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ON MARKOV NETWORKS

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Abstract. *The aim of this paper is to propose an approach based on statistics for studying some Markov networks. As an application, we quote an example related to distributed computing.*

1. MARKOV NETWORKS AND OPTIMAL SOLUTIONS

The stochastic models where optimization intervenes are usual (cf. for example, [1] and [3]) but their use is more interesting when the stochastic dependence is simple. Hence, the interest of Markov dependence.

Let $({}^kX_t)_{t\in IN}$, $k \in \{1, ..., \nu\}$, be ν homogeneous Markov chains with discrete time (IN denoting the set of positive integers), defined on a probability space $(\Omega, \mathcal{A}, \mathcal{P})$, the state space of the k^{th} chain being $({}^k\chi, {}^k\mathcal{B})$. We suppose that all these Markov chains verify the Doeblin condition (cf. [5]), and every one of them has only a single ergodic set without cyclically moving subsets. Let ${}^kP$ be the transition probability of the k^{th} chain and let ${}^kP_t$ be the t^{th} step transition probability, given recursively by

$$\forall x \in {}^k\chi, \ \forall B \in {}^k\mathcal{B}$$

$${}^kP_t(x, B) = \int_{{}^k\chi} {}^kP(x, dy)\, {}^kP_{t-1}(y, B)\,, \quad t \geq 2$$

and let ${}^k\pi$ be the stationary absolute probability given by

$$\lim_{t\to\infty} {}^kP_t(x, B) = {}^k\pi(B)\,.$$

Recall that $\forall B \in {}^k\mathcal{B}$,

$${}^{k}\pi(B) = \int_{{}^k\chi} {}^k\pi(dx)\, {}^kP(x, B)\,, \tag{1}$$

and that the k^{th} chain is said to be in permanent regime if ${}^k\pi$ is taken as initial absolute probability.

Suppose now that $\forall k \in \{1, ..., \nu\}$, ${}^kP$ depends on a parameter $\theta_k \in {}^k\Theta$. The Markov ν chains are said to constitute a network if there exists a finite number of relations $\mathcal{R}_h(\theta_1, ..., \theta_\nu)$, $h \in H$, between the $\theta_1, ..., \theta_\nu$.

The problem we have to examine consists of two stages:

According to the context, to choose a real function f of the $\theta_1, ..., \theta_\nu$ to be optimized; and to fin "solution (s)" $(\theta_1, ..., \theta_\nu)$ satisfying the relations $\mathcal{R}_h(\theta_1, ..., \theta_\nu)$, $h \in H$, which optimize the function $f(\theta_1, ..., \theta_\nu)$.

We propose a way for choosing the suitable function f:

Let $\{{}^kB_i,\ i \in {}^kI\}$ be a finite ${}^k\mathcal{B}$-measurable partition of ${}^k\chi$ and let ${}^k\mathcal{C}$ be the σ-algebra generated by $\{{}^kB_i,\ i \in {}^kI\}$.

Consider now $({}^kX_t(\omega))_{t \in [0,\, n] \cap IN}$ a fragment of the trajectory of the k^{th} chain (the sample) corresponding to the point $\omega \in \Omega$.

For $({}^kC\, {}^kC') \in ({}^k\mathcal{C})^2$, we denote by $n({}^kC \times {}^kC'; \omega)$ the number of direct transitions from ${}^kC$ to ${}^kC'$, and by $n({}^kC \times {}^kC')$ the corresponding random variable. We know (cf. [3]) that the mathematical expectation $E[n({}^kC \times {}^kC')]$ is $n \int_{{}^kC} {}^k\pi(dx, \theta_k)\, {}^kP(x, {}^kC'; \theta_k)$ and that $\frac{1}{n}\, {}^kn({}^kC \times {}^kC')$ is an almost surely consistent estimator (as $n \to \infty$) of $\int_{{}^kC} {}^k\pi(dx, \theta_k)^k P(x, {}^kC'; \theta_k)$.

As all useful information contained in the sample relatively to the partition $\{{}^kB_i,\ i \in {}^kI\}$ is given by $n({}^kC \times {}^kC'; \omega)$, $({}^kC, {}^kC') \in ({}^k\mathcal{C})^2$, f may be chosen as a function of $\frac{1}{n}E[n({}^kC \times {}^kC')]$, $({}^kC, {}^kC') \in ({}^k\mathcal{C})^2$.

More precisely, according to the context of the problem, in order to take advantage of information, we propose to choose a set of characteristic pairs $({}^kC, {}^kC') \in ({}^k\mathcal{C})^2$, namely ${}^k\mathcal{G} = \{({}^kC_1, {}^kC'_1), ..., ({}^kC_r, {}^kC'_r)\}$.

We then define f as function of the parameters $\theta_1, ..., \theta_\nu$:

$$f(\theta_1, ..., \theta_\nu) = \psi\Big(\int_{{}^kC} {}^k\pi(dx, \theta_k)\, {}^kP(x, {}^kC'; \theta_k)\,, \quad ({}^kC, {}^kC') \in {}^k\mathcal{G}\,,\ k \in \{1, ..., \nu\}\Big)\,.$$

f being chosen, we have to find the ν-uples $(\theta_1, ..., \theta_\nu) \in \mathcal{R}_h(\theta_1, ..., \theta_\nu)$, $h \in H$, which optimize f, or in the absence of such optimal solutions, to find solutions which make f as close as possible to its optimal value. For the convenience of the formulation, we can express the optimization in the form of a maximization.

2. APPLICATION TO A DISTRIBUTED COMPUTING PROBLEM

Consider the following network of ν homogeneous finite Markov chains. Its k^{th} Markov chain has ${}^k r$ states, ${}^k\chi = \{1, ..., {}^k r\}$, ${}^k r_1$, ${}^k r_2$, ${}^k r_3$ being integers > 0 such that ${}^k r_1 + {}^k r_2 + {}^k r_3 < {}^k r$. Let us denote

${}^k B_1 = \{1, ..., {}^k r_1\}$,

${}^k B_2 = \{{}^k r_1 + 1, ..., {}^k r_1 + {}^k r_2\}$,

${}^k B_3 = {}^k r_1 + {}^k r_2 + 1, ..., {}^k r_1 + {}^k r_2 + {}^k r_3\}$,

${}^k B_4 = {}^k r_1 + {}^k r_2 + {}^k r_3 + 1, ..., {}^k r\}$.

The entries of the transition matrix ${}^k P = ({}^k p_{ij})$ are described as follows

$\forall i \in {}^k B_1$, $\sum_{j \in {}^k B_1} {}^k p_{ij} = 1 - a_k$ and $\sum_{j \in {}^k B_2} {}^k p_{ij} = a_k$, where $a_k \in]0, 1[$;

$\forall i \in {}^k B_2$, $\sum_{j \in {}^k B_2} {}^k p_{ij} = 1 - b_k$ and $\sum_{j \in {}^k B_3} {}^k p_{ij} = b_k$, where $b_k \in]0, 1[$;

$\forall i \in {}^k B_3$, $\sum_{j \in {}^k B_3} {}^k p_{ij} = 1 - c_k$ and $\sum_{j \in {}^k B_4} {}^k p_{ij} = c_k$, where $c_k \in]0, 1[$;

$\forall i \in {}^k B_4$, $\sum_{j \in {}^k B_1} {}^k p_{ij} = 1$.

For the other (i, j), ${}^k p_{ij} = 0$.

The ν chains are connected into a network by the following relations:

$\forall (i, i') \in {}^k B_1 \times {}^{k+1} B_2$,

$$\sum_{j \in {}^k B_2} {}^k p_{ij} + \sum_{j \in {}^{k+1} B_3} {}^{k+1} p_{i'j} = 1 \text{ for } k \in \{1, ..., \nu - 1\} \qquad (2)$$

and $\forall (i, i') \in {}^\nu B_1 \times {}^1 B_2$,

$$\sum_{j \in {}^\nu B_2} {}^\nu p_{ij} + \sum_{j \in {}^1 B_3} {}^1 p_{i'j} = 1.$$

Proposition 1. *With the partition $\{{}^k B_1, {}^k B_2, {}^k B_3, {}^k B_4\}$ of ${}^k\chi$, the k^{th} Markov chain is lumpable. The lumped chain is an homogeneous Markov chain with four states; its transition matrix ${}^k M = ({}^k m_{su})$ is* *the following:* ${}^k m_{11} = 1 - a_k$, ${}^k m_{12} = a_k$, ${}^k m_{22} = 1 - b_k$, ${}^k m_{23} = b_k$, ${}^k m_{33} = 1 - c_k$, ${}^k m_{34} = c_k$, ${}^k m_{14} = 1$.

The other ${}^k m_{su}$ *are equal to zero.*

The ν *lumped chains are connected into a network by the relations*

$$\begin{cases} a_k + b_{k+1} - 1 = 0, & k \in \{1, ..., \nu - 1\}, \\ a_\nu + b_1 - 1 = 0, \end{cases} \tag{3}$$

the parameters being $\theta_k = (a_k, b_k)$.

Indeed, it is easy to complete the description of the matrix ${}^k P$ by writing: $\forall i \in {}^k B_1$, $\sum_{j \in {}^k B_3} {}^k p_{ij} = 0$, $\sum_{j \in {}^k B_4} {}^k p_{ij} = 0$, and so on. We then see that for every pair $({}^k B_s, {}^k B_u)$, $s, u \in \{1, 2, 3, 4\}$, $\sum_{j \in {}^k B_u} {}^k p_{ij}$, $i \in {}^k B_s$, depends uniquely on s (but does not depend on i individually). The common value of the sums $\sum_{j \in {}^k B_u} {}^k p_{ij}$, $i \in {}^k B_s$, is the ${}^k m_{su}$ of the transition matrix of the k^{th} lumped Markov chain (cf. [6]). Because of (2), the lumped chains are connected by relations (3).

We rediscover then the network of the dining philosophers problem studied in [2].

Proposition 2. *For every* $k \in \{1, ..., \nu\}$ *the initial* k^{th} *Markov chain satisfies the conditions of §1. With the partition* $\{{}^k B_1, {}^k B_2, {}^k B_3, {}^k B_4\}$ *and in permanent regime, we have*

$$E[n({}^k B_1 \times {}^k B_1)] = \frac{n(1 - a_k) b_k c_k}{D_k},$$

$$E[n({}^k B_2 \times {}^k B_2)] = \frac{n a_k (1 - b_k) c_k}{D_k},$$

$$E[n({}^k B_3 \times {}^k B_3)] = \frac{n a_k b_k (1 - c_k)}{D_k},$$

$$E[n({}^k B_1 \times {}^k B_2)] = E[n({}^k B_2 \times {}^k B_3)] =$$
$$E[n({}^k B_3 \times {}^k B_4)] = E[n({}^k B_4 \times {}^k B_1)] = \frac{n a_k b_k c_k}{D_k},$$

where $D_k = a_k b_k + b_k c_k + c_k a_k + a_k b_k c_k$. *The other* $E[n({}^k B_s \times {}^k B_u)]$ *are equal to zero.*

In particular,

$$E[n({}^kB_1 \times {}^k\chi)] = n\frac{b_k c_k}{D_k}\,,\ E[n({}^kB_2 \times {}^k\chi)] = n\frac{a_k c_k}{D_k}\,,$$
$$E[n({}^kB_3 \times {}^k\chi)] = n\frac{a_k b_k}{D_k}\,,\ E[n({}^kB_4 \times {}^k\chi)] = n\frac{a_k b_k c_k}{D_k}\,.$$

Indeed, with the indicated partition of ${}^k\chi$, the k^{th} lumped chain has only one ergodic set without cyclically moving subsets and satisfies the conditions of §1. Let us denote by ${}^k\mu = ({}^k\mu_1, {}^k\mu_2, {}^k\mu_3, {}^k\mu_4)$ its absolute stationary probability. Solving (1), i.e. ${}^k\mu.{}^kM = {}^k\mu$, we have

$${}^k\mu_1 = \frac{b_k c_k}{D_k},\ {}^k\mu_2 = \frac{a_k c_k}{D_k},\ {}^k\mu_3 = \frac{a_k b_k}{D_k},\ {}^k\mu_4 = \frac{a_k b_k c_k}{D_k}.$$

Then,

$$\begin{aligned}E[n({}^kB_s \times {}^kB_u)] &= n \sum_{i\in {}^kB_s} \sum_{i\in {}^kB_u} {}^k\pi_i\,.\,{}^kp_{ij} \\ &= n \sum_{i\in {}^kB_s} {}^k\pi_i\,.\sum_{i\in {}^kB_u} {}^kp_{ij} \\ &= n \sum_{i\in {}^kB_s} {}^k\pi_i\,.\,{}^km_{su} \quad \text{(because of the lumpability)} \\ &= n\,{}^k\mu_s\,.\,{}^k\mu_{su}\,.\end{aligned}$$

In particular, $\forall s \in \{1, 2, 3, 4\}$

$$E[n({}^kB_s \times {}^k\chi)] = n\sum_{u=1}^{4} {}^k\mu_s\,.\,{}^km_{su} = n\,{}^k\mu_s\,.$$

Let us examine now the problem of choosing a suitable and workable function f following our method indicated in §1. Let us recall that, because of its context (exposed in [2]), one "privileges" the access to state 3 of the k^{th} lumped chain, i.e. to the set ${}^kB_3$ of the k^{th} initial chain. Thus, we take

$${}^k\mathcal{G} = \{{}^kB_3 \times {}^k\chi\}.$$

For every k, we suggest maximizing $\frac{1}{n}E[n({}^kB_3 \times {}^k\chi)]$, viz. minimizing $\frac{n}{E[n({}^kB_3 \times {}^k\chi)]}$, so that, globally, under constrains (3), we minimize

$$\sum_{k=1}^{\nu} \frac{n}{E[n({}^kB_3 \times {}^k\chi)]} = \sum_{k=1}^{\nu} \frac{D_k}{a_k b_k}\,.$$

This leads to the same result as [2], found by another approach. Let the c_k's be fixed, and consider the θ_k's ($\theta_k = (a_k, b_k)$) as tuning parameters. f, as indicated in §1, is here the concave function

$$f[(a_1, b_1), ..., (a_\nu, b_\nu)] = -\sum_{k=1}^{\nu} \left[(1 + c_k) + \frac{c_k}{a_k} + \frac{c_k}{b_k}\right].$$

The Lagrange multipliers method used by [2] for lumped chains proves that there exists only one optimal solution, which is

$$((a_1, b_1), ..., (a_k, b_k), ..., (a_\nu, b_\nu))$$
$$= (\frac{1}{1+\rho_1}, \frac{\rho_\nu}{1+\rho_\nu}), \dots, (\frac{1}{1+\rho_k}, \frac{\rho_{k-1}}{1+\rho_{k-1}}), \dots, (\frac{1}{1+\rho_\nu}, \frac{\rho_{\nu-1}}{1+\rho_{\nu-1}}),$$

where $\rho_k = \sqrt{\frac{c_{k+1}}{c_k}}$, for $k \in \{1, ..., \nu - 1\}$ and $\rho_\nu = \sqrt{\frac{c_1}{c_\nu}}$.

We then infer the following result:

Suppose that the ${}^k p_{ij}$, $(i, j) \in [{}^k B_3 \times ({}^k B_3 \cup {}^k B_4)] \cup ({}^k B_4 \times {}^k B_1)$, be fixed, and consequently, so are the c_k's and suppose that the ${}^k p_{ij}$, $(i, j) \in [{}^k B_1 \times ({}^k B_1 \cup {}^k B_2)] \cup [{}^k B_2 \times ({}^k B_2 \cup {}^k B_3)]$, be tuning parameters. Then

Proposition 3. *The initial Markov network has the following optimal solutions:*

* *For the first chain,*
  the ${}^1 p_{ij}$, $(i, j) \in [{}^1 B_1 \times ({}^1 B_1 \cup {}^1 B_2)]$, *are such that*

$$\forall i \in {}^1 B_1, \quad \sum_{j \in {}^1 B_1} {}^1 p_{ij} = \frac{\rho_1}{1+\rho_1} \text{ and } \sum_{j \in {}^1 B_2} {}^1 p_{ij} = \frac{1}{1+\rho_1};$$

  the ${}^1 p_{ij}$, $(i, j) \in [{}^1 B_2 \times ({}^1 B_2 \cup {}^1 B_3)]$, *are such that*

$$\forall i \in {}^1 B_2, \quad \sum_{j \in {}^1 B_2} {}^1 p_{ij} = \frac{1}{1+\rho_\nu} \text{ and } \sum_{j \in {}^1 B_3} {}^1 p_{ij} = \frac{\rho_\nu}{1+\rho_\nu}.$$

* *For the* k^{th} *chain,* $k \in \{2, ..., \nu\}$,
  the ${}^k p_{ij}$, $(i, j) \in [{}^1 B_1 \times ({}^1 B_1 \cup {}^1 B_2)]$, *are such that*

$$\forall i \in {}^kB_1, \sum_{j\in {}^kB_1} {}^kp_{ij} = \frac{\rho_k}{1+\rho_k} \text{ and } \sum_{j\in {}^kB_2} {}^kp_{ij} = \frac{1}{1+\rho_k};$$

the ${}^kp_{ij}$, $(i, j) \in [{}^kB_2 \times ({}^kB_2 \cup {}^kB_3)]$, *are such that*

$$\forall i \in {}^kB_2, \sum_{j\in {}^kB_2} {}^kp_{ij} = \frac{1}{1+\rho_{k-1}} \text{ and } \sum_{j\in {}^kB_3} {}^kp_{ij} = \frac{\rho_{k-1}}{1+\rho_{k-1}}.$$

Among these optimal solutions is the following particular one:

- For the first chain

$$\forall (i, j) \in ({}^kB_1 \times {}^kB_1),\ {}^1p_{ij} = \frac{\rho_1}{{}^1r_1(1+\rho_1)};$$
$$\forall (i, j) \in ({}^kB_1 \times {}^kB_2),\ {}^1p_{ij} = \frac{1}{{}^1r_2(1+\rho_1)};$$
$$\forall (i, j) \in ({}^kB_2 \times {}^kB_2),\ {}^1p_{ij} = \frac{1}{{}^1r_2(1+\rho_\nu)};$$
$$\forall (i, j) \in ({}^kB_2 \times {}^kB_3),\ {}^1p_{ij} = \frac{\rho_\nu}{{}^1r_3(1+\rho_\nu)}.$$

- For the k^{th} chain, $k \in \{2, ..., \nu\}$,

$$\forall (i, j) \in ({}^kB_1 \times {}^kB_1),\ {}^kp_{ij} = \frac{\rho_k}{{}^kr_1(1+\rho_k)};$$
$$\forall (i, j) \in ({}^kB_1 \times {}^kB_2),\ {}^kp_{ij} = \frac{1}{{}^kr_2(1+\rho_k)};$$
$$\forall (i, j) \in ({}^kB_2 \times {}^kB_2),\ {}^kp_{ij} = \frac{1}{{}^kr_2(1+\rho_{k-1})};$$
$$\forall (i, j) \in ({}^kB_2 \times {}^kB_3),\ {}^kp_{ij} = \frac{\rho_{k-1}}{{}^kr_3(1+\rho_{k-1})}.$$

REFERENCES

1. C. Andrieu, *Sur certaines solutions fiables d'un problème stochastique de recherche optimale*, Operationsforsch. Stat., Ser. Optimization, **12** (1) (1981), 115 - 122.
2. M. Bui, *Réglage pour un fonctionnement optimal d'un réseau de processeurs en algorithmique distribuée*, Revue Roum. Math. Pures et App., **35** (3) (1990), 197 - 202.
3. Bui Trong Lieu, *Estimations dans les processus de Markov*, Publ. Inst. Stat. Univ. Paris XI, **2** (1962), 73 - 188.

4. Bui Trong Lieu, *Modèles stochastiques de redistribution et certain aspect optimal*, Optimization, **16** (1) (1985), 93 - 108.
5. J. L. Doob, *Stochastic processes*, Wiley, 1959.
6. J. G. Kemeny and J. L. Snell, *Finite Markov Chains*, Springer, 1983.

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