal{D}$ is provided with a norm $\|\,.\,\|_{\mathcal{D}}$. Then

$$\mu_{\mathcal{D}}(M) = [\inf\{\|\Delta\|_{\mathcal{D}}\,;\ \Delta \in \mathcal{D},\ \det(I_q - M\Delta) = 0\}]^{-1} \qquad (1)$$

is call the *μ-value* of M with respect to $\mathcal{D}$. If $\mathcal{D} = \mathbf{C}^{\ell\times q}$ and $\|\cdot\|_{\mathcal{D}}$ is the spectral norm then $\mu_{\mathcal{D}}(M)$ is the largest singular value of M. If $q = \ell$, $\mathcal{D} = \mathbf{C}I_q$ and $\|\cdot\|_{\mathcal{D}}$ is an arbitrary operator norm then $\mu_{\mathcal{D}}(M)$ coincides with the spectral radius $\rho(M)$. In general μ-values are difficult to determine, but there exist algorithms for computing upper and lower bounds of $\mu_{\mathcal{D}}(M)$ in the standard case where $\mathcal{D}$ is the set of all complex matrices having a fixed block-diagonal structure and $\|\cdot\|_{\mathcal{D}}$ is the spectral norm, see [6]. Very little is known about the real case (where $\mathcal{D} \subset \mathbf{R}^{\ell\times q}$). We will see that the situation is much easier, if M is nonnegative. For a nonempty subset $\mathcal{D} \subset \mathbf{C}^{\ell\times q}$ let us denote $\mathcal{D}_+ = \mathcal{D} \cap \mathbf{R}_+^{\ell\times q}$ and $\mathcal{D}_{\mathbf{R}} = \mathcal{D} \cap \mathbf{R}^{\ell\times q}$. Then, it is clear by definition that

$$\mu_{\mathcal{D}}(M) \geq \mu_{\mathcal{D}_{\mathbf{R}}}(M) \geq \mu_{\mathcal{D}_+}(M).$$

In general the above μ-values are different. However, for nonnegative matrices we have

Lemma 1. *Suppose $M \in \mathbf{R}_+^{q\times\ell}$, $\mathcal{D} \subset \mathbf{C}^{\ell\times q}$ and $\|\cdot\|_{\mathcal{D}}$ a norm on* span $\mathcal{D}$ *such that*

$$\Delta \in \mathcal{D} \text{ and } \Delta y = u \Rightarrow \exists \tilde{\Delta} \in \mathbf{C}^{\ell\times q} : \tilde{\Delta} y = u \text{ and } |\tilde{\Delta}| \in \mathcal{D} \text{ and } \| |\tilde{\Delta}| \|_{\mathcal{D}} \leq \|\Delta\|_{\mathcal{D}}. \quad (2)$$

If $\mathcal{D}$ is a cone then

$$\mu_{\mathcal{D}}(M) = \mu_{\mathcal{D}_{\mathbf{R}}}(M) = \mu_{\mathcal{D}_+}(M).$$

The proof is based on the Perron-Frobenius Theorem (see, e.g. [1]). We will see in the next section that condition (2) is satisfied for block-diagonal perturbation classes $\mathcal{D}$.

3. Consider a positive dynamical system described by the linear difference equation

$$x(t+1) = Ax(t), \quad t \in \mathbf{N} = \{0, 1, 2, \ldots\} \quad (3)$$

where A is a nonnegative $n \times n$-matrix. We assume that this system is *Schur stable*, i.e. $\rho(A) < 1$ where $\rho(A)$ is the spectral radius of A. Since a dynamical model is never an exact portrait of the real process, it is important to determine to which extent the stability of a given nominal system is preserved under various classes of perturbations. For the class of single perturbations $A \to A + D\Delta E$ this problem was considered in [4], using the state space approach developed first in [3]. In order to extend the results of [4] to more general perturbations classes, we consider arbitrary multiperturbations

$$A \to A + \sum_{i=1}^{N} D_i \Delta_i E_i \tag{4}$$

and arbitrary affine perturbations of A:

$$A \to A + \sum_{i=1}^{N} \delta_i A_i \,, \tag{5}$$

where the matrices D_i, A_i and E_i are given nonnegative matrices defining the structure of the perturbations and Δ_i (δ_i) are unknown matrices (scalars) representing the parameter uncertainty. The assumption that the structural matrices are nonnegative is quite natural for positive systems and is not too restrictive since the disturbances Δ_i, δ_i are not restricted to be nonnegative. It is easy to show that arbitrary affine perturbations of the types (4) and (5) can be represented by the following general uncertainty model

$$A \leadsto A(\Delta) = A + D\Delta E \,, \;\; \Delta \in \mathcal{D} \,, \tag{6}$$

where $D \in \mathbf{R}_+^{n\times \ell}$ and $E \in \mathbf{R}_+^{q\times n}$ are given matrices and $\mathcal{D} \subset \mathbf{C}^{\ell \times q}$ is a given subset of perturbation matrices. The *structure matrices* D, E and the *perturbation class* $\mathcal{D}$ together determine the structure of the perturbations $D\Delta E$. The *stability radius* of the system (3) with respect to the general class of perturbations (6) is defined by

$$r_{\mathcal{D}} = r_{\mathcal{D}}(A;\, D,\, E) = \inf\{\|\Delta\|_{\mathcal{D}};\ \Delta \in \mathcal{D},\ \rho(A + D\Delta E) \geq 1\}\,, \tag{7}$$

where $\|\,.\,\|_{\mathcal{D}}$ is a given norm on span $\mathcal{D}$ and, by definition, $\inf \emptyset = \infty$. In the particular cases when $\mathcal{D} = \mathbf{C}^{\ell\times q}$ (respectively, $\mathbf{R}^{\ell \times q}$ or $\mathbf{R}_+^{\ell\times q}$) the corresponding stability radii will be denoted by $r_{\mathbf{C}}$ (respectively, $r_{\mathbf{R}}$ and $r_{\mathbf{R}_+}$). We shall suppose that the perturbation class $\mathcal{D}$ satisfies the following

Assumption 2. *$\mathcal{D}$ is a block-diagonal perturbation class, i.e. there exist integers $\ell_i \geq 1$, $q_i \geq 1$ for $i \in \underline{N}$ and a subset $J \subset \underline{N}$ such that*

$$\mathcal{D} = \{\mathrm{diag}(\Delta_1, \ldots, \Delta_N);\ \Delta_i \in \mathcal{D}_i,\ i \in \underline{N}\}\,,$$

$$\mathcal{D}_i = \begin{cases} \mathbf{C}^{\ell_i \times q_i} & if\ i \in J \\ \mathbf{C} I_{q_i} & if\ i \in \underline{N} \setminus J \end{cases} \tag{8}$$

The vector spaces $\mathbf{C}^{\ell_i}$, $\mathbf{C}^{q_i}$ are provided with monotonic norms and $\mathcal{D}_i$ with the associated operator norm $\| \cdot \|_{\mathcal{D}_i}$, for each $i \in \underline{N}$. $\mathcal{D}$ is endowed with the norm

$$\|\Delta\|_{\mathcal{D}} = \|(\|\Delta_i\|_{\mathcal{D}_i})_{i\in\underline{N}}\|_{\mathbf{R}^N}, \tag{9}$$

where $\| \cdot \|_{\mathbf{R}^N}$ is a given monotonic norm on $\mathbf{R}^N$.

Let us define

$$\ell = \sum_{i=1}^{N} \ell_i\,,\ q = \sum_{i=1}^{N} q_i\,,\ \mathcal{D}_+ = \mathcal{D} \cap \mathbf{R}_+^{\ell\times q} \text{ and } \mathcal{D}_{\mathbf{R}} = \mathcal{D} \cap \mathbf{R}^{\ell\times q}. \tag{10}$$

The following two propositions are the main results of this section.

Proposition 3. *Suppose $\mathcal{D}$ satisfies Assumption 2, ℓ, q, $\mathcal{D}_+$, $\mathcal{D}_{\mathbf{R}}$ defined as (10) and $M \in \mathbf{R}_+^{q\times\ell}$. Then*

$$\mu_{\mathcal{D}}(M) = \mu_{\mathcal{D}_{\mathbf{R}}}(M) = \mu_{\mathcal{D}_+}(M).$$

This assertion follows from Lemma 1 by showing that, under the Assumption 2, (2) is satisfied.

Proposition 4. *Suppose $A \in \mathbf{R}_+^{n\times n}$ is Schur stable, $\mathcal{D}$ satisfies Assumption 2 and $D_i \in \mathbf{R}_+^{n\times\ell_i}$, $E_i \in \mathbf{R}_+^{q_i\times n}$, $i \in \underline{N}$. Then, with ℓ, q, $\mathcal{D}_+$, $\mathcal{D}_{\mathbf{R}}$ defined as (10),*

$$r_{\mathcal{D}}(A;\, D,\, E) = r_{\mathcal{D}_{\mathbf{R}}}(A;\, D,\, E) = r_{\mathcal{D}_+}(A;\, D,\, E), \tag{11}$$

where the structure matrices D, E are defined by

$$D = [D_1 ... D_N],\ E = [E_1^T ... E_N^T]^T. \tag{12}$$

The crucial point of the proof is the use of Hahn-Banach Theorem for constructing a destabilizing perturbation $\Delta \in \mathcal{D}$ which consists only of rank one blocks Δ_i and the fact that $\|\Delta_i\|_{\mathcal{D}_i} = \|\,|\Delta_i|\,\|_{\mathcal{D}_i}$ for operator norms $\| \cdot \|_{\mathcal{D}_i}$ induced by monotonic vector norms. There are simple examples illustrating that the nonnegativity of both system matrix A and the structure matrices D, E is essential for the validity of the previous propositions.

4. As is well-known, the *transfer matrix* plays an important role in deriving computable formulae for stability radii. For every triplet $(A, D, E) \in \mathbf{C}^{n\times n} \times \mathbf{C}^{n\times\ell} \times \mathbf{C}^{q\times n}$ the associated transfer matrix is defined by $G(s) = E(sI - A)^{-1}D$, $s \in \mathbf{C} \setminus \sigma(A)$, where $\sigma(A)$ is spectrum of A. In the case of single perturbations the stability radii are easily characterized via the transfer matrix. In fact, we have:

Theorem 5. *Suppose that* $(A, D, E) \in \mathbf{R}_+^{n\times n} \times \mathbf{R}_+^{n\times \ell} \times \mathbf{R}_+^{q\times n}$, $\rho(A) < 1$, $\mathbf{C}^\ell$, $\mathbf{C}^q$ *are provided with monotonic norms and* $\mathcal{D} = \mathbf{C}^{\ell\times q}$ *is endowed with the induced operator norm. Then*

$$r_{\mathbf{R}_+}(A; D, E) = r_{\mathbf{C}}(A; D, E) = r_{\mathbf{R}}(A; D, E) = \|G(1)\|^{-1}, \tag{13}$$

where, by definition, $0^{-1} = \infty$.

We note that a result similar to the last equality in (13) has been derived in [7] for nonnegative stable A and *arbitrary real* structure matrices D, E and the spectral perturbation norm. But in [7] the real stability radius is defined in a nonstandard way, namely as the norm of the smallest real perturbation Δ such that $A + D\Delta E$ is unstable *and nonnegative*. By the latter additional condition the admissible parameter perturbations depend on the nominal system which is an awkward assumption. Moreover, the equality of the real and the complex stability radius does not hold under the conditions of [7].

We now return to the block-diagonal perturbation classes $\mathcal{D}$ as described in Assumption 2. Let D, E be defines as in (12). Then, the transfer matrix associated with triplet (A, D, E) is

$$\begin{aligned} G(s) &= E(sI - A)^{-1}D = (G_{ij}(s))_{i,j\in\underline{N}}\,, \\ G_{ij}(s) &= E_i(sI - A)^{-1}D_j,\ i,\, j \in \underline{N}\,. \end{aligned} \tag{14}$$

Theorem 6. *Suppose* $A \in \mathbf{R}_+^{n\times n}$ *is Schur stable,* $D_i \in \mathbf{R}_+^{n\times \ell_i}$, $E_i \in \mathbf{R}_+^{q_i\times n}$, $i \in \underline{N}$ *and* D, E *are defined by (12). If* $\mathcal{D}$ *(8) is a class of block-diagonal perturbations and provided with the operator norm induced by a given pair of vector norms* $\|\cdot\|_{\mathbf{C}^\ell}$, $\|\cdot\|_{\mathbf{C}^q}$ *on* $\mathbf{C}^\ell$ *and* $\mathbf{C}^q$ *(where* ℓ, q *are defined by (10)), then*

$$\Big[\inf_{\alpha>0}\|F(\alpha)\|\Big]^{-1} \le r_{\mathcal{D}_+}(A; D, E)\,, \tag{15}$$

where $F(\alpha) := (\alpha_i G_{ij}(1)\alpha_j^{-1})_{i,j\in\underline{N}}$, $\alpha = (\alpha_1, \dots, \alpha_N) > 0$.

Moreover, if $\ell_i = q_i$ *for all* $i \in \underline{N}$ *and* $\mathbf{C}^q$, $\mathbf{C}^\ell$ *are provided with the same norm then*

$$\Big[\inf_{\alpha>0}\|F(\alpha)\|\Big]^{-1} \le r_{\mathcal{D}_+}(A; D, E) \le [\rho(G(1))]^{-1}\,. \tag{16}$$

Finally if $\mathcal{D}_i = \mathbf{C}I_{q_i}$ *for all* $i \in \underline{N}$, *i.e. the perturbations of A are of the form (5) with* $A_i = D_i E_i$ *then*

$$r_{\mathcal{D}_+}(A;\, D,\, E) = [\rho(G(1))]^{-1}. \tag{17}$$

Although arbitrary affine perturbations of the nominal system matrix can be represented in the form (5) it is more convenient in certain applications (e.g. in control, see[6]) to represent parameter uncertainties by multiperturbations (4). For these disturbance classes Theorem 6 only yields a lower bound (15) (which may be tight) and an upper bound (16) (which will in general not be tight). We conclude the paper by presenting another lower bound, which is less sharp but more easily computable than (15).

Proposition 7. *Suppose* $A \in \mathbf{R}_+^{n\times n}$ *is Schur stable and subjected to perturbations of the form (4) where* $D_i \in \mathbf{R}_+^{n\times \ell_i}$, $E_i \in \mathbf{R}_+^{q_i\times n}$, $i \in \underline{N}$ *are given, Suppose*

$$\mathcal{D} = \{\mathrm{diag}(\Delta_1, ..., \Delta_N)\,;\ \Delta_i \in \mathbf{C}^{\ell_i\times q_i},\ i \in \underline{N}\},$$

is provided with the norm (9) where $\|\cdot\|_{\mathbf{R}^N} = \|\cdot\|_\infty$. *Then (11) hold and*

$$r_{\mathcal{D}_+}(A;\, D,\, E) \geq \Big[\inf_{\alpha>0} \|F(\alpha)\|\Big]^{-1} \geq \big[\rho((\|G_{ij}(1)\|)_{i,j\in\underline{N}})\big]^{-1}. \tag{18}$$

The proof of the above assertion is based on the balancing result due to Stoer and Witzgall [9] which states that for a positive matrix $M \in \mathbf{R}_+^{N\times N}$

$$\min_{\alpha>0} \|\mathrm{diag}(\alpha_i)\, M\, \mathrm{diag}(\alpha_i^{-1})\| = \rho(M)\,, \tag{19}$$

where $\|\cdot\|$ is the operator norm induced by any p-norm on $\mathbf{R}^N$, $1 \leq p \leq \infty$ and the minimum is taken over all $\alpha = (\alpha_1, ..., \alpha_N) > 0$.

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Received December 28, 1995
Revised April 10, 1996

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