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WEAK COMPACTNESS AND CONVERGENCES IN BOCHNER AND PETTIS INTEGRATION

CHARLES CASTAING

INTRODUCTION

In this paper we give a brief survey of some recent weak compactness results in the theory of integration as well as some convergence results which arise from Mathematical Economics, Probability and Variational Analysis. Let $(\Omega, \mathcal{F}, \mu)$ be a complete probability space, E a Banach space, and $L^1_E(\mu)$ the Banach space of Bochner integrable functions equipped with its usual norm. In section 1, new characterizations of conditionally weakly compact (c.w.c.) (resp. relativcely weakly compact) (r.w.c.) subsets in Banach spaces via a class of regular method of summability (RMS) $a = (a_{pq})$ (cf. [24], [26]) are presented. A subset $K \subset E$ is c.w.c. (resp. r.w.c.) iff for any sequence (x_n) in K, there exists a subsequence (x_{n_k}) such that the sequence (s_k) with $s_k = \sum_{q=0}^{\infty} a_{k_q} x_{n_q}$ $(k \in N)$ is well-defined and weakly Cauchy (resp. weakly convergent). This characterization is equivalent to the following: for any sequence (x_n) in K, there exists a sequence $(\tilde{x}_n)$ with $\tilde{x}_n \in \text{co}\{x_m : m \geq n\}$, such that $(\tilde{x}_n)$ is weakly Cauchy (resp. weakly convergent). Also several criteria for c.w.c. and r.w.c. subsets in $L^1_E(\mu)$ are presented. In particular a bounded uniformly integrable and ball-conditionally weakly compact-tight subset in $L^1_E(\mu)$ is c.w.c. In section 2 new sequential weak compactness criteria for convex weakly compact valued scalarly integrable multifunctions are discussed. In section 3 we state some sequential weak compactness results for Pettis integrable functions with application to the existence of best approximations the space of Pettis integrable functions. In section 4 using a vector valued version of Komlos theorem due to Garling, we present a new version of Komlos - Slice convergence for integrable convex weakly compact valued multifunctions and also Banach-Saks property for bounded sequences in $L^1_H(\Omega, \mathcal{F}, \mu)$ where H is a separable Hilbert space.

This paper also contains several types of convergence in $L^1_E(\mu)$ with applications to Mathematical Economics and Minimization problems.

Most of our proofs are detailed and easy, except for some of them which rely on deeper results due to Garling [29], Rosenthal ([43], [44]) and Talagrand [47].

NOTATIONS AND PRELIMINARIES

We will use the following notions and notations. We denote by

– $(\Omega, \mathcal{F}, \mu)$ a complete probability space.

– E a separable Banach space,

– E' the topological dual of E, E'_s (resp. E'_τ) (resp. E'_b) the vector space E' equipped with the $\sigma(E', E)$ (resp. Mackey) (resp. norm) topology.

– $\overline{B}_E$ (resp. $\overline{B}_{E'}$) is the closed ball of center 0 and radius 1 in E (resp. E').

– $cwk(E)$ the collection of non empty convex weakly compact subsets of E.

– $\mathcal{R}_{wk}(E)$ (resp. $\mathcal{R}_{cwc}(E)$) the collection of borelian subsets of E such that its intersection with any ball of E is relatively weakly (conditionally weakly) compact.

– $\delta^*(., A)$ is support function of a subset A of E.

– $L^1_E(\mu)$ is the space of Bochner integrable mappings $u : \Omega \to E$ and $L^\infty_{E'_s}(\mu)$ is the topological dual of $L^1_E(\mu)$ (cf. A. and C. Ionescu Tulcea [34]).

– $P^1_E(\mu)$ is the space of Pettis functions $u : \Omega \to E$.

– If X is a topological space, $\mathcal{B}(X)$ is the Borel tribe of X.

– A multifunction $\Gamma : \Omega \to \mathcal{B}(X)$ is measurable if its graph $Gr(\Gamma)$ belongs to $\mathcal{F} \otimes \mathcal{B}(X)$.

– $\mathcal{P}^1_{cwk(E)}(\mu)$ is the space of all scalarly-integrable multifunctions.

– $\mathcal{L}^1_{cwk(E)}(\mu)$ is the space of all scalarly integrable multifunctions $\Gamma : \Omega \to cwk(E)$ such that the scalar function $|\Gamma| : \omega \to \sup\{\|x\| : x \in \Gamma(\omega)\}$ is integrable, such a Γ is said to be *integrably bounded*.

– A subset of $\mathcal{H}$ of $\mathcal{L}^1_{cwk(E)}(\mu)$ is *bounded* if the set $\{|X| : X \in \mathcal{H}\}$

is bounded in $L^1_{\mathbf{R}}(\mu)$; $\mathcal{H}$ is *uniformly integrable* if the set $\{|X| : X \in \mathcal{H}\}$ is uniformly integrable in $L^1_{\mathbf{R}}(\mu)$.

– A multifunction $M : \mathcal{F} \to cwk(E)$ is a *multimeasure* if, for any $x' \in E'$, $\delta^*(x', M(.))$ is a scalar measure; M is a *multimeasure of bounded variation* if there exists a finite positive measure ν defined on $\mathcal{F}$ such that $M(A) \subset \nu(A)\overline{B}_E$ for all $A \in \mathcal{F}$.

– A mapping $m : \mathcal{F} \to E$ is a *selection measure* of a multimeasure $M : \mathcal{F} \to cwk(E)$ if m is a vector measure satisfying $m(A) \in M(A)$ for all $A \in \mathcal{F}$.

– A mapping $l : L^{\infty}_{E'_b}(\mu) \to \mathbf{R}$ is said to be *additive* if for any pair (f_1, f_2) in $L^{\infty}_{E'_b}(\mu)$ with disjoint supports, $l(f_1 + f_2) = l(f_1) + l(f_2)$; l is said to be *absolutely continuous* if it admits the integral representation

$$\forall u \in L^{\infty}_{E'_b}(\mu),\ l(u) = \int_{\Omega} \delta^*(u(\omega), X(\omega))\, \mu(d\omega)$$

with $X \in \mathcal{L}^1_{cwk(E)}(\mu)$; such a X is said to be the *density* of l.

– $Si(\mathbf{N})$ is the set of strictly increasing mapping from $\mathbf{N}$ to $\mathbf{N}$.

– A subset $\mathcal{H}$ of $L^1_E(\mu)$ is $\mathcal{R}_{wk}(E)$ (resp. $\mathcal{R}_{cwc}(E)$) *tight* if for every $\varepsilon > 0$, there exists a measurable multifunction $\Gamma_\varepsilon : \Omega \to \mathcal{R}_{wk}(E)$ (resp. $\mathcal{R}_{cwc}(E)$) such that $\forall u \in \mathcal{H},\ \mu[\{\omega \in \Omega : u(\omega) \notin \Gamma_\varepsilon(\omega)\}] < \varepsilon$.

– If (x_n) is a sequence in E, $w - Ls\{x_n\}$ is defined by

$$w - Ls\{x_n\} := \bigcap_{n=1}^{\infty} \overline{\{x_k : k \geq n\}}^{\sigma}$$

where $\overline{\{.\}}^{\sigma}$ denotes the closure for the $\sigma(E, E')$ topology.

1. WEAK COMPACTNESS AND CONDITIONALLY WEAK COMPACTNESS IN BANACH SPACES AND IN L^1_E

The material of this section is borrowed from Benabdellah-Castaing ([6], [7]). We aim to present some recent compactness results which are based on the results of Rosenthal ([43], [44]) and Talagrand [47].

An infinite matrix $(a_{pq})_{(p,q)\in\mathbf{N}\times\mathbf{N}}$ is called a *regular method of summability* (RMS) if

$$\sup_{p\in\mathbf{N}} \sum_{q=0}^{\infty} |a_{pq}| < +\infty\,, \tag{1.1}$$

$$\forall q \in \mathbf{N}, \ \lim_{p\to\infty} a_{pq} = 0, \tag{1.2}$$

$$\lim_{p\to\infty} \sum_{q=0}^{\infty} a_{pq} = 1. \tag{1.3}$$

It is easy to check that $a = (a_{pq})$ is a RMS iff for any sequence (x_n) in a Banach space E, converging to $x \in E$, then the sequence (x'_n) given by $x'_n = \sum_{q=0}^{\infty} a_{nq}x_q$, converges to x. A sequence (x_n) in a Banach space is called *summable* with respect to a RMS $a = (a_{pq})$ if the sequence (x'_n) given by $x'_n = \sum_{q=0}^{\infty} a_{nq}x_q$ is well-defined and converges for the norm of E. An RMS $a = (a_{pq})$ is positive if, $\forall p, \ q, \ a_{pq} \geq 0$.

Let us mention first an easy lemma before we state the main results.

Lemma 1.1. *Let (a_{pq}) be a positive RMS and let (x_n) be a sequence in* $\mathbf{R}$ *such that the series $\sum_{q=0}^{\infty} a_{pq}x_q$ are convergent. Then we have*

(1) $\liminf_{p\to\infty} x_p \leq \liminf_{p\to\infty} \sum_{q=0}^{\infty} a_{pq}x_q$.

In particular, if $\tilde{x}_n \in co\{x_k : k \geq n\}$, $\forall n$, then we have

(2) $\liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} \tilde{x}_n$.

Proof. (1) let $(x_n) \in \mathbf{R}$ such that the series $u_p := \sum_{q=0}^{\infty} a_{pq}x_q$ are convergent in $\mathbf{R}$. Let $r < \liminf_{n\to\infty} x_n = \sup_n \inf_{k\geq n} x_k$.

Then there exists a positive integer n_0 such that $k \geq n_0$ implies $r < x_k$. Hence $\forall q \geq n_0$, $a_{pq}r \leq a_{pq}x_q$. Therefore $(\sum_{q=n_0}^{\infty} a_{pq})r \leq \sum_{q=n_0}^{\infty} a_{pq}x_q$. Consequently we get

(*) $(\sum_{q=0}^{\infty} a_{pq})r - (\sum_{q=0}^{n_0-1} a_{pq})r \leq u_p - \sum_{q=0}^{n_0-1} a_{pq}x_q$.

Since $\lim_{p\to\infty} \sum_{q=0}^{\infty} a_{pq} = 1$, and $\lim_{p\to\infty} \sum_{q=0}^{n_0-1} a_{pq} = 0$ by virtue of properties (1.2) and (1.3) of the RMS and since $\lim_{p\to\infty} \sum_{q=0}^{n_0-1} a_{pq}x_q = 0$, then by taking the lim inf in (∗), we obtain

$$r \leq \liminf_{p\to\infty}\Big(u_p - \sum_{q=0}^{n_0-1} a_{pq}x_q\Big) = \liminf_{p\to\infty} u_p$$

It follows that $\liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} \sum_{q=0}^{\infty} a_{nq}x_q$.

(2) is easy consequence of (1).

Now we are able to produce the main results of this section.

Theorem 1.2. *Let K be a subset of a Banach space E and let $a = (a_{pq})$ be a positive RMS. Then the following are equivalent:*

(1) *K is conditionally weakly compact.*

(2) *Given any sequence $(x_n)_n \subset K$, there exists a subsequence $(x_{n_k})_k$ such that the sequence $(s_k)_k$ with $s_k = \sum_{q=0}^{\infty} a_{kq}x_{n_q}$ $(k \in \mathbf{N})$ is well-defined and weakly Cauchy.*

(3) *Given any sequence $(x_n)_n \subset K$, there exists a sequence $(\tilde{x}_n)_n$ with $\tilde{x}_n \in \mathrm{co}\{x_m : m \geq n\}$ such that $(\tilde{x}_n)$ is weakly Cauchy.*

Proof. The implication (1) → (2) follows easily from properties of the RMS. Let us prove (2) ⇒ (3).

Since K satisfies (2), K is bounded. Indeed it is enough to check that, $\forall x' \in E'$, we have $\delta^*(x', K) = \sup_{x\in K}\langle x', x\rangle < +\infty$. Take a sequence $(u_n) \subset K$ such that $\lim_{n\to\infty}\langle x', u_n\rangle = \delta^*(x', K)$. By (2), there exists a subsequence $(u_{n_k})_k$ of $(u_n)_n$ such that the sequence $(v_p)_p$ with $v_p := \sum_{q=0}^{\infty} a_{pq}u_{n_q}$ is well-defined and weakly Cauchy. Hence the sequence $(\langle x', v_p\rangle)_p$ with $\langle x', v_p\rangle = \sum_{q=0}^{\infty} a_{pq}\langle x', u_{n_q}\rangle$ converges in $\mathbf{R}$ to a point $v_{x'}$. Clearly by obvious properties of the RMS, we have

$$\delta^*(x', K) = \lim_{p\to\infty}\langle x', u_{n_p}\rangle = \liminf_{p\to\infty}\sum_{q=0}^{\infty} a_{pq}\langle x', u_{n_q}\rangle = v_{x'} < +\infty\,.$$

Now set $M := \sup\{\|x\| : x \in K\}$ and let us prove that K satisfies (3).

Let $(x_n) \subset K$ and let $s_k = \sum_{k=0}^{\infty} a_{kq}x_{n_q}$ given by (2). For each $x' \in E'$, let $r_{x'} = \lim_{k\to\infty}\langle x', s_k\rangle$. According to properties (1.1) and (1.2) of the RMS, it is easy to construct two strictly increasing sequences of positive integers (N_p) and (p_k) such that

$$\forall p,\ \forall k \geq 1,\ \sum_{q>N_p} a_{pq} \leq 2^{-p} \text{ and } \sum_{q=0}^{k-1} a_{p_k q} \leq 2^{-k} \tag{1.2.1}$$

For every $k \geq 1$, set $\lambda_k := \sum_{q=k}^{N_{p_k}} a_{p_k q}$. Then by (1.2.1), we obtain

$$0 \leq \sum_{q=0}^{\infty} a_{p_k q} - \lambda_k \leq 2^{-k} + 2^{-p_k}$$

Consequently by property (1.3) of the RMS, we deduce that $\lim_{k\to\infty} \lambda_k = 1$. Set

$$\forall k,\ \lambda_q^k := \frac{1}{\lambda_k} a_{p_k q} \text{ and } \tilde{x}_k := \sum_{q=k}^{N_{p_k}} \lambda_q^k x_{n_q}$$

Then it is clear that $\tilde{x}_k \in \operatorname{co}\{x_{n_q} : q \geq k\}$. Moreover, for every k, we have

$$\begin{aligned}
&|\langle x', \tilde{x}_k\rangle - s_{x'}| \\
&= \Big|\frac{1}{\lambda_k}\big[\langle x', s_{p_k}\rangle - \langle x', \sum_{q=0}^{k-1} a_{p_k q} x_{n_q} + \sum_{q>N_{p_k}} a_{p_k q} x_{n_q}\rangle\big] - s_{x'}\Big| \\
&\leq \Big|\frac{1}{\lambda_k}\langle x', s_{p_k}\rangle - s_{x'}\Big| + \frac{M\|x'\|}{\lambda_k}(2^{-k} + 2^{-pk})
\end{aligned}$$

Hence it follows that $\lim_{k\to\infty}\langle x', \tilde{x}_k\rangle = s_{x'}$. Whence $(\tilde{x}_k)$ is weakly Cauchy and satisfies $\tilde{x}_k \in \operatorname{co}\{x_m : m \geq k\}$, $\forall k$.

Now it remains to prove (3) $\Rightarrow$ (1). By using Lemma 1.1, we can show similarly as in the previous implication that K is bounded. Assume by contradiction that K is not conditionally weakly compact. Then according to a result of H.P. Rosenthal [43], there exist $r \in \mathbf{R}$, $\delta > 0$ and a sequence $(x_n)_n \subset K$ such that the sequence $(A_n, B_n)_{n\in\mathrm{N}}$ defined by

$$A_n = \{x' \in \overline{B}_{E'} : \langle x', x_n\rangle \geq r + \delta\} \text{ and } B_n = \{x' \in \overline{B}_{E'} : \langle x', x_n\rangle \leq r\}$$

is independent. By (3), there exists $\tilde{x}_n \in \operatorname{co}\{x_m : m \geq n\}$ $(n \in \mathrm{N})$ such that $(\tilde{x}_n)$ is weakly Cauchy. Each $\tilde{x}_n$ has the form $\tilde{x}_n = \sum_{i=n}^{m_n} \lambda_i^n x_i$

with $\lambda_i^n \geq 0$, $\sum_{i=n}^{m_n} \lambda_i^n = 1$, $m_n > n$. Let $n_0 = 0$, $n_1 = m0 + 1, \ldots, n_{k+1} = m_{n_k} + 1$. Then (n_k) is a strictly increasing sequence such that for all $i \neq j$, $[n_i, m_{n_i}] \cap [n_j, m_{n_j}] = \emptyset$.

Now let us consider the following sets

$$\widetilde{A}_k := \bigcap_{i=n_k}^{m_{n_k}} A_i \text{ and } \widetilde{B}_k := \bigcap_{i=n_k}^{m_{n_k}} B_i$$

Then $(\widetilde{A}_k, \widetilde{B}_k)$ is a sequence of disjoint pairs of subsets in $\overline{B}_{E'}$ and is independent. Indeed, let I and J be two finite, nonempty, disjoint subsets of $\mathbf{N}$. Then we have

$$(\bigcap_{k\in I} \widetilde{A}_k) \cap (\bigcap_{k\in J} \widetilde{B}_k) = (\bigcap_{i\in I'} A_i) \cap (\bigcap_{i\in J'} B_i) \tag{1.2.2}$$

where $I' := \bigcup_{k\in I} [n_k, m_{n_k}]$ and $J' := \bigcup_{k\in J} [n_k, m_{n_k}]$ are disjoint. Consequently, the intersection in (1.2.2) is nonempty. On the other hand, for every k, we have

$$x' \in \widetilde{A}_k \Rightarrow \langle x', \tilde{x}_{n_k} \rangle = \sum_{i=n_k}^{m_{n_k}} \lambda_i^{n_k} \langle x', x_i \rangle \geq \sum_{i=n_k}^{m_{n_k}} \lambda_i^{n_k} (r + \delta) = r + \delta$$

and $x' \in \widetilde{B}_k \Rightarrow \langle x', \tilde{x}_{n_k} \rangle \leq \sum_{i=n_k}^{m_{n_k}} \lambda_i^{n_k} r = r.$

By invoking again Rosenthal [43], we conclude that $(\tilde{x}_{n_k})$ is equivalent to the unit vector basis of l^1. This contradicts the fact that $(\tilde{x}_n)$ is weakly Cauchy, thus proving the implication (3) $\Rightarrow$ (1).

Here is an analogous criterion for relative weakly compact subset in a Banach space where equivalence (1) $\Leftrightarrow$ (3) was stated by Ülger [48] and Diestel-Ruess-Schachermayer [23] by different methods.

Theorem 1.3. *Let K be a subset of a Banach space E and let $a = (a_{pq})$ be a positive RMS. Then the following are equivalent:*

(1) *K is relatively weakly compact.*

(2) *Given any sequence $(x_n)_n$ in K, there exists a subsequence $(x_{n_k})_k$ such that the sequence $(s_k)_k$ with $s_k = \sum_{q=0}^{\infty} a_{kq} x_{n_q}$, $k \in \mathbf{N}$) is well-defined and weakly convergent.*

(3) *Given any sequence* $(x_n)_n$ *in* K, *there exists a sequence* $(\tilde{x}_n)$ *with* $\tilde{x}_n \in \mathrm{co}\{x_m : m \geq n\}$ *such that* $(\tilde{x}_n)$ *is weakly convergent.*

(4) *Given any sequence* $(x_n)_n$ *in* K *there exists* y *such that,* $\forall x' \in E'$,

$$\liminf_{n\to\infty} \langle x', x_n \rangle \leq \langle x', y \rangle .$$

Proof. The proof of implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) follows from the arguments we used in the proof of Theorem 1.2.

(3) $\Rightarrow$ (4) is an immediate consequence of Lemma 1.1 applied to the sequences $(\langle x', x_n \rangle)$ and $(\langle x', \tilde{x}_n \rangle)$.

(4) $\Rightarrow$ (1) follows from a classical characterization of relatively sequentially weakly compact subset in normed spaces (see e.g. Holmes [32] §18.A).

Remark. It would be interesting to address the following question: what happens if one replace "weakly relatively compactness" by "norm relatively compactness" in the statement of Theorem 1.3.

The following example shows that, in general, the statement of Theorem 1.3 is not true if one replace "weakly" by "norm". Let $E = c_0$ and let $K = \{e_n : n \in \mathbf{N}\}$ be the unit vector basis of c_0. Then K is not relatively compact for the norm topology since for $n \neq m$, $\|e_n - e_m\|_\infty = 1$, although K satisfies the following property: given any sequence $(x_n)_n \subset K$, there exists a sequence $(\tilde{x}_n)$ with $\tilde{x}_n \in \mathrm{co}\{x_m : m \geq n\}$ $(n \in \mathbf{N})$ such that $(\tilde{x}_n)$ converges for the norm topology. Indeed set $X = \{x_n : n \in \mathbf{N}\}$. If X is finite, there exists $m \in \mathbf{N}$ and a subsequence $(x_{n_k})_k$ of $(x_n)_n$ such that, $\forall k$, $x_{n_k} = e_m$, so that in this case, we can take, $\forall k$, $\tilde{x}_k = x_{n_k} = e_m$. If X is infinite, there exist two subsequences $(x_{p_k})_k$ and $(e_{q_k})_k$ of $(x_n)_n$ and (e_n) respectively such that, $\forall k$, $x_{p_k} = e_{q_k}$. Set $\tilde{x}_k = \frac{1}{k+1}\sum_{i=k}^{2k} e_{q_i}$, $\forall k$, then $\tilde{x}_k \in \mathrm{co}\{e_{q_i} : i \geq k\} \subset \mathrm{co}\{x_n : n \geq k\}$ and $(\tilde{x}_k)_k$ converges to 0 for the norm topology.

Now we are able to present weak compactness criteria and convergence results in $L^1_E(\mu)$.

We begin by recalling a celebrated result due to Talagrand ([47], Theorem 1).

Theorem 1.4. *Let* (u_n) *be a bounded sequence in* $L^1_E(\mu)$. *Then there exists a sequence* $(\tilde{u}_n)$ *with* $\tilde{u}_n \in \mathrm{co}\{u_m : m \geq n\}$ *and two sets* A *and* B *in* $\mathcal{F}$ *with* $\mu(A \cup B) = 1$ *such that*

(a) for each ω *in* A, *the sequence* $(\tilde{u}_n(\omega))$ *is weakly Cauchy,*

(b) for each ω in B, there exists and integer k such that the sequence $(\tilde{u}_n(\omega))_{n\geq k}$ is equivalent to the vector unit basis of l^1.

Remark. Although the thesis is more general than the one given in [47], Theorem 1), in which (u_n) is bounded in $L^\infty_E(\mu)$, Theorem 1.4 is an easy consequence of Theorem 1 in ([47]). In the same vein, Diestel-Ruess-Schachermayer obtained a refinement of Talagrand's theorem by another method (see [23], Lemma 2.5). Indeed let $v_n = \|u_n(.)\|$, $\forall n$. Then (v_n) is a bounded sequence in $L^1_{\mathbf{R}}(\mathcal{F})$. By ([15], Theorem 3.1 and Remarks, p. 60-61), there is a sequence $(\tilde{v}_n)$ with $\tilde{v}_n \in \mathrm{co}\{v_m : m \geq n\}$ such that $(\tilde{v}_n)$ converges almost everywhere to some $v \in L^1_{\mathbf{R}}(\mathcal{F})$. Each $\tilde{v}_n$ has the form $\tilde{v}_n = \sum\limits_{k=n}^{\nu_n} \lambda^n_k v_k$ with $0 \leq \lambda^n_k \leq 1$ and $\sum\limits_{k=n}^{\nu_n} \lambda^n_k = 1$. Extracting a subsequence if necessary and modifying the v_k, $k \in \mathbf{N}$, on a negligible set we may suppose that $(\tilde{v}_n(\omega))_n$ converges to $v(\omega)$ for all $\omega \in \Omega$. Set

$$\forall \omega \in \Omega,\ M(\omega) := 1 + \sup_n \tilde{v}_n(\omega) \quad \text{and}$$

$$h_n(\omega) := \frac{1}{M(\omega)} \sum_{k=n}^{\nu_n} \lambda^n_k u_k(\omega)\,.$$

Then we can apply Talagrand's theorem to (h_n). There is a sequence $(\tilde{h}_n)$ with $\tilde{h}_n \in \mathrm{co}\{h_m : m \geq n\}$ which satisfies conditions (a) and (b) of Theorem 1.4. Now it is enough to set $\tilde{u}_n(\omega) = M(\omega)\tilde{h}_n(\omega)$, $\forall \omega \in \Omega$.

Now we state our first result which is a direct application of Theorem 1.2. and Talagrand's results [47].

Theorem 1.5. *Let $\mathcal{H}$ be a bounded subset of $L^1_E(\mu)$. Then the following are equivalent:*

(1) $\mathcal{H}$ is conditionally weakly compact.

(2) $\mathcal{H}$ is uniformly integrable and given any sequence $(f_n) \subset \mathcal{H}$, there exists a sequence $(\tilde{f}_n)$ with $\tilde{f}_n \in \mathrm{co}\{f_m : m \geq n\}$ such that $(\tilde{f}_n(\omega))_n$ is weakly Cauchy in E for a.e. $\omega \in \Omega$.

Proof. Let us prove (1) $\Rightarrow$ (2). It is well-known that conditionally weakly compact subsets of $L^1_E(\mu)$ are uniformly integrable (see e.g. [21]). Now let (f_n) be any sequence in $\mathcal{H}$. Then by Theorem 1.4, there exists a sequence $(\tilde{f}_n)$, with $\tilde{f}_n \in \mathrm{co}\{f_m : m \geq n\}$, and two sets, A, B in $\mathcal{F}$ with $\mu(A \cup B) = 1$, such that

(a) for each ω in A, $(\tilde{f}_n(\omega))_n$ is weakly Cauchy in E,

(b) for each ω in B, there exists an integer k, such that the sequence $(\tilde{f}_n(\omega))_{n\geq k}$ is equivalent to the vector unit basis of l^1.

Suppose that the measure of subset B of Ω is strictly positive. Then by Talagrand ([47]), Lemma 4), there exists k such that the sequence $(\tilde{f}_n)_{n\geq k}$ is equivalent to the vector unit basis of l^1. But this contradicts the fact that (f_n) is c.w.c since $(\tilde{f}_n)$ lies in the set co($\mathcal{H}$), which is c.w.c. (see [44] or [12] Theorem 5.E). Therefore $\mu(B) = 0$, and for a.e. $w \in \Omega$, the sequence $(\tilde{f}_n(\omega))$ is weakly Cauchy.

Let us prove now (2) $\Rightarrow$ (1). By Theorem 1.2, it is enough to check that given $(f_n) \subset \mathcal{H}$ and $(\tilde{f}_n)$ as in (2), the sequence $(\tilde{f}_n)$ is weakly Cauchy in $L^1_E(\mu)$. Let $g \in L^\infty_{E'}[E]$. Since $(\tilde{f}_n(\omega))_n$ is weakly Cauchy in E for a.e. $\omega \in \Omega$, the sequence $(\langle g(\omega), \tilde{f}_n(\omega)\rangle)_n$ converges a.e. Let

$$\varphi(\omega) := \lim_{n\to\infty} \langle g(\omega), \tilde{f}_n(\omega)\rangle \text{ for } \omega \notin N,$$

where N is a negligible set and $\varphi(\omega) = 0$ for $\omega \in N$. Then by Fatou's lemma, $\varphi \in L^1_{\mathbf{R}}(\mu)$ and since $(\langle g, \tilde{f}_n\rangle)_n$ is uniformly integrable, by Vitali's theorem, we get

$$\lim_{n\to\infty} \int_\Omega \langle g, \tilde{f}_n\rangle d\mu = \int_\Omega \varphi d\mu$$

thus proving that (2) $\Rightarrow$ (1).

Concerning r.w.c. subsets in $L^1_E(\mu)$ we recall the following which is essentially due to Ülger [48] and relies on the equivalence (1) $\Leftrightarrow$ (3) in Theorem 1.3.

Theorem 1.6. (Ülger-Diestel-Ruess-Schachermayer [23]). *Let E be a Banach space and $\mathcal{H}$ be a subset of $L^1_E(\mu)$. Then the following are equivalent:*

(1) *$\mathcal{H}$ is relatively weakly compact.*

(2) *$\mathcal{H}$ is uniformly integrable and given any sequence (u_n) in $\mathcal{H}$, there is a sequence $(\tilde{u}_n)$ with $\tilde{u}_n = \text{co}\{u_m : m \geq n\}$, $\forall n$, such that $(\tilde{u}_n(\omega))$ is weakly convergent in E for almost all $\omega \in \Omega$.*

The following result is mentioned in Diestel ([22], p. 237). We provide the proof here for the sake of completeness.

Proposition 1.7. *Let E be an arbitrary Banach space, K a nonempty subset of E. Then the following are equivalent:*

(1) *K is conditionally weakly compact.*

(2) *For every $\varepsilon > 0$, there exists a conditionally weakly compact set K_ε such that*

$$K \subset K_\varepsilon + \varepsilon \overline{B}_E .$$

Proof. (1) $\Rightarrow$ (2) being obvious, let us prove (2) $\Rightarrow$ (1). Let (ε_p) be a decreasing sequence of strictly positive numbers with $\lim_{p\to\infty} \varepsilon_p = 0$, and (K_p) be a sequence of conditionally weakly compact subsets in E such that

$$\forall p,\ K \subset K_p + \varepsilon_p \overline{B}_E \tag{1.7.1}$$

We have to show that, given any sequence $(x_n) \subset K$, there exists a weakly Cauchy subsequence. By (1.7.1), for every n, there exists $y_p^n \in K_p$ such that $\|x_n - y_n^p \leq \varepsilon_p$.

Since each K_p is c.w.c., the sequence $(y_p^n)_n$ admits a weakly Cauchy subsequence. Then by induction we find a sequence (φ_n) in $Si(\mathrm{N})$ such that

$$\forall p,\ (y^p_{\varphi_0 \circ \ldots \circ \varphi_p(n)})_n \text{ is weakly Cauchy in } E . \tag{1.7.2}$$

Let us consider the diagonal sequence $\psi(n) := \varphi_0 \circ \ldots \circ \varphi_n(n)$, $\forall n$, and let us prove that $(x_{\psi(n)})$ is weakly Cauchy. Let $\varepsilon > 0$ be fixed. Choose p such that $\varepsilon_p < \frac{\varepsilon}{4}$. Then for any $x' \in \overline{B}_{E'}$, and for $m > k > p$, we have

$$\begin{aligned} |\langle x', x_{\psi(m)} - x_{\psi(k)}\rangle| &\leq |\langle x', x_{\psi(m)} - y^p_{\psi(m)}\rangle| + |\langle x', x_{\psi(k)} - y^p_{\psi(k)}\rangle| \\ &\quad + |\langle x', y^p_{\psi(m)} - y^p_{\psi(k)}\rangle| \\ &\leq 2\varepsilon_p + |\langle x', y^p_{\psi(m)} - y^p_{\psi(k)}\rangle| \end{aligned}$$

Since by (1.7.2), $(y^p_{\varphi_0 \circ \ldots \circ \varphi_p(n)})_n$ is weakly Cauchy, so is $(y^p_{\psi(n)})_n$. Therefore $\lim_{\substack{m\to\infty \\ k\to\infty}} \langle x', y^p_{\psi(m)} - y^p_{\psi(k)}\rangle = 0$. Hence there exists $p_\varepsilon > p$ such that $m > k > p_\varepsilon$ implies $|\langle x', x_{\psi(m)} - x_{\psi(k)}\rangle| \leq 2.\frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon$, proving that $(x_{\psi(n)})_n$ is weakly Cauchy.

We need a couple of notions which are inspired by ([1] and [47]) before we state our c.w.c. criteria in $L^1_E(\mu)$. Let us recall that $\mathcal{R}_{cwc}(E)$ (resp. $\mathcal{R}_{wk}(E)$) is the class of subsets $K \in \mathcal{B}(E)$ such that, their intersection with any ball is c.w.c (rep. r.w.c) in E. An element $K \in \mathcal{R}_{cwc}(E)$ (resp. $\mathcal{R}_{wk}(E)$) is called ball-c.w.c. (resp. ball-r.w.c). It is clear that $\mathcal{R}_{cwc}(E)$ and $\mathcal{R}_{wk}(E)$ are stable under finite unions and that they contains the empty set $\emptyset$.

A subset $\mathcal{H} \subset L^1_E(\mu)$ is called $\mathcal{R}_{cwc}(E)$-tight (reps. $\mathcal{R}_{wk}(E)$-tight) if, for every $\varepsilon > 0$, there exists a measurable multifunction L_ε from Ω into $\mathcal{R}_{cwc}(E)$ (resp. $\mathcal{R}_{wk}(E)$) such that

$$\forall u \in \mathcal{H},\ \mu[\{\omega \in \Omega : u(\omega) \notin L_\varepsilon(\omega)\}] < \varepsilon$$

A subset $\mathcal{H} \subset L^1_E(\mu)$ has the *conditionally weak Talagrand property*, shortly, conditionally WTP, (resp. weak *Talagrand property*, shortly, WTP) if, for any sequence $(f_n) \subset \mathcal{H}$, there exists a sequence (g_n) with $g_n \in \mathrm{co}\{f_m : m \geq n\}$, $\forall n$, such that, for a.e. $\omega \in \Omega$, $(g_n(\omega))_n$ is weakly Cauchy (resp. weakly convergent) in E.

There is a folklore Lemma which characterizes the above tightness notion.

Lemma 1.8. *Let E be a separable Banach space. Let $\mathcal{R}$ be a class of borelian subsets of E such that: $\emptyset \in \mathcal{R}$, $A, B \in \mathcal{R} \Rightarrow A \cup B \in \mathcal{R}$. Let $\mathcal{H}$ be a subset of $L^1_E(\mu)$. Then the following are equivalent:*

(1) *For any $\varepsilon > 0$, there exists a measurable multifunction $L_\varepsilon : \Omega \to \mathcal{R}$ such that*

$$\forall u \in \mathcal{H},\ \mu[\{\omega \in \Omega : u(\omega) \notin L_\varepsilon(\omega)\}] < \varepsilon\,.$$

(2) *There exists a $\mathcal{F} \otimes \mathcal{B}(E)$-measurable integrand $\varphi : \Omega \times E \to [0, +\infty]$ such that for all $\omega \in \Omega$ and all $r \geq 0$, $\{x \in E : \varphi(\omega, x) \leq r\} \in \mathcal{R}$ and that*

$$\sup_{u \in \mathcal{H}} \int_\Omega \varphi(\omega, u(\omega))\mu(d\omega) < +\infty\,.$$

(3) *There exists a $\mathcal{F} \otimes \mathcal{B}(E)$-measurable integrand $\varphi : \Omega \times E \to [0, +\infty]$ such that for all $\omega \in \Omega$ and all $r \in \mathbf{R}^+$, $\{x \in E : \varphi(\omega, x) \leq r\} \in \mathcal{R}$ and that*

$$\lim_{\lambda \to +\infty} \sup_{u \in \mathcal{H}} \mu[\{\omega \in \Omega : \varphi(\omega, u(\omega)) \geq \lambda\}] = 0\,.$$

Proof. (1) $\Rightarrow$ (2). Let $\varepsilon_p = 3^{-p}$ ($p \in \mathbf{N}$). By (1) there exists a measurable multifunction $L_p : \Omega \to \mathcal{R}$ such that

$$\forall u \in \mathcal{H},\ \mu[\{\omega \in \Omega : u(\omega) \notin L_p(\omega)\}] < \varepsilon_p$$

Let us consider the multifunctions $K_n : \Omega \to \mathcal{B}(E)$ $(n \in \mathbf{N} \cup \{\infty\})$ given by:

$$\forall \omega \in \Omega,\ K_0(\omega) = L_0(\omega),\ K_n(\omega) = L_n(\omega) \setminus K_{n-1}(\omega),\ \forall n \geq 1$$

and $$K_\infty(\omega) = E \setminus \bigcup_{n\in\mathbf{N}} K_n(\omega) = E \setminus \bigcup_{n\in\mathbf{N}} L_n(\omega).$$

Then it is obvious that each K_n $(n \in \mathbf{N} \cup \{\infty\})$ is measurable and the sequence $(Gr(K_n))_{n\in\mathbf{N}\cup\{\infty\}}$ is a $\mathcal{F} \otimes \mathcal{B}(E)$-measurable partition of $\Omega \times E$. Set

$$\varphi(\omega, x) = \begin{cases} 2^n & \text{if } (\omega, x) \in Gr(K_n),\ n \in \mathbf{N} \\ +\infty & \text{if } (\omega, x) \in Gr(K_\infty) \end{cases}$$

We claim that φ is $\mathcal{F} \otimes \mathcal{B}(E)$-measurable integrand which satisfies condition (2). Indeed, let $r \geq 0$. If $r < 1$, $\{x \in E : \varphi(\omega, x) \leq r\}$ is empty; if $r \geq 1$, let m be the unique integer such that $m \leq \frac{\log r}{\log 2} < m + 1$. Then

$$\{(\omega, x) \in \Omega \times E : \varphi(\omega, x) \leq r\} = \bigcup_{n=0}^{m} Gr(K_n) \in \mathcal{F} \otimes \mathcal{B}(E)$$

Similarly for all $\omega \in \Omega$, we have

$$\{x \in E : \varphi(\omega, x) \leq r\} = \bigcup_{n=0}^{m} K_n(\omega) = \bigcup_{n=0}^{m} L_n(\omega) \in \mathcal{R}.$$

It remains to check that $\sup_{u\in\mathcal{H}} \int_\Omega \varphi(\omega, u(\omega))\mu(d\omega) < +\infty$.

For each $u \in \mathcal{H}$ and each $n \in \mathbf{N} \cup \{\infty\}$, set

$$\Omega_n^u = \{\omega \in \Omega : u(\omega) \in K_n(\omega)\}$$

Then $(\Omega_n^u)_{n\in\mathbf{N}\cup\{\infty\}}$ is a $\mathcal{F}$-measurable partition of Ω with $\mu(\Omega_n^u) < \varepsilon_{n-1}$, $\forall n \in \mathbf{N}^*$ and $\mu(\Omega_\infty^u) = 0$. Consequently we have

$$\int_\Omega \varphi(\omega, u(\omega))\mu(d\omega) = \sum_{n=0}^{\infty} \int_{\Omega_n^u} \varphi(\omega, u(\omega))\mu(d\omega) = \sum_{n=0}^{\infty} 2^n \mu(\Omega_n^u)$$

$$\leq 1 + \sum_{n=1}^{\infty} \frac{2^n}{3^{n-1}} < +\infty$$

thus proving the implication (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3) follows immediately from Tchebyschev's inequality. Let us prove (3) $\Rightarrow$ (1). For every $\varepsilon > 0$, there exists $\lambda_\varepsilon > 0$ such that $\sup_{u\in\mathcal{H}} \mu[\{\omega \in \Omega : \varphi(\omega, u(\omega)) > \lambda_\varepsilon\}] < \varepsilon$. Since ε is $\mathcal{F} \otimes \mathcal{B}(E)$-measurable, the multifunction $L_\varepsilon(\omega) := \{x \in E : \varphi(\omega, x) \le \lambda_\varepsilon\}$, $\forall \omega \in \Omega$, is measurable and takes it values in $\mathcal{R}$ by (3). Since we have

$$\forall u \in \mathcal{H},\ \mu[\{\omega \in \Omega : u(\omega) \notin L_\varepsilon(\omega)\}] = \mu[\{\omega \in \Omega : \varphi(\omega, u(\omega)) > \lambda_\varepsilon\}] < \varepsilon$$

(3) $\Rightarrow$ (1) is proved.

Now we are able to present our second conditionally weakly compact criterion in $L^1_E(\mu)$.

Theorem 1.9. *Let E be a separable Banach space. Assume that $\mathcal{H}$ is uniformly integrable and $\mathcal{R}_{cwc}(E)$-tight subset of $L^1_E(\mu)$. Then $\mathcal{H}$ is conditionally weakly compact in $L^1_E(\mu)$.*

Proof. Let $\varepsilon > 0$ be fixed. Since $\mathcal{H}$ is uniformly integrable, there exists $\delta > 0$ and $\alpha > 0$ such that

$$\sup_{u\in\mathcal{H}} \int_{[|u|>\alpha]} |u|\,d\mu < \frac{\varepsilon}{2}$$

and

$$\forall B \in \mathcal{F},\ \mu(B) \le \delta \Rightarrow \sup_{u\in\mathcal{H}} \int_B |u|\,d\mu < \frac{\varepsilon}{2}$$

By our assumption there exists a measurable multifunction $L_\delta : \Omega \to \mathcal{R}_{cwc}(E)$ such that

$$\forall u \in \mathcal{H},\ \mu[\{\omega \in \Omega : u(\omega) \notin L_\delta(\omega)\}] < \delta$$

For each $u \in \mathcal{H}$, set

$$A_u = [|u| \le \alpha],\ \ B_u = \{\omega \in \Omega : u(\omega) \in L_\delta(\omega)\}$$

Then we have

$$u = 1_{A_u \cap B_u} u + 1_{A_u^c \cap B_u} u + 1_{B_u^c} u$$

and

$$\|1_{A_u^c \cap B_u} u + 1_{B_u^c} u\|_1 \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$

Set $\mathcal{H}_\varepsilon = \{1_{A_u \cap B_u} u : u \in \mathcal{H}\}$. Then it is obvious that

$$\mathcal{H} \subset \mathcal{H}_\varepsilon + \varepsilon \overline{B}_{L^1_E(\mu)}\,.$$

Now we claim that $\mathcal{H}_\varepsilon$ is conditionally weakly compact in $L^1_E(\mu)$. Let $(u_n)_n \subset \mathcal{H}$ and $v_n := 1_{A_{u_n} \cap B_{u_n}} u_n$, $\forall n$. Then

$$v_n(\omega) \in (L_\delta(\omega) \cup \{0\}) \cap \eta \overline{B}_E$$

for all $\omega \in \Omega$. Moreover $G_\delta(\omega) := (L_\delta(\omega) \cup \{0\} \cap \eta \overline{B}_E$ is conditionally weakly compact in E because $L_\delta(\omega) \in \mathcal{R}_{cwc}(E)$. By Talagrand's theorem ([47], Theorem 1), there exist $A \in \mathcal{F}$ and a sequence $(\tilde{v}_n)$ with $\tilde{v}_n \in \mathrm{co}\{v_m : m \geq n\}$, $\forall n$, such that

(a) $\forall \omega \in A$, $(\tilde{v}_n(\omega))_n$ is weakly Cauchy in E

(b) for a.e. $\omega \in A^c$, there exists k such that $(\tilde{v}_n(\omega))_{n \geq k}$ is equivalent to the unit vector basis of l^1.

Now, $\forall \omega \in \Omega$, $\tilde{v}_n(\omega) \in \mathrm{co} G_\delta(\omega)$ and $\mathrm{co}(G_\delta(\omega))$ is conditionally weakly compact (see [44], or [12] Theorem 5.E). Hence $\mu(A^c) = 0$. So we conclude that $(\tilde{v}_n(\omega))_n$ is weakly Cauchy for a.e. $\omega \in \Omega$. By virtue of Theorem 1.5.,$\mathcal{H}_\varepsilon$ is conditionally weakly compact in $L^1_E(\mu)$. Since $\mathcal{H} \subset \mathcal{H}_\varepsilon + \varepsilon \overline{B}_{L^1_E(\mu)}$, then by Proposition 1.7, $\mathcal{H}$ is conditionally weakly compact too, thus completing the proof.

Remark. Theorem 1.9 is a slight refinement of some results obtained by Pisier [42] and Bourgain [11].

Similarly we have the following criterion for relatively weakly compact subsets of $L^1_E(\mu)$ (see [1], Theorem 6, p. 174 for proof).

Theorem 1.10. *Let E be a separable Banach space. Let $\mathcal{H}$ be a uniformly integrable and $\mathcal{R}_{wk}(E)$-tight subset of $L^1_E(\mu)$. Then $\mathcal{H}$ is relatively weakly compact in $L^1_E(\mu)$.*

The following result provides the connections between "tightness notions" and "Talagrand's properties".

Theorem 1.11. *Let E be a separable Banach space. If $\mathcal{H}$ is a bounded $\mathcal{R}_{cwc}(E)$ (resp. $\mathcal{R}_{wk}(E)$)-tight subset of $L^1_E(\mu)$, then $\mathcal{H}$ is conditionally WTP (resp. WTP) in $L^1_E(\mu)$.*

Proof. We have only to prove the thesis for the $\mathcal{R}_{cwc}(E)$-tight case, since the proof of $\mathcal{R}_{wk}(E)$-tight case is similar by invoking Theorem 1.10.

Let $(u_n) \subset \mathcal{H}$. By Biting lemma ([28], [45]) there exists an increasing sequence (A_k) in $\mathcal{F}$ with $\lim_{k\to\infty} \mu(A_k) = 1$ and a subsequence (u_{n_k}) such that $(1_{A_k} u_{n_k})$ is uniformly integrable in $L^1_E(\mu)$, and that $(1_{A_k^c} u_{n_k})$ converges to 0 a.e. Set $\mathcal{K} = \{1_{A_k} u_{n_k} : k \in \mathbf{N}\}$.

We claim that $\mathcal{K}$ is $\mathcal{R}_{cwc}(E)$-tight. Let $\varepsilon > 0$. By our assumption, there exists a measurable multifunction $L_\varepsilon : \Omega \to \mathcal{R}_{cwc}(E)$ such that

$$\forall u \in \mathcal{H},\ \mu[\{\omega \in \Omega : u(\omega) \notin L_\varepsilon(\omega)\}] < \varepsilon .$$

Set $G_\varepsilon(\omega) := L_\varepsilon(\omega) \cup \{0\}$, $\forall \omega \in \Omega$. Then G_ε is a measurable multifunction from Ω to $\mathcal{R}_{cwc}(E)$ such that

$$\forall k \in \mathbf{N},\ \mu[\{\omega \in \Omega : (1_{A_k} u_{n_k}(\omega) \notin G_\varepsilon(\omega)\}]$$

$$= \mu[\{\omega \in A_k : u_{n_k}(\omega) \notin L_\varepsilon(\omega)\}] < \varepsilon .$$

Hence $\mathcal{K}$ is $\mathcal{R}_{cwc}(E)$-tight as desired. Since $\mathcal{K}$ is uniformly integrable, by Theorem 1.9, $\mathcal{K}$ is c.w.c. in $L^1_E(\mu)$. By virtue of Theorem 1.2, there exists a sequence (v_p) with $v_p \in \mathrm{co}\{1_{A_k} u_{n_k} : k \geq p\}$, $\forall p$, such that, for a.e. $\omega \in \Omega$, $(v_p(\omega))_p$ is weakly Cauchy in E. Each v_p has form $v_p = \sum_{k=p}^{\nu_p} \lambda^p_k 1_{A_k} u_{n_k}$, with $\lambda^p_k \geq 0$, $\sum_{k=p}^{\nu_p} \lambda^p_k = 1$. Set $\tilde{u}_p = \sum_{k=p}^{\nu_p} \lambda^p_k u_{n_k}$, $\forall p$.

Then $\tilde{u}_p = v_p + w_p$, where $w_p := \sum_{k=p}^{\nu_p} \lambda^p_k 1_{A^c_k} u_{n_k}$ with $w_p \to 0$ a.e. since $1_{A^c_k} u_{n_k} \xrightarrow[k]{} 0$ a.e. We deduce that for a.e. $\omega \in \Omega$, the sequence $(\tilde{u}_p(\omega))$ is weakly Cauchy in E, thereby proving the Theorem.

Theorem 1.12. *Let $\mathcal{H}$ be a bounded subset of $L^1_E(\mu)$. Then the following are equivalent:*

(1) $\mathcal{H}$ has the weak Talagrand property (WTP).

(2) Given any sequence (u_n) in $\mathcal{H}$, there are an increasing sequence (A_k) in $\mathcal{F}$ with $\lim_{k\to\infty} \mu(A_k) = 1$ and a subsequence (u_{n_k}) of (u_n) such that $(1_{A_k} u_{n_k})_k$ is relatively weakly compact in $L^1_E(\mu)$ and that $(1_{A^c_k} u_{n_k})_k$ converges a.e. to zero.

(3) Given any sequence (u_n) in $\mathcal{H}$, there exists a sequence $(\tilde{u}_n)$ with $\tilde{u}_n \in \mathrm{co}\{u_m : m \geq n\}$, $\forall n$, and $u_\infty \in L^1_E(\mu)$ such that $(\tilde{u}_n)$ converges a.e. to u_∞ for the norm topology of E.

Proof. (1) $\Rightarrow$ (2). By Biting lemma ([28, [45]) there exist an increasing sequence (A_k) in $\mathcal{F}$ with $\lim_{k\to\infty} \mu(A_k) = 1$ and a subsequence (u_{n_k}) of (u_n) such that $(1_{A_k} u_{n_k})_k$ is uniformly integrable in $L^1_E(\mu)$ and that $(1_{A^c_k} u_{n_k})_k$ converges to zero a.e. Now we claim that the set $\mathcal{K} := \{1_{A_k} u_{n_k} : k \in \mathbf{N}\}$ has the (WTP). Indeed, by (1) there exists a subsequence $(u_{n_{k_p}})$ of (u_{n_k}) and a sequence (v_p) with $v_p \in \mathrm{co}\{u_{n_{k_j}} : j \geq p\}$,

$\forall p$, such that for a.e. $\omega \in \Omega$, $(v_p(\omega))_p$ converges weakly to $v(\omega)$ in E. Each v_p has form $v_p = \sum_{j=p}^{\nu_p} \lambda_j^p u_{n_{k_j}}$ with $\lambda_j^p \geq 0$ and $\sum_{j=p}^{\nu_p} \lambda_j^p = 1$. Set $w_p = \sum_{j=p}^{\nu_p} \lambda_j^p 1_{A_{k_j}} u_{n_{k_j}}$, $\forall p$. Then it is easily seen that $w_p(\omega) \to v(\omega)$ weakly a.e. in E. As $w_p \in \text{co}\{1_{A_{k_j}} u_{n_{k_j}} : j \geq p\}$, $\forall p$, $\mathcal{K}$ has the (WTP). Since $\mathcal{K}$ is uniformly integrable, by Ülger-Diestel-Ruess-Schachermayer Theorem (Theorem 1.6), we conclude that $\mathcal{K}$ is r.w.c. in $L^1_E(\mu)$.

(2) $\Rightarrow$ (3). Let (A_k) and (u_{n_k}) according to (2). By Mazur's lemma, we may assume, by extracting a subsequence if necessary, that there exists a sequence (v_k) with $v_k \in \text{co}\{1_{A_m} u_{n_m} : m \geq k\}$, $\forall k$, such that $(v_k)_k$ converges a.e. to an element $v_\infty \in L^1_E(\mu)$. Each v_k has the form $v_k = \sum_{j=k}^{\nu_k} \lambda_j^k 1_{A_j} u_{n_j}$, with $0 \leq \lambda_j^k \leq 1$, $\sum_{j=k}^{\nu_k} \lambda_j^k = 1$. Let $\tilde{u}_k = \sum_{j=k}^{\nu_k} \lambda_j^k u_{n_j}$, $\forall k$. Then $(\tilde{u}_k)$ has the desired properties. (3) $\Rightarrow$ (1) is obvious.

Corollary 1.13. *Let K be a convex bounded WTP set in $L^1_E(\mu)$ which is closed for the topology of the convergence in measure. Let $J : K \to [0, +\infty[$ be a convex lower semicontinuous function for the topology of convergence in measure. Then J reaches its minimum on K.*

The preceding corollary generalizes a result due to Levin [38]. (see [15] for details an references).

Let us mention the following consequence of Theorem 1.12.

Proposition 1.14. *Let $\mathcal{H}$ be a bounded WTP set in $L^1_E(\mu)$. then the following are equivalent:*

(1) *$\forall v \in L^\infty_{E'_s}(\mu)$, $\{\langle v(\cdot), u(\cdot)\rangle : u \in \mathcal{H}\}$ is uniformly integrable in $L^1_{\mathbf{R}}(\mu)$.*

(2) *$\mathcal{H}$ is relatively weakly compact in $L^1_E(\mu)$.*

Proof. (2) $\Rightarrow$ (1) being obvious, it is enough to prove that (1) $\Rightarrow$ (2). We may suppose that E is separable. Let (u_n) be a sequence in $\mathcal{H}$. By Theorem 1.12, there are $u_\infty \in L^1_E(\mu)$ and a sequence $(\tilde{u}_n)$ with $\tilde{u}_n \in \text{co}\{u_m : m \geq n\}$, such that $(\tilde{u}_n)$ converges a.e. to u_∞ for the norm topology of E. By (1), $\forall v \in L^\infty_{E'_s}(\mu)$, the sequence $(\langle v, \tilde{u}_n\rangle)_n$ is uniformly integrable, then by Vitali's theorem $\lim_{n\to\infty} \int_\Omega \langle v, \tilde{u}_n\rangle d\mu =$

$\int_\Omega \langle v, u_\infty \rangle d\mu$. By virtue of Theorem 1.3, we conclude that $\mathcal{H}$ is relatively weakly compact in $L^1_E(\mu)$.

Now we present some nice properties of bounded WTP sequence in $L^1_E(\mu)$.

Theorem 1.15. *Let (u_n) be a bounded WTP sequences in $L^1_E(\mu)$. Then the following properties hold:*

(a) There exist an increasing sequence (A_p) in $\mathcal{F}$ with $\lim_p \mu(A_p) = 1$, a subsequence (u_{n_k}) of (u_n), a sequence $(\tilde{u}_n)$ with $\tilde{u}_n \in \mathrm{co}\{u_{n_k} : k \geq n\}$ and $u_\infty \in L^1_E(\mu)$ such that, $\forall p$, $(u_{n_k|A_p})_k\, \sigma(L^1, L^\infty)$ converges to $u_\infty|A_p$ and that $\tilde{u}_n(\omega))$ converges in norm to $u_\infty(\omega)$ for a.e. $\omega \in \Omega$.

(b) If (v_n) is a bounded sequence in $L^\infty_{E'_s}(\mu)$ converging in measure to $v_\infty \in L^\infty_{E'_s}(\mu)$ for the norm topology of the strong dual of E and if the sequence $(\langle v_n, u_n \rangle^-)_n$ is uniformly integrable in $L^1_{\mathbf{R}}(\mu)$, then we have

$$\liminf_{n\to\infty} \int_\Omega \langle v_n, u_n \rangle d\mu \geq \int_\Omega \langle v_\infty, u_\infty \rangle d\mu \,.$$

(c) If $\varphi : \Omega \times E \to [0, +\infty[$ is an $\mathcal{F} \otimes \mathcal{B}(E)$-measurable integrand such that, $\forall \omega \in \Omega$, $\varphi(\omega, .)$ is convex lower semicontinuous on E, then we have

$$\liminf_{n\to\infty} \int_\Omega \varphi(\omega, u_n(\omega))\mu(d\omega) \geq \int_\Omega \varphi(\omega, u_\infty(\omega))\mu(d\omega) \,.$$

Proof. (a) Repeating the Biting lemma ([28],[45]), we find an increasing sequence (A_p) in $\mathcal{F}$ with $\lim_{p\to\infty} \mu(A_p) = 1$ and a subsequence (u'_n) of (u_n) such that, for each p, $(u'_{n|A_p})$ is uniformly integrable. Since (u_n) is WTP, then $(u'_{n|A_p})_n$ is uniformly integrable and WTP in $L^1_E(A_p)$. By virtue of Theorem 1.3, $\forall p$, $(u'_{n|A_p})_n$ is relatively weakly compact. Consequently, by a straightforward diagonal procedure, there are $u_\infty \in L^1_E(\mu)$ and a subsequence (u_{n_k}) such that, for every p, $(u_{n_k|A_p})_k\, \sigma(L^1, L^\infty)$ converges to $u_{\infty|A_p}$.

Since $(u_{n_k})_k$ is WTP, by Theorem 1.12 there exist $v_\infty \in L^1_E(\mu)$ and $(\tilde{u}_n)$ with $\tilde{u}_n \in \mathrm{co}\{u_{n_k} : k \geq n\}$ such that $(\tilde{u}_n)$ converges a.e. to v_∞ for the norm topology of E.

For any fixed p, and $B \in \mathcal{F} \cap A_p$ and any $x' \in E'$, we have

$$\int_B \langle x', v_\infty \rangle d\mu = \lim_{n\to\infty} \int_B \langle x', \tilde{u}_n \rangle = \lim_{k\to\infty} \int_B \langle x', u_{n_k} \rangle d\mu = \int_B \langle x', u_\infty \rangle d\mu \,.$$

Hence $\langle x', v_\infty \rangle = \langle x', u_\infty \rangle$ a.e. on A_p, so $(\tilde{u}_n(\omega))$ converges in norm to $u_\infty(\omega)$ for a.e. $\omega \in \Omega$. This proves Assertion (a).

Assertion (b) follows from the arguments given in [15]. Let us check (c). We may suppose that $a := \liminf_n \int_\Omega \varphi(\omega, u_n(\omega))\mu(d\omega)$ is finite and by extracting a subsequence that $a = \lim_{n\to\infty} \int_\Omega \varphi(\omega, u_n(\omega))\mu(d\omega)$. Let $(\tilde{u}_n)$ and $u_\infty \in L^1_E(\mu)$ given by Assertion (a). Each $\tilde{u}_n$ has the form $\tilde{u}_n(\omega) = \sum_{j=n}^{\nu_n} \lambda_j^n u_{n_j}(\omega)$ with $0 \le \lambda_j^n \le 1$ and $\sum_{j=n}^{\nu_n} \lambda_j^n = 1$. By convexity, we have

$$\forall \omega, \forall n, \varphi(\omega, \tilde{u}_n(\omega)) \le \sum_{j=n}^{\nu_n} \lambda_j^n \varphi(\omega, u_{n_j}(\omega)) .$$

Hence

$$\limsup_n \int_\Omega \varphi(\omega, \tilde{u}_n(\omega))\mu(d\omega) \le a .$$

By lower semicontinuity of $\varphi(\omega, .)$ and by Fatou's lemma, we get

$$\liminf_{n\to\infty} \int_\Omega \varphi(\omega, \tilde{u}_n(\omega))\mu(d\omega) \ge \int_\Omega \varphi(\omega, u_\infty(\omega))\mu(d\omega) .$$

Hence

$$\liminf_n \int_\Omega \varphi(\omega, u_n(\omega))\mu(d\omega) \ge \int_\Omega \varphi(\omega, u_\infty(\omega))\mu(d\omega) .$$

Remarks. (1) Properties (a) and (b) yield a version of Fatou's lemma in Mathematical Economics. See [15] for a complete bibliography of this subject.

(2) Property (c) is a lower semicontinuity result. It turns out that (c) allows to state a minimization problem as in the corollary of Theorem 2.9. The details are left to the reader.

(3) If E is separable and if (un) is bounded and $\mathcal{R}_{wk}(E)$-tight, then one can check that $u_\infty(\omega) \in \overline{co}w - Ls\{u_n(\omega)\}$ a.e. We refer the reader to Amrani-Castaing-Valadier ([1]), Theorem 8) for details.

There is a variant of Theorem 1.15.

Theorem 1.16. *Assume that E'_b is separable. Let (u_n) be a bounded sequence in $L^1_E(\mu)$ such that*

(i) $\forall A \in \mathcal{F}$, $\mathcal{H}_A := \bigcup_n \{ \int_A u_n d\mu \}$ is relatively weakly compact.

(ii) Any vector measure $m : \mathcal{F} \to E$ with bounded variation such that, $\forall A \in \mathcal{F}$, $m(A) \in \overline{co}(\mathcal{H}_A)$, admits a density in $L^1_E(\mu)$.

Then properties (1), (2), (3) in Theorem 1.12 hold.

Proof. We sketch only the proof. It is enough to repeat the arguments of the proof of Theorem 1.12 by noting that, for each p, $(u'_{n|A_p})$ is relatively $\sigma(L^1, L^\infty)$ compact. See ([17], Theorem 3.1).

2. SEQUENTIAL WEAK COMPACTNESS IN SET-VALUED INTEGRATION

We recall and summarize some results which will be used in the sequel.

Theorem 2.1. *Let E be a separable Banach space and let $\Gamma : \Omega \to cwk(E)$ be a scalarly integrable multifunction such that the set $\{\delta^*(x', X) : x' \in \overline{B}_{E'}\}$ is uniformly integrable. Then the set S_Γ^{Pe} of all Pettis integrable selections of X is nonempty and the multivalued integral $\int_A X d\mu := \left\{ \int_A f d\mu : f \in S_\Gamma^{Pe} \right\}$ is convex weakly compact.*

Proof. By criteria of Pettis integrability ([30, [33], [40]) S_Γ^{Pe} is nonempty and the weak compactness of the integral $\int_A X d\mu$ follows from ([19], Theorem V-14).

The following result ([16], Lemma 4.1) is a key ingredient of several proofs of the results that we present below and we provide the proof for the sake of completeness.

Lemma 2.2. *Let E be a separable Banach space and $(X_n)_{n\in\mathbf{N}}$ a sequence of scalarly integrable $cwk(E)$-valued multifunctions satisfying:*

(1) $\{\delta^*(x', X_n) : x' \in \overline{B}_{E'},\ n \in \mathbf{N}\}$ *is uniformly integrable.*

(2) *For every $A \in \mathcal{F}$ the set $\mathcal{H}_A := \bigcup_{n\in\mathbf{N}} \int_A X_n d\mu$ is relatively weakly compact.*

Then there is a subsequence $(X_{n_k})_{k\in\mathbf{N}}$ such that

$$\forall A \in \mathcal{F},\ \forall x' \in E',\ \lim_{k\to\infty} \int_A \delta^*(x', X_{n_k}) d\mu$$

exists in $\mathbf{R}$.

Proof. We may suppose that for every $A \in \mathcal{F}$, there exists a convex weakly compact subset $\mathcal{K}_A$ such that $\mathcal{H}_A \subset \mathcal{K}_A$. Let D' be a countable dense sequence in E'_τ for the Mackey topology (see [19], Lemma III-32) and let $\mathcal{A} = \sigma(A_i,\ i \in \mathbf{N})$ be the σ-algebra generated by $(X_n)_{n\in\mathbf{N}}$. By

Theorem 2.1, for each $n \in \mathbf{N}$ and for each $A \in \mathcal{F}$, $\int_A X_n d\mu$ is a convex weakly compact subset of K_A. Further observe that for each i, the set $\mathcal{C}_i = \{C \in cwk(E) : C \subset K_{A_i}\}$ is compact for the Hausdorff topology associated to the weak topology. Then by (2) and by extracting diagonal subsequences, we find a subsequence $(X_{n_k})_{k\in\mathbf{N}}$ such that for any fixed $i \in \mathbf{N}$, $\left(\int_{A_i} X_{n_k} d\mu\right)$ converges to an element $C_i \in \mathcal{C}_i$ for this topology which coincides with the topology of pointwise convergence of their support functions on D'. By Strassen's theorem ([19], Theorem V-14), it follows that

$$\lim_{k\to\infty} \delta^*(x', \int_{A_i} X_{n_k} d\mu) = \lim_{k\to\infty} \int_{A_i} \delta^*(x', X_{n_k}) d\mu = \delta^*(x', C_i)$$

for all $x' \in D'$ and for all $i \in \mathbf{N}$. Now since D' is dense for the Mackey topology, the preceding equalities are valid for every $x' \in E'$. Let $A \in \mathcal{A}$ and $\varepsilon > 0$. Since the set $\{\delta^*(x', X_{n_k}) : x' \in \overline{B}_{E'},\ k \in \mathbf{N}\}$ is uniformly integrable, there is a measurable set A_i such that

$$\int_{A_i \Delta A} |\delta^*(x', X_{n_k})| d\mu \le \varepsilon$$

so that

$$\left| \int_A \delta^*(x', X_{n_k}) d\mu - \int_{A_i} \delta^*(x', X_{n_k}) d\mu \right| \le \int_{A_i \Delta A} |\delta^*(x', X_{n_k})| d\mu \le \varepsilon$$

for all $x' \in \overline{B}_{E'}$ and for all $k \in \mathbf{N}$. It follows that $\lim_{k\to\infty} \int_A \delta^*(x', X_{n_k}) d\mu$ exists in $\mathbf{R}$. Consequently, for any $x' \in E'$ and for any positive $\mathcal{A}$-measurable and bounded function h

$$\lim_{k\to\infty} \int_\Omega h\delta^*(x', X_{n_k}) d\mu$$

exists in $\mathbf{R}$. Now let h be any positive $\mathcal{F}$-measurable and bounded function and let $E^{\mathcal{A}}h$ the conditional expectation of h, then we have

$$\lim_{k\to\infty} \int_\Omega h\delta^*(x', X_{n_k}) d\mu = \lim_{k\to\infty} \int_\Omega E^{\mathcal{A}} h\delta^*(x', X_{n_k}) d\mu .$$

Here is an integral representation theorem for sublinear continuous functions defined on $L^\infty_{E'_b}(\mu)$.

Theorem 2.3. *Assume that E' is strongly separable. Let $l : L^{\infty}_{E'_b}(\mu) \to \mathbf{R}$ be a sublinear continuous mapping. Assume that the following conditions are satisfied:*

(i) For any pair (f_1, f_2) in $L^{\infty}_{E'_b}(\mu)$ with disjoint supports,

$$l(f_1 + f_2) = l(f_1) + l(f_2).$$

(ii) For any increasing sequence (A_n) in $\mathcal{F}$ with $A_n \uparrow A$ and for any $x' \in E'$,

$$\lim_{n\to\infty} l(\chi_{A_n} x') = l(\chi_A x').$$

(iii) For any fixed $A \in \mathcal{F}$, $x' \mapsto l(\chi_A x')$ is $\tau(E', E)$ continuous on E'.

(iv) Any vector measure $m : \mathcal{F} \to E$ with bounded variation verifying

$$\forall (A, x') \in \mathcal{F} \times E',\ \langle x', m(A)\rangle \le l(\chi_A x')$$

admits a density in $L^1_E(\mu)$.

Then there exists a unique $X \in \mathcal{L}^1_{cwk(E)}(\mu)$ such that

$$\forall u \in L^{\infty}_{E'_b}(\mu),\ l(u) = \int_\Omega \delta^*(u(\omega), X(\omega))\mu(d\omega).$$

Proof. Since the proof is rather long, we sketch only the main steps. See Castaing-Clauzure [17]. Note that for any $X \in \mathcal{L}^1_{cwk(E)}(\mu)$ the function

$$u \mapsto \int_\Omega \delta^*(u(\omega), X(\omega))\mu(d\omega)$$

is a sublinear continuous mapping from $L^{\infty}_{E'_b}(\mu)$ to $\mathbf{R}$ which satisfies all the conditions of our theorem.

Conversely let $l : L^{\infty}_{E'_b}(\mu) \to \mathbf{R}$ satisfying (i), (ii), (iii), (iv).

Step 1. It is obvious that for any fixed $A \in \mathcal{F}$, $x' \mapsto l(\chi_A x')$ is support function of a convex weakly compact set $M(A)$ in E. Set

$$\delta^*(x', M(A)) = l(\chi_A x'),\ \forall (A, x') \in \mathcal{F} \times E'.$$

By ([17], Prop. 2.1), the multifunction $M : \mathcal{F} \to cwk(E)$ is a multimeasure of bounded variation, that is $A \mapsto \delta^*(x', M(A))$ is a scalar measure and there exists a finite positive measure ν such that $M(A) \subset \nu(A)\overline{B}_{E'}$ for all $A \in \mathcal{F}$. Moreover ν is absolutely continuous with respect to μ.

Step 2. Let S_M be the set of all selections measure of M. Then for every $A \in \mathcal{F}$ and every $x' \in E'$ we have $\langle x', M(A)\rangle \leq \delta^*(x', M(A))$. Since m is of bounded variation, by (iv) there is $f_m \in L^1_E(\mu)$ such that

$$\forall A \in \mathcal{F},\ m(A) = \int_A f_m d\mu\,.$$

Set

$$\mathcal{H} = \{f_m : m \in S_M\}$$

and

$$\forall A \in \mathcal{F},\ \mathcal{H}_A := \left\{\int_A f_m d\mu : f_m \in L^1_E(\mu),\ m \in S_M\right\}.$$

Then $\mathcal{H}_A = M(A)$ and it is easily checked that $\mathcal{H}$ is uniformly integrable because we have

$$\lim_{\mu(A)\to 0}\ \sup_{m\in S_M} \int_A |f_m| d\mu \leq \lim_{\mu(A)\to 0} \nu(A) = 0\,.$$

As E' is strongly separable, $\mathcal{H}$ satisfies all the conditions of Theorem 4.1 in Castaing-Clauzure [17]. Hence $\mathcal{H}$ is relatively weakly compact in $L^1_E(\mu)$. Since $\mathcal{H}$ is closed and convex, $\mathcal{H}$ is weakly compact. Moreover, for any $A \in \mathcal{F}$ and any pair (u, v) in $\mathcal{H}$, we have $\chi_A u + \chi_{A^c} v \in \mathcal{H}$, then by a well-known result (cf. e.g.[36]), there exists $X \in \mathcal{L}^1_{cwk(E)}(\mu)$ such that $\mathcal{H} = S^1_X$ where S^1_X is the set of all integrable selections of X. It follows that

$$\forall A \in \mathcal{F},\ M(A) = \int_A X d\mu := \left\{\int_A f d\mu : f \in S^1_X\right\}.$$

Equivalently we have

$$\forall A \in \mathcal{F},\ \forall x' \in E',\ l(\chi_A x') = \delta^*(x', M(A)) = \int_A \delta^*(x', X) d\mu\,.$$

To finish the proof, it is enough to repeat the arguments given ([17], Prop. 2.2) to obtain

$$\forall u \in L^\infty_{E'_b}(\mu),\ l(u) = \int_\Omega \delta^*(u(\omega), X(\omega))\mu(d\omega)\,.$$

The following is a weak sequential compactness result in $\mathcal{L}^1_{cwk(E)}(\mu)$.

Proposition 2.4. *Suppose that E' is strongly separable and E has Radon-Nikodym property and (X_n) is a bounded sequence in $\mathcal{L}^1_{cwk(E)}(\mu)$ satisfying:*

(1) $\{\delta^*(x', X_n) : x' \in \overline{B}_{E'},\ n \in \mathbf{N}\}$ *is uniformly integrable;*

(2) *for every* $A \in \mathcal{F}$, $\mathcal{H}_A := \bigcup_n \int_A X_n d\mu$ *is relatively weakly compact in E, then there exist a subsequence* (X_{n_k}) *and* $X_\infty \in \mathcal{L}^1_{cwk(E)}(\mu)$ *such that*

$$\lim_{n\to\infty} \int_A \delta^*(x', X_{n_k})d\mu = \int_A \delta^*(x', X_\infty)d\mu$$

for every $A \in \mathcal{F}$ *and every* $x' \in E'$.

Proof. By (1), (2) and Lemma 2.2 there exist a subsequence (X_{n_k}) of (X_n) such that

$$\lim_{k\to\infty} \int_A \delta^*(x', X_{n_k})d\mu \tag{2.4.1}$$

exists in $\mathbf{R}$. Now we complete the proof by adapting the arguments in ([16]), Theorem 4.1). Note that for each $n \in \mathbf{N}^*$, the mapping $l_n : u \mapsto \int_\Omega \delta^*(u(\omega), X_n(\omega))\mu(d\omega)$ is additive sublinear and continuous for the norm of $L^\infty_{E'_b}$ since we have

$$|l_n(u)| \le \|u\|_\infty \sup_{n\in\mathbf{N}} \int_\Omega |X_n| d\mu \tag{2.4.2}$$

for all $u \in L^\infty_{E'_b}$. By (2.4.2) (l_n) is relatively compact in the space of all continuous mappings from $L^\infty_{E'_b}$ to $\mathbf{R}$ endowed with the topology of pointwise convergence. Hence there exists a filter $\mathcal{U}$ finer than the Frechet filter and a mapping $l_\infty : L^\infty_{E'_b} \mapsto \mathbf{R}$ such that

$$\forall u \in L^\infty_{E'_b},\ l_\infty(u) = \lim_{\mathcal{U}} \int_\Omega \delta^*(u, X_n)d\mu\,. \tag{2.4.3}$$

It is obvious that l_∞ is sublinear additive and by (2.4.1) satisfies the inequality

$$\forall (A, x') \in \mathcal{F} \times \overline{B}_{E'},\ l_\infty(\chi_A x') \le \sup_{x'\in\overline{B}_{E'}, n\in\mathbf{N}} \int_A \delta^*(x', X_n)d\mu\,. \tag{2.4.4}$$

By (2.4.4) we see that, for any fixed $x' \in E'$ and for any sequence (A_n) such that $A_n \uparrow A$, we have $\lim_{n\to\infty} l_\infty(\chi_{A_n} x') = l_\infty(\chi_A x')$. Moreover it is clear that $l_\infty(\chi_A x') \le \delta^*(x', \mathcal{H}_A)$ for all $(A, x') \in \mathcal{F} \times E'$ so that for every $A \in \mathcal{F}$, $x' \to l_\infty(\chi_A x')$ is continuous on E' for the Mackey topology. Hence we can apply the integral representation Theorem 2.3 to l_∞ which provides $X_\infty \in \mathcal{L}^1_{cwk(E)}$ such that

$$\forall u \in L^{\infty}_{E'_b},\ l_{\infty}(u) = \int_{\Omega} \delta^*(u,\ X_{\infty})d\mu . \tag{2.4.6}$$

Then (2.4.1), (2.4.3) and (2.4.6) yield

$$\forall (A, x') \in \mathcal{F} \times E',\ \lim_{k\to\infty} \int_A \delta^*(x', X_{n_k})d\mu = \int_A \delta^*(x', X_{\infty})d\mu$$

as stated.

Now we aim to extend the preceding compactness results to scalarly integrable multifunctions. It is convenient to introduce the following limiting notions.A sequence (X_n) of scalarly integrable $cwk(E)$-valued multifunctions *scalarly Mazurconverges* to a scalarly integrable $cwk(E)$-valued multifunction Y if there is a sequence (Y_n) of convex combinations of (X_n) of the form

$$Y_n = \sum_{i=n}^{\nu_n} \lambda_i^n X_i, \text{ with } 0 \le \lambda_i^n \le 1 \text{ and } \sum_{i=n}^{\nu_n} \lambda_i^n = 1$$

which scalarly converges almost everywhere to Y, that is the support functions of Y_n converge almost everywhere to the support functions of Y. We will write $Y_n \in \text{co}\{X_m : m \ge n\}$. If the Cesaro sums $\frac{1}{n} \sum_{m=1}^{n} X_i$ scalarly converge almost everywhere to Y, we say that X_n scalarly C-converges to Y almost everywhere.

We need first an easy lemma.

Lemma 2.5. *Let $(X_n)_{n\in\mathbf{N}}$ be a sequence of scalarly integrable cwk(E)-valued multifunctions such that*

(1) $\{\delta^*(x', X_n) : x' \in \overline{B}_{E'},\ n \in \mathbf{N}\}$ *is uniformly integrable.*

(2) *For every measurable set A, $\bigcup_n \int_A X_n d\mu$ is relatively weakly compact in E.*

(3) *For every subsequence (Y_n) of (X_n) there is a sequence $(\widetilde{Y}_n)$ with $\widetilde{Y}_n \in \text{co}\{Y_m : m \ge n\}$ which scalarly converges to a scalarly integrable $cwk(E)$-valued multifunction.*

Then there exist a subsequence (X_{n_k}) and $\widetilde{Y} \in P^1_{cwk(E)}(\mu)$ such that

$$\lim_{n\to\infty} \int_A \delta^*(x', X_{n_k})d\mu = \int_A \delta^*(x', \widetilde{Y})d\mu$$

for all $A \in \mathcal{A}$ and for all $x' \in E'$.

Proof. By (1) and (2), we may apply Lemma 2.2 to (X_n). Then there is a subsequence still denoted by (X_n) such that

$$\forall A \in \mathcal{F},\ \forall x' \in E',\ \lim_{n\to\infty} \int_A \delta^*(x', X_n)d\mu$$

exists in $\mathbf{R}$. By (3) there is a sequence $(\widetilde{Y}_n)$ with $\widetilde{Y}_n \in \mathrm{co}\{X_m : m \geq n\}$ which scalarly converges almost everywhere to a $cwk(E)$-valued scalarly integrable multifunction $\widetilde{Y}$. Since each $\widetilde{Y}_n$ has the form

$$\widetilde{Y}_n = \sum_{m=n}^{\nu_n} \lambda_m^n X_m\,,\ \text{with } 0 \leq \lambda_m^n \leq 1 \text{ and } \sum_{m=n}^{\nu_n} \lambda_m^n = 1$$

then the sequence $(\widetilde{Y}_n)$ satisfies also the UI condition (1), namely

$$\{\delta^*(x', \widetilde{Y}_n) : x' \in \overline{B}_{E'},\ n \in \mathbf{N}\}$$

is uniformly integrable. Hence by Lebesgue-Vitali's theorem we get

$$\lim_{n\to\infty} \int_A \delta^*(x', X_n)d\mu = \lim_{n\to\infty} \int_A \delta^*(x', \widetilde{Y}_n)d\mu = \int_A \delta^*(x', \widetilde{Y})d\mu.$$

Remark. Lemma 2.5 holds if we replace (3) by: Every subsequence (Y_n) of (X_n) admits a subsequence which scalarly C-converges to $Y \in P^1_{cwk(E)}(\mu)$.

Now we are ready to produce weak sequential compactness results for scalarly integrable $cwk(E)$-valued multifunctions.

Theorem 2.6. *Let X be a scalarly integrable $cwk(E)$-valued multifunction such that $\{\delta^*(x', X) : x' \in \overline{B}_{E'}\}$ is uniformly integrable. Let $(X_n)_{n\in\mathbf{N}}$ be a sequence in $\mathcal{P}^1_{cwk}(\mu)$ such that $\forall n$, $\forall \omega$, $X_n(\omega) \subset X(\omega)$. Then there are a subsequence (X_{n_k}) and $X_\infty \in \mathcal{P}^1_{cwk}(\mu)$ such that*

$$\lim_{k\to\infty} \int_A \delta^*(x', X_{n_k})d\mu = \int_A \delta^*(x', X_\infty)d\mu.$$

Proof. Let (e'_p) be a dense sequence in E' for the Mackey topology. Applying Komlos theorem [34] and using a standard diagonal process there exist a subsequence $(X_{n_k})_{k\in N}$ and a sequence $(\varphi_p)_{p\in\mathbf{N}}$ in $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ such that

$$\lim_{k\to\infty} \frac{1}{n}\sum_{k=1}^{n} \delta^*(e'_p, X_{n_k}(\omega)) = \varphi_p(\omega) \text{ a.e.} \qquad (*)$$

Since $\frac{1}{n}\sum_{k=1}^{n} X_{n_k}(\omega) \subset X(\omega)$ for all $n \in \mathbf{N}^*$ and for all $\omega \in \Omega$, by $(*)$ we may apply ([14], Lemma 3.3) providing a scalarly integrable $cwk(E)$-valued multifunction Y such that $(\frac{1}{n}\sum_{k=1}^{n} X_{n_k}(\omega))_{n\in\mathbf{N}^*}$ scalarly converges to Y a.e., that is (X_{n_k}) scalarly C-converges a.e. to Y. Finally, by Theorem 2.1, for every $A \in \mathcal{F}$, $\mathcal{H}_A := \bigcup_{n\in\mathbf{N}} \int_A X_n d\mu$ is relatively weakly compact because it is included in the convex weakly compact $\int_A X d\mu$. So we can finish the proof by using Lemma 2.5.

3. WEAK COMPACTNESS FOR BOCHNER AND PETTIS E-VALUED FUNCTIONS

The material in this section is borrowed from Amrani-Castaing [2]. We recall the following basic result ([30], [33], [40]) and we provide an alternative proof for the convenience of the reader.

Theorem 3.1. *Let E be a Banach space, $(f_n)_{n\in\mathbf{N}}$ a sequence of E-valued Pettis functions and $f : \Omega \to E$ a scalarly integrable function satisfying:*

(1) *$\{\langle x', f\rangle : x' \in \overline{B}_{E'}\}$ is uniformly integrable.*

(2) *For every $x' \in E'$, $\langle x', f_n\rangle$ converges $\sigma(L^1, L^\infty)$ to $\langle x', f\rangle$.*

Then f is Pettis-integrable.

Proof. By (2) for every $x' \in E'$ and for every $A \in \mathcal{F}$, we have

$$\lim_{n\to\infty} \langle x', \int_A f_n d\mu\rangle = \lim_{n\to\infty} \int_A \langle x', f_n\rangle d\mu = \int_A \langle x', f\rangle d\mu .$$

So, in order to prove the theorem, it is sufficient to show that for every $A \in \mathcal{F}$, the sequence $(\int_A f_n d\mu)_{n\in\mathbf{N}}$ is relatively weakly compact in E. By the Eberlein-Smulyan-Grothendieck theorem ([31], Corollary 1 of Theorem 7) it is equivalent to prove: for every sequence $(x'_k)_{k\in\mathbf{N}}$ in

$\overline{B}_{E'}$ and for every subsequence $(f_{n_m})_{m\in\mathbf{N}}$ of $(f_n)_{n\in\mathbf{N}}$, we have

$$\alpha := \lim_{k\to\infty}\lim_{m\to\infty}\langle x'_k, \int_A f_{n_m}\,d\mu\rangle = \beta := \lim_{m\to\infty}\lim_{k\to\infty}\langle x'_k, \int_A f_{n_m}\,d\mu\rangle \tag{3.1.1}$$

provided theses limits exist. First by (2) we have

$$\lim_{m\to\infty}\langle x'_k, \int_A f_{n_m}\,d\mu\rangle = \lim_{m\to\infty}\langle x'_k, \int_A f_{n_m}\rangle d\mu = \int_A \langle x'_k, f\rangle d\mu\,. \tag{3.1.2}$$

By Komlos theorem [37], applied to the sequence $(\langle x'_k, f\rangle)_{k\in\mathbf{N}}$ there exists a sequence $(y'_n)_{n\in\mathbf{N}}$ with $y'_n = \frac{1}{n}\sum_{i=1}^{n} x'_{k_i}$ and a real valued integrable function h such that $\langle y'_n, f\rangle$ converges to h almost everywhere. So by (3.1.2) and (1) we have

$$\alpha = \lim_{k\to\infty}\int_A \langle x'_k, f\rangle d\mu = \lim_{n\to\infty}\int_A \langle y'_n, f\rangle d\mu = \int_A h\,d\mu\,. \tag{3.1.3}$$

Let y'_0 be a weak* cluster point of $(y'_n)_{n\in\mathbf{N}}$, then for every $m\in\mathbf{N}$, we have

$$\begin{aligned}\lim_{k\to\infty}\langle x'_k, \int_A f_{n_m}\,d\mu\rangle &= \lim_{k\to\infty}\langle y'_n, \int_A f_{n_m}\,d\mu\rangle = \langle y'_0, \int_A f_{n_m}\,d\mu\rangle \\ &= \int_A \langle y'_0, f_{n_m}\rangle d\mu\,.\end{aligned} \tag{3.1.4}$$

Taking the limit when $m\to\infty$ in the last integral in (3.1.4) and using (2) we obtain

$$\beta = \lim_{m\to\infty}\int_A \langle y'_0, f_{n_m}\rangle d\mu = \int_A \langle y'_0, f\rangle d\mu\,. \tag{3.1.5}$$

Since $\langle y'_n, f\rangle$ converges to h almost everywhere and y'_0 is a weak* cluster point of $(y'_n)_{n\in\mathbf{N}}$, $h = \langle y'_0, f\rangle$ almost everywhere. Returning to (3.1.1) and using (3.1.3), (3.1.4), (3.1.5) we get $\alpha = \beta$.

Theorem 3.2. *If E is a separable Banach space and $\mathcal{H}$ is a subset of $P^1_E(\mu)$ satisfying: (1) $\{\langle x', f\rangle : x'\in \overline{B}_{E'}, f\in\mathcal{H}\}$ is uniformly integrable; (2) given any sequence (f_n) in $\mathcal{H}$, there are a sequence $(\tilde{f}_n)$ with*

$\tilde{f}_n \in \mathrm{co}\{f_k : k \geq n\}$ and $\tilde{f}_\infty \in P^1_E(\mu)$ such that, $\forall x' \in E', \langle x', \tilde{f}_n\rangle$ converges $\sigma(L^1, L^\infty)$ to $\langle x', f_\infty\rangle$, then $\mathcal{H}$ is relatively sequentially compact for the topology of pointwise convergence on $L^\infty_{\mathbf{R}} \otimes E'$.

Proof. Step 1: Let $(f_n)_{n\in\mathbf{N}}$ in $\mathcal{H}$. We note that by (1) $\mathcal{H}_A$ is bounded for every measurable subset A. Now we claim that, $\forall A \in \mathcal{A}$, $\mathcal{H}_A := (\int_A f_n d\mu)_{n\in\mathbf{N}}$ is relatively weakly compact or equivalently $\mathcal{K}_A := \overline{\mathrm{co}}\mathcal{H}_A$ is weakly compact. By James's theorem [35] it is enough to prove that for every $x' \in E'$, there exist $\varsigma \in \mathcal{K}_A$ such that

$$\langle x', \varsigma\rangle = \sup_{x\in\mathcal{K}_A} \langle x', x\rangle = \delta^*(x', \mathcal{K}_A) = \delta^*(x', \mathcal{H}_A).$$

Let $(f_{n_k})_{k\in\mathbf{N}}$ be a subsequence of $(f_n)_{n\in\mathbf{N}}$ such that

$$\lim_{k\to\infty} \langle x', \int_A f_{n_k} d\mu\rangle = \delta^*(x', \mathcal{H}_A).$$

Let $(\tilde{f}_n)_{n\in\mathbf{N}}$ and $\tilde{f}_\infty \in P^1_E(\mu)$ associated to $(f_{n_k})_{k\in\mathbf{N}}$ by (2). Since each $\tilde{f}_n$ has the form $\tilde{f}_n = \sum_{i=n}^{\nu_n} \lambda_i^n f_{n_i}$ with $0 \leq \lambda_i^n \leq 1$ and $\sum_{i=n}^{\nu_n} \lambda_i^n = 1$, then we have

$$\begin{aligned}\delta^*(x', \mathcal{K}_A) &= \lim_{k\to\infty} \langle x', \int_A f_{n_k} d\mu\rangle = \lim_{n\to\infty} \langle x', \sum_{i=n}^{\nu_n} \lambda_i^n \int_A f_{n_i} d\mu\rangle \\ &= \langle x', \int_A \tilde{f}_\infty d\mu\rangle \leq \delta^*(x', \mathcal{K}_A).\end{aligned}$$

So the claim is true. Note that in this step, it is not necessary to suppose that E is separable.

Step 2: Since $(\int_A f_n d\mu)_{n\in\mathbf{N}}$ is relatively weakly compact we may apply Lemma 2.2 in section 2 which provides a subsequence still denoted by $(f_{n_k})_{k\in\mathbf{N}}$, such that for every measurable set A and every $x' \in E'$, $\lim_{k\to\infty} \int_A \langle x', f_{n_k}\rangle d\mu$ exists in $\mathbf{R}$. Let $(\tilde{f}_n)_{n\in\mathbf{N}}$ and $\tilde{f}_\infty \in P^1_E(\mu)$ associated to $(f_{n_k})_{k\in\mathbf{N}}$ by (2). Then we have

$$\lim_{k\to\infty} \int_A \langle x', f_{n_k}\rangle d\mu = \lim_{n\to\infty} \int_A \langle x', \tilde{f}_n\rangle d\mu = \int_A \langle x', \tilde{f}_\infty\rangle d\mu$$

so that by standard arguments we get

$$\lim_{k\to\infty} \int h\langle x', f_{n_k}\rangle d\mu = \int h\langle x', f_\infty\rangle d\mu$$

for all $h \in L^\infty_{\mathbf{R}}$ and $x' \in E'$.

Corollary 3.3. *If E is a separable Banach space and $\mathcal{H}$ is a subset of $P^1_E(\mu)$ satisfying: (1) $\{\langle x', f\rangle : x' \in \overline{B}_{E'}, f \in \mathcal{H}\}$ is uniformly integrable; (2) given any sequence (f_n) in $\mathcal{H}$, there is a sequence $(\tilde{f}_n)$ with $\tilde{f}_n \in co\{f_k : k \geq n\}$ such that $(\tilde{f}_n)$ weakly converges in E almost everywhere, then $\mathcal{H}$ is relatively sequentially compact for the topology of pointwise convergence on $L^\infty_{\mathbf{R}} \otimes E'$.*

Proof. Let $(f_n)_{n\in\mathbf{N}}$ in $\mathcal{H}$ and let $(\tilde{f}_n)$ given by (2). Let us consider $\tilde{f}_\infty(\omega) :=$ weak-$\lim_n \tilde{f}_n(\omega)$ for $\omega \notin N$ where N is a negligible set and $\tilde{f}_\infty(\omega) = 0$ for $\omega \in N$. By (1) and ([39], Remark 1, p. 162) $\tilde{f}_\infty$ is Pettis integrable and by Lebesgue-Vitali's theorem we have

$$\lim_{n\to\infty} \int_A \langle x', \tilde{f}_n\rangle d\mu = \int_A \langle x', \tilde{f}_\infty\rangle d\mu$$

for every $A \in \mathcal{F}$ and every $x' \in E'$. So Corollary 3.3 follows from Theorem 3.2.

Corollary 3.3 allows to deduce a recent weak compactness result in [16].

Proposition 3.4. *Let E be a separable Banach space. Let $\Gamma : \Omega \mapsto cwk(E)$ be a scalarly integrable multifunction and S_Γ the set of scalarly integrable selection of Γ. If $\{\langle x', f\rangle : x' \in \overline{B}_{E'}, f \in S_\Gamma\}$ is uniformly integrable, then the set S^{Pe}_Γ is nonempty and sequentially compact for the topology of pointwise convergence on $L^\infty_{\mathbf{R}} \otimes E'$.*

Proof. Nonemptiness of S^{Pe}_Γ is ensured by hypothesis and ([30], [33], [40]). Now let $(f_n)_{n\in\mathbf{N}} \subset S^{Pe}_\Gamma$ and let $(e'_p)_{p\in\mathbf{N}}$ be a dense sequence in $\overline{B}_{E'}$ for the Mackey topology. Since for each $p \in \mathbf{N}$, the sequence $(\langle e'_p, f_n\rangle)_{n\in\mathbf{N}}$ is uniformly integrable, by Komlos theorem [37], then by an obvious diagonal process, there exist a subsequence $(f_{n_k})_{k\in\mathbf{N}}$ and a sequence $(\varphi_p)_{p\in\mathbf{N}}$ in $L^1_{\mathbf{R}}(\Omega, \mathcal{F}, \mu)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \langle e'_p, f_{n_k}(\omega)\rangle = \varphi_p(\omega) \text{ a.e.}$$

Since $\frac{1}{n} \sum_{k=1}^{n} f_{n_k}(\omega) \in \Gamma(\omega)$ for all $n \in \mathbf{N}^*$ and for all $\omega \in \Omega$, and $\Gamma(\omega)$ is convex weakly compact, it is not difficult to see that $(\frac{1}{n} \sum_{k=1}^{n} f_{n_k}(\omega))_{n\in\mathbf{N}^*}$ weakly converges a.e. So by Corollary 3.3 and, using the fact that Γ is scalarly integrable with convex weakly compact values, we conclude that S_Γ^{Pe} is sequentially compact for the topology of pointwise convergence on $L^\infty_{\mathbf{R}} \otimes E'$. □

To end this section we mention two applications of the preceding techniques to best approximation in $P^1_E(\mu)$ and to weak compactness in $L^1_E(\mu)$.

Proposition 3.5. *Let $\mathcal{B}$ be a sub-σ-algebra of $\mathcal{F}$ and let E be a separable Banach space and let $\Gamma : \Omega \to cwk(E)$ be a scalarly $\mathcal{B}$-measurable and integrable multifunction such that the set $\{\langle x', f\rangle : x' \in \overline{B}_{E'}, f \in S_\Gamma^{Pe}(\mathcal{B})\}$ is uniformly integrable. Then $S_\Gamma^{Pe}(\mathcal{B})$ is proximinal in $P^1_E(\Omega, \mathcal{F}, \mu)$.*

Proof. Let $f \in P^1_E(\Omega, \mathcal{F}, \mu)$ and let (f_n) be a minimizing sequence in $S_\Gamma^{Pe}(\mathcal{B})$, that is

$$\lim_{n\to\infty} \|f - f_n\|_{Pe} := \lim_{n\to\infty} \sup_{x' \in \overline{B}_{E'}} \int_\Omega |\langle x', f - f_n\rangle| d\mu$$
$$= \inf_{g \in S_\Gamma^{Pe}(\mathcal{B})} \|f - g\|_{Pe}$$

where $\|.\|_{Pe}$ denotes the Pettis norm. By Corollary 3.3 we may suppose that (f_n) converges $\sigma(P^1_E(\Omega, \mathcal{F}, \mu), L^\infty_{\mathbf{R}} \otimes E')$ to a Pettis integrable function $f_\infty \in S_\Gamma^{Pe}(\mathcal{B})$. We claim that (f_n) converges $\sigma(P^1_E(\Omega, \mathcal{F}, \mu), L^\infty_{\mathbf{R}} \otimes E')$ to f_∞. Indeed, let $x' \in E'$ and $h \in L^\infty_{\mathbf{R}}(\mathcal{F})$. Denote by $E^{\mathcal{B}}h$ the conditional expectation of h with respect to $\mathcal{B}$. Then we have

$$\lim_{n\to\infty} \int_\Omega h\langle x', f_n\rangle d\mu = \lim_{n\to\infty} \int_\Omega E^{\mathcal{B}} h\langle x', f_n\rangle d\mu$$
$$= \int_\Omega E^{\mathcal{B}} h\langle x', f_\infty\rangle d\mu$$
$$= \int_\Omega h\langle x', f_\infty\rangle d\mu .$$

In particular, for every $x' \in \overline{B}_{E'}$ we have

$$\int_\Omega |\langle x', f - f_\infty\rangle| d\mu \le \liminf_{n\to\infty} \int_\Omega |\langle x', f - f_n\rangle| d\mu$$
$$\le \liminf_{n\to\infty} \sup_{x' \in \overline{B}_{E'}} \int_\Omega |\langle x', f - f_n\rangle| d\mu$$
$$\le \lim_{n\to\infty} \|f - f_n\|_{Pe}$$
$$= \inf_{g \in S_\Gamma^{Pe}(\mathcal{B})} \|f - g\|_{Pe} .$$

By taking the supremum over $\overline{B}_{E'}$ in the preceding inequality, we get

$$\|f - f_\infty\|_{Pe} = \inf_{g \in S_\Gamma^{Pe}(B)} \|f - g\|_{Pe}.$$

The following is a criteria of sequential weak compactness in L^1_E.

Proposition 3.6. *Let E be a separable Banach space and $\mathcal{H}$ a subset of $L^1_E(\mu)$. Assume that: (1) $\{\langle x', f\rangle : x' \in \overline{B}_{E'}, f \in \mathcal{H}\}$ is uniformly integrable; (2) given any sequence (f_n) in $\mathcal{H}$, there are a sequence $(\tilde{f}_n)$ with $\tilde{f}_n \in co\{f_k : k \geq n\}$ and $\tilde{f}_\infty \in L^1_E(\mu)$ such that, $\forall x' \in E'$, $\langle x', \tilde{f}_n\rangle$ converges $\sigma(L^1, L^\infty)$ to $\langle x', \tilde{f}_\infty\rangle$, then $\mathcal{H}$ is relatively sequentially $\sigma(L^1_E, L^\infty_{\mathbf{R}} \otimes E')$ compact.*

In particular, if E' is strongly separable and $\mathcal{H}$ is a uniformly integrable subset of $L^1_E(\mu)$ satisfying (2), then $\mathcal{H}$ is relatively sequentially $\sigma(L^1_E, L^\infty_{E'})$ compact.

Proof. Apply mutatis mutandis the proof of Theorem 3.2 and use the norm separability of E' and a result in ([13]) which says that on the unit ball of $L^\infty_{E'_b}$ the topology of convergence in measure coincide with the topology of uniform convergence on uniformly integrable sets in L^1_E. We omit the details which are left to the reader.

4. CONVERGENCE OF CONVEX WEAKLY COMPACT RANDOM SETS IN SUPER-REFLEXIVE BANACH SPACE

We shall assume that E is a separable super-reflexive Banach space. We recall the following vector E-valued version of Komlos theorem [37] due to Garling ([29], Theorem 6, p. 310).

Theorem 4.1. *Suppose that E is super-reflexive and (f_n) is a bounded sequence in L^1_E. Then there is a subsequence $(g_k) = (f_{n_k})$ and f in L^1_E such that*

$$(1/l)\sum_{j=1}^{l} g_{k_j}(\omega)) \to f(\omega)$$

a.e., for each subsequence (g_{k_l}).

We will use the following limiting notions. Also we shall use the following limiting notions. If $C_1, C_2, \ldots, C_n, \ldots$ and C_∞ are nonempty closed convex subsets of E, C_n *Mosco converges* to C_∞ (shortly $C_\infty = M - \lim_n Cn$) if the two following inclusions are satisfied:

$$C_\infty \subset s\text{-}li\, C_n := \{x \in E : \|x - x_n\| \to 0;\ x_n \in C_n\}$$

$$w - ls\, C_n := \{x \in E : x_{n_k} \to x \text{ weakly};\ x_{n_k} \in C_{n_k}\} \subset C_\infty .$$

Given two nonempty subsets B and C in E, the gap between B and C is defined by:

$$D(B,C) = \inf\{\|x-y\| : x \in B, y \in C\}.$$

The slice topology τ_s on $cc(E)$ (nonempty closed convex subsets of E) is the weakest topology τ on $cc(E)$ such that for each nonempty bounded closed convex subset B of E, the function $C \mapsto D(B,C)$ is τ-continuous (see, [8], Theorem 5.3).

A sequence (f_n) in $L^1_E(\Omega, \mathcal{F}, \mu)$ *Komlos converges* to $f_\infty \in L^1_E(\Omega, \mathcal{F}, \mu)$, if there is a subsequence $(f_{\beta(n)})$ of (f_n) such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} f_{\gamma(j)} = f_\infty$$

a.e., for each subsequence $(f_{\gamma(n)})$ of $(f_{\beta(n)})$.

Given a sub σ-algebra $\mathcal{B}$ of $\mathcal{F}$ and $X \in \mathcal{L}^1_{cwk(E)}$, the conditional expectation $E^{\mathcal{B}}X$ is a $\mathcal{B}$-measurable $cwk(E)$-valued multifunction which enjoys the following property:

$$\forall B \in \mathcal{B},\ \forall x' \in E',\ \int_B \delta^*(x', E^{\mathcal{B}}X)d\mu = \int_B \delta^*(x', X)d\mu .$$

We refer to [19] for details.

The following is a version of Komlos-slice convergence theorem for convex weakly compact random sequences (X_n) in $\mathcal{L}^1_{cwk(E)}$.

Theorem 4.2. (Castaing-Ezzaki [18]) *Suppose that E is a separable super-reflexive Banach space and $(X_n)_{n\in\mathbb{N}^*}$ is a uniformly integrable sequence in $\mathcal{L}^1_{cwk(E)}$. Then the following hold:*

(a) There exists a subsequence $(X_{\alpha(n)})$ and $X_\infty \in \mathcal{L}^1_{cwk(E)}$ such that

$$\forall A \in \mathcal{F}, \forall x' \in E',\ \lim_{n\to\infty} \int_A \delta^*(x', X_{\alpha(n)})d\mu = \int_A \delta^*(x', X_\infty)d\mu . \tag{4.2.1}$$

(b) There is a subsequence $(X_{\beta(n)})$ of $(X_{\alpha(n)})$ such that

$$M-\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}X_{\gamma(j)}(\omega)=X_\infty(\omega) \tag{4.2.2}$$

a.e., for each subsequence $(X_{\gamma(n)})$ of $(X_{\beta(n)})$.

(c) Assume further that $(\mathcal{F}_n)_{n\in\mathbf{N}^*}$ is an increasing sequence of sub-σ-algebras of $\mathcal{F}$ with $\mathcal{F}_\infty := \sigma(\bigcup_n \mathcal{F}_n)$, Y is a positive random variable such that $E^{\mathcal{F}_1}Y < +\infty$ and that $|X_n| \le Y$ for all $n \in \mathbf{N}$ and a.e. $\omega\in\Omega$, then (4.2.2) implies

$$\lim_{n\to\infty} D\big(B,\, E^{\mathcal{F}_n}\big[\frac{1}{n}\sum_{j=1}^{n}X_{\gamma(j)}\big]\big) = D(B,\, E^{\mathcal{F}_\infty}X_\infty) \tag{4.2.3}$$

for any bounded closed convex subset B of E and a.e. $\omega\in\Omega$.

Proof. Statement (a) follows directly from Theorem 2.4. Let us prove (b). Let $D_1^* := (e_k^*)_{k\in\mathbf{N}^*}$ be a dense sequence in $\overline{B}_{E'}$ for the norm topology. Let $(X_{\alpha(n)})$ and X_∞ as in (a). For each k, we pick a maximum integrable selection $\sigma^k_{\alpha(n)}$ of $X_{\alpha(n)}$ associated to e_k^*. Using (4), (a), Theorem 4.1 (Garling's theorem) and extracting diagonal sequences, we find a subsequence $(X_{\beta(n)})$, a subsequence $(\sigma^k_{\beta(n)})$, $\sigma^k_\infty \in L^1_E$ and $\varphi\in L^1_{\mathbf{R}^+}$ such that, for each k the following hold:

$$\lim_{n\to\infty}\delta^*\big(e_k^*, \frac{1}{n}\sum_{j=1}^{n}X_{\gamma(j)}(\omega)\big)=\delta^*(e_k^*, X_\infty(\omega)) \tag{4.2.4}$$

almost everywhere, for each subsequence $(X_{\gamma(n)})$ of $(X_{\beta(n)})$;

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\sigma^k_{\gamma(j)}(\omega)=\sigma^k_\infty(\omega) \tag{4.2.5}$$

almost everywhere, for each subsequence $(\sigma^k_{\gamma(n)})$ of $(\sigma^k_{\beta(n)})$; and that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}|X_{\gamma(j)}|(\omega)=\varphi(\omega) \tag{4.2.6}$$

almost everywhere, for each subsequence $(X_{\gamma(n)})$ of $(X_{\beta(n)})$. Apart from the use of weak compactness result in (a) Komlos arguments in (4.2.4) and (4.2.6) are not new since they have been already used in a series of papers by Balder (see e.g. [3]) whereas Garling's theorem is first used in (4.2.5). For simplicity we set

$$S_n(\omega) := \frac{1}{n}\sum_{j=1}^{n} X_{\gamma(j)}(\omega)$$

for all $n \in \mathbf{N}^*$ and for all $\omega \in \Omega$. Then by (4.2.6) we have

$$\sup_n |S_n(\omega)| < +\infty \tag{4.2.7}$$

a.e. so that by the norm separability of E' (4.2.4) holds for all $x^* \in \overline{B}_{E'}$, that is, there exists a negligible set N such that

$$\lim_{n\to\infty} \delta^*\big(x^*, \frac{1}{n}\sum_{j=1}^{n} X_{\gamma(j)}(\omega)\big) = \delta^*(x^*, X_\infty(\omega)) \tag{4.2.8}$$

for all $(\omega, x^*) \in (\Omega \setminus N) \times \overline{B}_{E'}$. Now by obvious properties of (σ_n^k), (4.2.4), (4.2.5) and the definition of $s\text{-}li\frac{1}{n}\sum_{j=1}^{n} X_{\gamma(j)}(\omega)$ we have

$$\delta^*(e_k^*, X_\infty(\omega)) = \langle e_k^*, \sigma_\infty^k(\omega)\rangle \le \delta^*(e_k^*, s\text{-}li\frac{1}{n}\sum_{j=1}^{n} X_{\gamma(j)}(\omega)) \tag{4.2.9}$$

almost everywhere, for all k. Hence (4.2.2) follows from (4.2.8) and (4.2.9).

Now using (4.2.8), (4.2.9) we will prove (4.2.3) that is a new formulation of Komlos-slice convergence in $\mathcal{L}^1_{cwk(E)}$ involving conditional expectation.

Step1. Claim: (*) $\forall x^* \in \overline{B}_{E'}$, $\lim_{n\to\infty} E^{\mathcal{F}_n}\delta^*(x^*, S_n(\omega)) = E^{\mathcal{F}_\infty}\delta^*(x^*, X_\infty(\omega))$ a.e. on Ω.

Since $E^{\mathcal{F}_1}Y < +\infty$ and $|X_n| \le Y$, for all $n \in \mathbf{N}$ and a.s., then using (4.1.8) and a new version of dominated convergence theorem for conditional expectation of real-valued integrable random variable ([27], Chap V, Lemma 2.4) we deduce that $\forall x^* \in D_1^*$, there exists a negligible set N_{x^*} in Ω such that

$$\begin{aligned}\lim_{n\to\infty} E^{\mathcal{F}_n}\delta^*(x^*, S_n(\omega)) &= \lim_{n\to\infty} \delta^*(x^*, (E^{\mathcal{F}_n}S_n)(\omega)) \\ &= E^{\mathcal{F}_\infty}\delta^*(x^*, X_\infty(\omega)) \\ &= \delta^*(x^*, (E^{\mathcal{F}_\infty}X_\infty)(\omega))\end{aligned}$$

for all $\omega \in \Omega \setminus N_{x^*}$. Set $N = \bigcup_{x^* \in D_1^*} N_{x^*}$. For any $(\omega, x^*) \in (\Omega \setminus N) \times D_1^*$, we have

$$\lim_{n\to\infty} \delta^*(x^*, (E^{\mathcal{F}_n} S_n)(\omega)) = \delta^*(x^*, (E^{\mathcal{F}_\infty} X_\infty)(\omega)) . \qquad (4.2.10)$$

Let $\omega \in \Omega \setminus N$, $x^* \in \overline{B}_{E^*}$ and let $\varepsilon > 0$. There exists x_k^* in D_1^* such that $\|x - x_k^*\| \le \varepsilon$. Hence

$$\begin{aligned}
&|\delta^*(x^*, (E^{\mathcal{F}_n} S_n)(\omega)) - \delta^*(x^*, (E^{\mathcal{F}_\infty} X_\infty)(\omega))| \\
&\le |\delta^*(x^*, (E^{\mathcal{F}_n} S_n)(\omega)) - \delta^*(x_k^*, (E^{\mathcal{F}_n} S_n)(\omega))| \\
&\quad + |\delta^*(x_k^*, (E^{\mathcal{F}_n} S_n)(\omega)) - \delta^*(x_k^*, (E^{\mathcal{F}_\infty} X_\infty)(\omega))| \\
&\quad + |\delta^*(x_k^*, (E^{\mathcal{F}_\infty} X_\infty)(\omega)) - \delta^*(x^*, (E^{\mathcal{F}_\infty} X_\infty)(\omega))| \\
&\le \max(\delta^*(x^* - x_k^*, (E^{\mathcal{F}_n} S_n)(\omega)), \delta^*(x_k^* - x^*, (E^{\mathcal{F}_n} S_n)(\omega)) \\
&\quad + |\delta^*(x_k^*, (E^{\mathcal{F}_n} S_n)(\omega)) - \delta^*(x_k^*, (E^{\mathcal{F}_\infty} X_\infty)(\omega))| \\
&\le \max(\delta^*(x_k^* - x^*, (E^{\mathcal{F}_\infty} X_\infty)(\omega)), \delta^*(x^* - x_k^*, (E^{\mathcal{F}_\infty} X_\infty)(\omega)) \\
&\le 2\|x^* - x_k^*\| E^{\mathcal{F}_1} Y(\omega) + |\delta^*(x_k^*, (E^{\mathcal{F}_n} S_n)(\omega)) - \delta^*(x_k^*, (E^{\mathcal{F}_\infty} X_\infty)(\omega))| \\
&\le 2\varepsilon E^{\mathcal{F}_1} Y(\omega) + |\delta^*(x_k^*, (E^{\mathcal{F}_n} S_n)(\omega)) - \delta^*(x_k^*, (E^{\mathcal{F}_\infty} X_\infty)(\omega))| .
\end{aligned}$$

Since by (4.2.10) the last term of the right side of the preceding inequality goes to 0 when n goes to infinity, the Claim follows.

Step 2. Claim: (**) $D(B, E^{\mathcal{F}_\infty} X_\infty) \le \liminf_{n\to\infty} D(B, E^{\mathcal{F}_n} S_n)$ for all bounded closed convex subset B in E and a.e. $\omega \in \Omega$.
By (*) we have

$$\begin{aligned}
\liminf_{n\to\infty} D(B, E^{\mathcal{F}_n} S_n) &= \liminf_{n\to\infty} \sup_{x^* \in \overline{B}_{E'}} \{-\delta^*(x^*, E^{\mathcal{F}_n} S_n) - \delta^*(-x^*, B)\} \\
&\ge \sup_{x^* \in \overline{B}_{E'}} \liminf_{n\to\infty} \{-\delta^*(x^*, E^{\mathcal{F}_n} S_n) - \delta^*(-x^*, B)\} \\
&= \sup_{x^* \in \overline{B}_{E'}} \{-\limsup_{n\to\infty} \delta^*(x^*, E^{\mathcal{F}_n} S_n) - \delta^*(-x^*, B)\} \\
&= \sup_{x^* \in \overline{B}_{E'}} \{-\delta^*(x^*, E^{\mathcal{F}_\infty} X_\infty) - \delta^*(-x^*, B)\} \\
&= D(B, E^{\mathcal{F}_\infty} X_\infty)
\end{aligned}$$

for any bounded closed convex subset B in E and a.e. $\omega \in \Omega$, thus proving the *liminf part* (**). Let us prove now the *limsup part*.

Step 3. Claim: (***) $\limsup_{n\to\infty} D(B, E^{\mathcal{F}_n} S_n) \le D(B, E^{\mathcal{F}_\infty} X_\infty)$ for all bounded closed convex subset B in E and a.e. $\omega \in \Omega$.

Let $(g_k)_{k\in\mathbf{N}}$ be a Castaing's representation of $E^{\mathcal{F}_\infty}X_\infty$ in $S^1_{E^{\mathcal{F}_\infty}X_\infty}$. By (4.2.9) and ([27], Chap V, Theorem 3.1), we have

$$E^{\mathcal{F}_\infty}X_\infty(.) \subset s\text{-}liE^{\mathcal{F}_n}S_n(.)$$

almost surely on Ω. Hence there exists a negligible set M in Ω such that

$$\forall\omega \in \Omega\setminus M,\ \forall k\in\mathbf{N},\ g_k(\omega)\in s\text{-}li(E^{\mathcal{F}_n}S_n)(\omega)\,.$$

Hence

$$\forall\omega\in\Omega\setminus M,\ \forall x\in E,\ \forall k\in\mathbf{N},\ \limsup_{n\to\infty} d(x,(E^{\mathcal{F}_n}S_n)(\omega))\le d(x,g_k(\omega)),$$

which implies that

$$\limsup_{n\to\infty} d(x,(E^{\mathcal{F}_n}S_n)(\omega))\le \inf_{k\in\mathbf{N}} d(x,g_k(\omega)) = d(x,(E^{\mathcal{F}_\infty}X_\infty)(\omega))$$

a.e. on Ω. Now let B a bounded closed convex subset of E, then

$$\begin{aligned}\limsup_{n\to\infty} D(B,(E^{\mathcal{F}_n}S_n)(\omega)) &= \limsup_{n\to\infty}\inf_{x\in B} d(x,(E^{\mathcal{F}_n}S_n)(\omega))\\ &\le \inf_{x\in B}\limsup_{n\to\infty} d(x,(E^{\mathcal{F}_n}S_n)(\omega))\\ &\le \inf_{x\in B} d(x,(E^{\mathcal{F}_\infty}X_\infty)(\omega))\\ &= D(B,(E^{\mathcal{F}_\infty}X_\infty)(\omega))\end{aligned}$$

almost surely on Ω. Hence (4.2.3) follows from (**) and (***).

Remark. The preceding result holds for a decreasing sequence $(\mathcal{B}_n)_{n\in\mathbf{N}^*}$ of sub-σ-algebras of $\mathcal{F}$ with $\mathcal{B}_\infty := \bigcap_n \mathcal{B}_n$. In this case we assume that $E^{\mathcal{B}_\infty}Y<\infty$ and $|X_n|\le Y$ for all $n\in\mathbf{N}^*$ and a.e. $\omega\in\Omega$. Then

$$\lim_{n\to\infty} D(B,\ E^{\mathcal{B}_n}[\frac{1}{n}\sum_{j=1}^{n}X_{\gamma(j)}]) = D(B,\ E^{\mathcal{B}_\infty}X_\infty)$$

for any bounded closed convex subsets of E and a.e. $\omega\in\Omega$.

To end this paper we will discuss some Banach-Saks properties with respect to a RMS (a_{pq}).

The remainder of this section is borrowed from Benabdellah ([6], [7]).

Let E be a Banach space. Let $a = (a_{pq})$ be a RMS. The Banach space E has the *Banach-Saks* (resp. *weak Banach-Saks*) *property with respect* to the RMS (a_{pq}) if any bounded (resp. weakly null) sequence in E, has a summable subsequence with respect to (a_{pq}) (cf. [24], p. 75; [26], p. 232). Analyzing Theorem 1.2 and 1.3 reveals that theses properties characterize relative weakly compact and conditionally weakly compact subsets in E. Hence it is noteworthy to study theses properties and their implications on convergence problems for bounded sequences in $L^1_E(\mu)$.

We need first a lemma.

Lemma 4.3. *Let H be a Hilbert space and (a_{pq}) be a RMS such that*

$$\lim_{n\to\infty} \sum_{q=0}^{\infty} |a_{pq}|^2 = 0 \qquad (*)$$

If (x_n) is a weakly null sequence in H, then there exists $\varphi \in Si(\mathbf{N})$ such that

$$\lim_{p\to\infty} \sup_{\psi\in \mathrm{Si}(\mathbf{N})} \|\sum_{q=0}^{\infty} a_{pq} x_{\varphi\circ\psi(q)}\| = 0\,.$$

Proof. W.l.o.g., we may suppose that $\|x_n\| \le 1$ for all n. Let $(\varepsilon_n)_{n\ge 1}$ be a decreasing sequence in $\mathbf{R}^{+*}$ such that $\sum_{n=1}^{\infty} \varepsilon_n^2 < +\infty$. Set $M = \sup_p \sum_{q=0}^{\infty} |a_{pq}| < +\infty$ and $n_0 = 0$. Choose $n_1 > n_0$ such that

$$|\langle x_{n_0}, x_{n_1}\rangle| < \frac{\varepsilon_1}{M}$$

Take $n_2 > n_1$ such that

$$|\langle x_{n_0}, x_{n_2}\rangle| < \frac{\varepsilon_2}{M} \text{ and } |\langle x_{n_1}, x_{n_2}\rangle| < \frac{\varepsilon_2}{M}$$

Then by induction, there exists a finite sequence with $n_k > n_{k-1} > \cdots > n_0$ such that

$$\forall j < k,\ |\langle x_{n_j}, x_{n_k}\rangle| < \frac{\varepsilon_k}{M}$$

Take $\varphi(k) := n_k$, $\forall k$. We shall show that φ has the desired property. Let $\psi \in \mathrm{Si}(\mathbf{N})$. For every $k \in \mathbf{N}$, we have

$$\begin{aligned}\|\sum_{i=0}^{\infty} a_{ki}x_{\varphi\circ\psi(i)}\|^2 &= \sum_{i=0}^{\infty}|a_{ki}|^2\|x_{\varphi\circ\psi(i)}\|^2\\ &\quad + 2\sum_{j<l} a_{kj}a_{kl}\langle x_\varphi\circ\psi(j), x_{\varphi\circ\psi(l)}\rangle\\ &\leq \sum_{i=0}^{\infty}|a_{ki}|^2 + 2\sum_{j<l}|a_{kj}a_{kl}|\frac{\varepsilon_{\varphi\circ\psi(l)}}{M}\\ &\leq \sum_{i=0}^{\infty}|a_{ki}|^2 + \frac{2}{M}\sum_{l=1}^{\infty}\sum_{j=0}^{l-1}|a_{kj}a_{kl}|\varepsilon_l\end{aligned}$$

since $\varepsilon_{\varphi\circ\psi(l)} \leq \varepsilon_l$, $\forall l$. On the other hand by Hölder inequality

$$\begin{aligned}\sum_{l=1}^{\infty}\sum_{j=0}^{l-1}|a_{kj}a_{kl}|\varepsilon_l &= \sum_{l=1}^{\infty}|a_{kl}|\varepsilon_l(\sum_{j=0}^{l-1}|a_{kj}|) \leq \sum_{l=1}^{\infty}|a_{kl}|\varepsilon_l M\\ &\leq M\sqrt{\sum_{l=1}^{\infty}\varepsilon_l^2}\sqrt{\sum_{l=1}^{\infty}|a_{kl}|^2}\end{aligned}$$

set $L := \sum_{l=1}^{\infty}\varepsilon_l^2$. Then we obtain

$$\|\sum_{i=0}^{\infty} a_{ki}x_{\varphi\circ\psi(i)}\|^2 \leq \sum_{i=0}^{\infty}|a_{ki}|^2 + 2\sqrt{L}\sqrt{\sum_{l=1}^{\infty}|a_{kl}|^2}$$

Since by our assumption $\lim_{p\to\infty}\sum_{q=0}^{\infty}|a_{pq}|^2 = 0$, the assertion we are after follows from the preceding inequality.

Remark 4.4. Let us consider the two following (RMS):

$$a_{pq} = \begin{cases}\frac{1}{p+1} & \text{if } 0 \leq q \leq p\\ 0 & \text{if } q > p\end{cases}$$

$$b_{pq} = \begin{cases}2^{p-q} & \text{if } q > p\\ 0 & \text{if } q \leq p\end{cases}$$

It is easy to check that (a_{pq}) and (b_{pq}) are (RMS). Moreover, for all p, we have

$$\sum_{q=0}^{\infty}|a_{pq}|^2 = \frac{1}{p+1}$$

$$\sum_{q=0}^{\infty} |b_{pq}|^2 = \sum_{q>p} 4^{p-q} = \frac{1}{3} > 0$$

Then (a_{pq}) satisfies the condition

$$\lim_{p\to\infty} \sum_{q=0}^{\infty} |a_{pq}|^2 = 0 \qquad (*)$$

whereas (b_{pq}) does not satisfy $(*)$.

Now it is worth to observe that the (RMS) which satisfy the condition $(*)$ are those for which the spaces $L^1(S, \Sigma, \nu)$, where (S, Σ, ν) is a Probability space, have the weak Banach-Saks property. Indeed let $a = (a_{pq})$ be a RMS that does not satisfy $(*)$.

Let $\Omega = [0, 1]$ and μ be the Lebesgue measure on Ω. Let us consider the sequence (r_n) of Rademacher functions on $[0, 1]$. It is well-known that (r_n) is an orthonormal system in the Hilbert space $L^2([0, 1])$ and $r_n \to 0$ for $\sigma(L^1, L^\infty)$ topology. Suppose by contradiction that there exists a subsequence (r_{n_k}) of (r_n) which is summable with respect to the RMS (a_{pq}) in $L^1([0, 1])$. Then the sequence (s_p) with $s_p := \sum_{q=0}^{\infty} a_{pq} r_{n_q}$ converges to 0 for the norm of L^1, hence converges to 0 in measure. Since (s_p) is uniformly integrable in $L^2([0, 1])$, $s_p \to 0$ for the norm of $L^2([0, 1])$. As (r_n) is an orthonormal system in $L^2([0, 1])$, we deduce that

$$\|s_p\|_2^2 = \| \sum_{q=0}^{\infty} a_{pq} r_{n_q} \|_2^2 = \sum_{q=0}^{\infty} |a_{pq}|^2$$

This contradicts the fact that $a = (a_{pq})$ does not satisfy $(*)$. Hence $L^1([0, 1])$ does not satisfy the weak Banach-Saks property with respect to the RMS (a_{pq}).

Now we are able to produce the following result which generalizes the Szlenk's one to (a_{pq})-summability in L^1_H where H is a Hilbert space.

Theorem 4.5. *Let H be a Hilbert space. Let $a = (a_{pq})$ be a RMS.*

(1) *If (a_{pq}) satisfies the property*

$$\lim_{p\to\infty} \sum_{q=0}^{\infty} |a_{pq}|^2 = 0 \qquad (*)$$

then, for any weakly null sequence (u_n) in $L^1_H(\mu)$, there exists $\psi \in Si(\mathbb{N})$ such that

$$\lim_{p\to\infty} \sup_{\varphi\in \mathrm{Si}(\mathbf{N})} \Big\|\sum_{q=0}^{\infty} a_{pq} u_{\psi\circ\varphi(q)}\Big\|_1 = 0 .$$

(2) *Conversely, if all the spaces $L^1_{\mathbf{R}}(S,\Sigma,\nu)$ have the weak Banach-Saks property with respect to the RMS $a = (a_{pq})$, then a satisfies*

$$\lim_{p\to\infty} \sum_{q=0}^{\infty} |a_{pq}|^2 = 0 . \qquad (*)$$

The assertion (2) follows from the above remark 4.4.

Proof. We shall divide the proof in two steps.

Step1. Claim: For any $\varepsilon > 0$, there exists $\psi \in \mathrm{Si}(\mathbf{N})$ such that

$$\limsup_{p\to\infty} \sup_{\varphi\in \mathrm{Si}(\mathbf{N})} \Big\|\sum_{q=0}^{\infty} a_{pq} u_{\psi\circ\varphi(q)}\Big\|_1 \le \varepsilon .$$

W.l.o.g. we may suppose that $\|u_n\|_1 \le 1$ for all n.

Let $M > \max(1, \sup_p \sum_{q=0}^{\infty} |a_{pq}|)$ and let $\varepsilon > 0$. As (u_n) is uniformly integrable, there is $\alpha > 0$ such that

$$\sup_n \int_{[\|u_n\|\ge\alpha]} \|u_n\| d\mu \le \frac{\varepsilon}{3M}$$

Set $A_n := [\|u_n\| \ge \alpha]$, $u'_n := 1_{A_n} u_n$ and $u''_n := 1_{A_n^c} u_n$. Since $\|u''_n\| \le \alpha$ a.e., there exists $v \in L^\infty_H(\mu)$ such that $\|v\| \le \alpha$ a.e. and a subsequence $(u''_{\psi(k)})$, $\psi \in \mathrm{Si}(\mathbf{N})$, such that $(u''_{\psi(k)})$ converges $\sigma(L^\infty_H, L^1_H)$ to v. Hence $u'_{\psi(k)} = u_{\psi(k)} - u''_{\psi(k)}$ converges $\sigma(L^1_H, L^\infty_H)$ to $-v$. Moreover it is obvious that, $\forall k$, $\|u'_{\psi(k)}\|_1 \le \frac{\varepsilon}{3M}$, hence $\|v\|_1 \le \frac{\varepsilon}{3M}$. As $(u''_{\psi(k)} - v)\, \sigma(L^2_H, L^2_H)$ converges to 0, then in view of Lemma 4.3, we may suppose that

$$\lim_{p\to\infty} \sup_{\sigma\in \mathrm{Si}(\mathbf{N})} \Big\|\sum_{q=0}^{\infty} a_{pq} (u''_{\psi\circ\sigma(q)} - v)\Big\|_2 = 0 .$$

There is $p_\varepsilon \in \mathbf{N}$ such that $p \ge p_\varepsilon$ implies

$$\sup_{\sigma\in \mathrm{Si}(\mathbf{N})} \Big\|\sum_{q=0}^{\infty} a_{pq} (u''_{\psi\circ\sigma(q)} - v)\Big\|_2 \le \frac{\varepsilon}{3}$$

Then for all $p \ge p_\varepsilon$ and $\varphi \in \mathrm{Si}(\mathbf{N})$, we have

$$\|\sum_{q=0}^{\infty} a_{pq}u_{\psi\circ\varphi(q)}\|_1 \leq \|\sum_{q=0}^{\infty} a_{pq}u'_{\psi\circ\varphi(q)}\|_1 + \|\sum_{q=0}^{\infty} a_{pq}(u''_{\psi\circ\varphi(q)} - v)\|_1$$

$$+ \|(\sum_{q=0}^{\infty} a_{pq})v\|_1$$

$$\leq \sum_{q=0}^{\infty} |a_{pq}|\frac{\varepsilon}{3M} + |\sum_{q=0}^{\infty} a_{pq}|.\|v\|_1 + \|\sum_{q=0}^{\infty} a_{pq}(u''_{\psi\circ\varphi(q)} - v)\|_2$$

$$\leq M\frac{\varepsilon}{3M} + M\frac{\varepsilon}{3M} + \frac{\varepsilon}{3} = \varepsilon$$

thus proving our claim.

Step 2. Let (u_n) be a weakly null sequence in $L^1_H(\mu)$ with $\|u_n\|_1 \leq 1$, $\forall n$. According to the first step, we find, by induction, $\varphi_0, \ldots, \varphi_k$ in Si(N) such that

$$\limsup_{p\to\infty} \sup_{\sigma\in \mathrm{Si}(\mathbf{N})} \|\sum_{q=0}^{\infty} a_{pq}u_{\psi_k\circ\sigma(q)}\|_1 \leq 2^{-k} \tag{4.5.1}$$

with $\psi_k := \varphi_0 \circ \ldots \circ \varphi_k$. Let us consider the diagonal sequence $\psi(k) := \psi_k(k), \forall k$ and let us show that $\psi(.)$ has the required property in Theorem 4.5. Let $\theta \in \mathrm{Si}(\mathbf{N})$ and $k \in \mathbf{N}$ be fixed. Define

$$\varphi(n) := \begin{cases} n & \text{if } n \leq k \\ \varphi_{k+1} \circ \ldots \circ \varphi_{\theta(n)}(\vartheta(n)) & \text{if } n \geq k+1 \end{cases}$$

Then $\varphi \in \mathrm{Si}(\mathbf{N})$ and, $\forall q \geq k+1$, $\psi \circ \theta(q) = \psi_k \circ \varphi(q)$. Moreover we have

$$\|\sum_{q=0}^{\infty} a_{pq}u_{\psi\circ\theta(q)}\|_1 \leq \|\sum_{q=0}^{k} a_{pq}u_{\psi(\theta(q))}\|_1 + \|\sum_{q=k+1}^{\infty} a_{pq}u_{\psi_k\circ\theta(q)}\|_1$$

$$\sum_{q=0}^{k} |a_{pq}| + \|\sum_{q=0}^{\infty} a_{pq}u_{\psi_k\circ\varphi(q)} - \sum_{q=0}^{k} a_{pq}u_{\psi_k\circ\varphi(q)}\|_1$$

$$\leq 2\sum_{q=0}^{k} |a_{pq}| + \sup_{\sigma\in\mathrm{Si}(\mathbf{N})} \|\sum_{q=0}^{\infty} a_{pq}u_{\psi_k\circ\sigma(q)}\|_1 \tag{4.5.2}$$

By (4.5.1) and (4.5.2), it follows that

$$\limsup_{p\to\infty} \sup_{\theta\in\mathrm{Si}(\mathbf{N})} \|\sum_{q=0}^{\infty} a_{pq}u_{\psi\circ\theta(q)}\|_1 \leq 2\lim_{p\to\infty} \sum_{q=0}^{k} |a_{pq}| + 2^{-k} = 2^{-k} \tag{4.5.3}$$

Since k is a arbitrary, assertion (1) follows immediately from (4.5.3).

Corollary 4.6. *Let H be a Hilbert space and $a = (a_{pq})$ be a RMS which satisfies property $(*)$. Let (u_n) be a bounded sequence in $L^1_H(\mu)$. Then there exist $\psi \in Si(\mathbf{N})$ and $u \in L^1_H(\mu)$ such that, for all $\varphi \in Si(\mathbf{N})$, the sequence $(\sum_{q=0}^{\infty} a_{pq} u_{\psi\circ\varphi(q)})_p$ converges in measure to u.*

Proof. By Theorem 1.12, we may suppose that there exists an increasing sequence (A_n) in $\mathcal{F}$ with $\lim_{n\to\infty} \mu(A_n^c) = 0$ such that $(1_{A_n} u_n)$ $\sigma(L^1, L^\infty)$-converges to $u \in L^1_H(\mu)$ and $(1_{A_n^c} u_n)$ converges μ-a.e. to 0. Now we apply Theorem 4.5 to the weakly null sequence $v_n = 1_{A_n} u_n - u$. Then there exists $\psi \in \mathrm{Si}(\mathbf{N})$ such that, $\forall \varphi \in \mathrm{Si}(\mathbf{N})$, the sequence $(\sum_{q=0}^{\infty} a_{pq} u_{\psi\circ\varphi(q)})_p$ converges in $L^1_H(\mu)$ to 0. Let $\varphi \in \mathrm{Si}(\mathbf{N})$ be fixed and set $\theta = \psi \circ \varphi$. Then

$$\sum_{q=0}^{\infty} a_{pq} u_{\theta(q)} = \sum_{q=0}^{\infty} a_{pq} 1_{A_{\theta(q)}} u_{\theta(q)} + \sum_{q=0}^{\infty} a_{pq} 1_{A^c_{\theta(q)}} u_{\theta(q)}$$

$$= \Big(\sum_{q=0}^{\infty} a_{pq}\Big) u + \sum_{q=0}^{\infty} a_{pq} v_{\theta(q)} + \sum_{q=0}^{\infty} a_{pq} 1_{A^c_{\theta(q)}} u_{\theta(q)}$$

As (a_{pq}) is a RMS, the sequence $((\sum_{q=0}^{\infty} a_{pq})u)_p$ pointwisely converges to u and the sequence $(\sum_{q=0}^{\infty} a_{pq} 1_{A^c_{\theta(q)}} u_{\theta(q)})_p$ converges μ-a.e. to 0. Hence $(\sum_{q=0}^{\infty} a_{pq} u_{\theta(q)})_p$ converges in measure to u.

Open Problem. It might be interesting to obtain a multivalued version of Komlos theorem for integrable closed convex random sets X_n, that is $d(0, X_n(.))$ is bounded in $L^1_{\mathbf{R}}$ and SLLN for closed convex valued martingales in separable super-reflexive Banach spaces. The first problem for convex weakly compact random sets is studied in this paper (cf. Theorem 4.2), whereas the second one was stated in Ezzaki [27] for convex weakly compact valued martingales in p-smooth separable Banach spaces. To end this paper we would like to address the following question: What happens if one replaces the Cesaro sums

$$\frac{1}{n}\sum_{j=1}^{n} X_j$$

in vector-valued Komlos version theorem ([29], Theorem 6) by the following ones

$$\sum_{q=0}^{\infty} a_{pq} X_q$$

where (X_n) is a bounded sequence in L^1_E and $(a_{pq})_{(p,q)\in \mathbf{N}\times\mathbf{N}}$ is a positive regular method of summability satisfying suitable conditions (see [24]; [26], Cor. 2.17). Taking account into the above mentioned results, we suspect that

$$(\sum_{q=0}^{\infty} a_{pq} X_{\gamma(q)})_p$$

converges in probability to $X_\infty \in L^1_E$, that is

$$\lim_{p\to\infty} \|X_\infty - \sum_{q=0}^{\infty} a_{pq} X_{\gamma(q)}\| = 0$$

in probability where $(X_{\gamma(n)})$ is a subsequence of (X_n). This conjecture is a sort of Banach-Saks property for bounded sequences in L^1_E with respect to a regular method of summability.

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Département de Mathématiques
Université Montpellier II
F - 34095, Montpellier cedex 5
France.