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Short Communication

A SMOOTHNESS CHARACTERIZATION FOR HYPERBOLIC TRIGONOMETRIC POLYNOMIAL APPROXIMATIONS

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1. In this note we give a characterization of smoothness properties which govern a preassigned degree of the best multivariate approximation by trigonometric polynomials (t. p.) with frequencies from so called hyperbolic crosses (h. c.) for the case when these h. c. are, in general, the intersection of an infinite number of single ones. Being different from the case of finite intersections of single h. c., this case can not be reduced to the case of single h. c. Our note is directly related to [1-4]. In particular, the results in this note generalize those in [4].

2. Let A be a compact subset of $\mathbf{R}^d_+ := \{x \in \mathbf{R}^d : x_j \geq 0\}$. The set

$$\Gamma_A(t) := \{k \in \mathbf{Z}^d : \prod_{j \in J_\alpha} |k_j|^{\alpha_j} < t,\, \alpha \in A\},\ t > 0,$$

is called hyperbolic cross where $J_\alpha := \{j : \alpha_j \neq 0\}$. We let

$$E^A_t(f)_p := \inf_{g \in \mathcal{P}^A_t} \|f - g\|_p\,,\ 1 \leq p \leq \infty, \tag{1}$$

denote the error in the best $L_p(\mathbf{T}^d)$ - approximation of f by elements from $\mathcal{P}^A_t$ the $L_p(\mathbf{T}^d)$-closure of the span of the harmonic $e^{i\langle k, \cdot\rangle}$, $k \in \Gamma_A(t)$, where $\mathbf{T}^d := [-\pi, \pi]^d$ is the d-dimensional torus and $\|\cdot\|_p$ the p-integral norm of $L_p(\mathbf{T}^d)$ with the change to the sup norm when $p = \infty$.

Let the vector $\alpha^* = (\alpha^*_1, \dots, \alpha^*_d)$ be defined by

$$\alpha^*_j := \max_{\alpha \in A} \alpha_j\,.$$

The most important case of the approximation (1) is that when the h.c. $\Gamma_A(t)$ is finite subset of $\mathbf{Z}^d$ for each $t > 0$, i.e. the subspace $\mathcal{P}_t^A$ consists of all t.p. with frequencies from Γ_t^A. This occurs if and only if the vector α^* has only positive coordinates. However, we would like to emphasize that the results of the present note will be stated without any requirement on finiteness of $\Gamma_A(t)$.

We are interested in characterization of the smoothness properties of f which give a preassigned degree of $E_t^A(f)_p$. We let Φ denote the set of all functions $\varphi \in C([0, 1])$ such that $\varphi(t) > 0$ for $t > 0$, $\varphi(0) = 0$, and φ is nondecreasing on $[0, \tau]$ for some $0 < \tau \le 1$. The degree of $E_t^A(f)_p$ which we will consider, are of the form $\varphi(1/t)$ for $\varphi \in \Phi$, satisfying certain conditions of regularity (see Conditions (BS) and (Z_θ) below). Let H_p^A be the space of all functions $f \in L_p(\mathbf{T}^d)$ for which the quasinorm

$$|f|_{H_p^A} := \sup_{h\in[0,\pi]^d} \|\Delta_h^r f\|_p/\Omega_A(h)$$

is finite for some r with $r_j > \alpha_j^*$ where Δ_h^r is the r-th mixed difference operator (see a definition below),

$$\Omega_A(h) := \inf_{\alpha\in A} \prod_{j=1}^d h_j^{\alpha_j}.$$

If α^* is a vector with positive coordinates, then being a degree of the above mentioned form, the function $\varphi_A(1/t) := t^{-1}\{\log^{d-1} t\, \omega^s(1/t)\}^{1/p^*}$ is the degree of $E_t^B(f)_p$, $1 < p < \infty$, on the unit ball of H_p^A, where $p^* := \min(p, 2)$, s is a certain nonnegative integer not greater than $d-1$, $\omega(\cdot)$ a certain modulus of continuity, B a certain compact subset of $\mathbf{R}_+^d$, which are constructed from A (see [2]).

Let us introduce spaces of functions f with common degree of $E_t^A(f)_p$. If $\varphi \in \Phi$ and $0 < q \le \infty$, we let $\mathcal{E}_{p,q}^{A,\varphi}$ denote the space of all functions $f \in L_p(\mathbf{T}^d)$ such that quasinorm

$$|f|_{\mathcal{E}_{p,q}^{A,\varphi}} := \begin{cases} \left(\sum_{n=0}^{\infty}\{E_{2^n}^A(f)_p/\varphi(2^{-n})\}^q\right)^{1/q}, & q<\infty \\ \sup_{0\le n<\infty} \{E_{2^n}^A(f)_p/\varphi(2^{-n})\}, & q=\infty \end{cases}$$

is finite.

3. We now give definitions of new moduli of smoothness. For a nonnegative integer r, the univariate difference operator Δ_h^r, $h \in \mathbf{T}$, is defined inductively by $\Delta_h^r := \Delta_h^1 \Delta_h^{r-1}$, starting from the operators

$$\Delta_h^0 f := f\,, \quad \Delta_h^1 f := f(\cdot + h/2) - f(\cdot - h/2)\,.$$

The univariate integral operator I_h^r is defined in the same way, starting from the operators

$$I_h^0 f := f\,, \quad I_h^1 := h^{-1} g\,,$$

where g is the primitive with zero mean value of f, i.e.

$$g' = f\,, \quad \int_{-\pi}^{\pi} g(x)dx = 0\,.$$

For $r \in \mathbf{Z}_+^d := \{k \in \mathbf{Z}^d : k_j \geq 0\}$, we let the multivariate mixed difference operator Δ_h^r, $h \in \mathbf{T}^d$, be defined by

$$\Delta_h^r f := \Delta_{h_1}^{r_1} \Delta_{h_2}^{r_2} \cdots \Delta_{h_d}^{r_d} f\,,$$

where the univariate operator $\Delta_{h_j}^{r_j}$ is applied to the variable x_j. The multivariate mixed integral operator I_h^r is defined similarly.

For a pair $\gamma = (r, \beta) \in \mathbf{Z}_+^d \times \mathbf{Z}_+^d$, the operator $D_t^\gamma = D_t^\gamma(A)$ is defined by

$$D_t^\gamma f := \int_{\delta V_A(t)} \Delta_h^r I_h^\beta f \prod_{j \in J_{\alpha^*}} h_j^{-1} dh\,,$$

where $\delta V_A(t) := V_A(t) \setminus V_A(t/2)$ is the augmented of the hyperbolic set

$$V_A(t) := \{h \in \mathbf{T}^d : h_j \geq 0,\ \prod_{j \in J_\alpha} h_j^{\alpha_j} < 2t,\ \alpha \in A\},\ t > 0\,.$$

We define the modulus of smoothness $\Omega_A^\gamma(f, \cdot)_p$ by

$$\Omega_A^\gamma(f, \delta)_p := \sup_{t \leq \delta} \|D_t^\gamma f\|_p\,, \quad \delta > 0\,.$$

A slight modification of this definition was given in [4] for single h. c.

We now introduce Besov spaces of common smoothness. If $\varphi \in \Phi$ and $0 < q \leq \infty$, we let $\mathcal{B} := \mathcal{B}_{p,q}^{A,\gamma,\varphi}$ denote the Besov space of all functions $f \in L_p(\mathbf{T}^d)$ such that the quasinorm

$$|f|_B := \begin{cases} \Big(\sum_{n=0}^{\infty} \{\Omega_A^\gamma(f, 2^{-n})_p/\varphi(2^{-n})\}^q \Big)^{1/q}, & q < \infty \\ \sup_{0\le n<\infty} \{\Omega_A^\gamma(f, 2^{-n})_p/\varphi(2^{-n})\}, & q = \infty \end{cases}$$

is finite.

4. We will require some conditions of regularity on φ. Namely, we say that $\varphi \in \Phi$ satisfies Condition (BS) if

$$\int_0^t \varphi(x)\,\frac{dx}{x} \le C\,\varphi(t)\,,$$

and Condition (Z_θ), $\theta > 0$, if

$$\int_0^t \varphi(x)\,x^{-\theta}\,\frac{dx}{x} \le C\,\varphi(t)\,t^{-\theta}\,.$$

We will need also some restriction on $\gamma = (r, \beta)$ for the modulus of smoothness $\Omega_A^\gamma(f, .)_p$. We say that the pair $\gamma = (r, \beta)$ satisfies Condition (R) with respect to A if $J_r = J_\beta = J_{\alpha^*}$ and $1 < \beta_j < r_j$, $j \in J_{\alpha^*}$. For $\gamma = (r, \beta)$ satisfying Condition (R), we define $\rho(A, \gamma) := \min\{(r_j - \beta_j)/\alpha_j^* : j \in J_{\alpha^*}\}$ and $\nu(A, \gamma)$ as the number of $j \in J_{\alpha^*}$ such that $(r_j - \beta_j)/\alpha_j^* = \rho(A, \gamma)$. Denote by card Γ the cardinality of a set Γ and recall that $p^* := \min(p, 2)$ for $1 < p < \infty$.

Theorem 1. *Let $1 < p < \infty$, $0 < q \le \infty$, and let A be a compact subset of $\mathbf{R}_+^d$. Then for any $\theta > 0$ and any natural number $\nu \le \operatorname{card} J_{\alpha^*}$, we can constructively find a pair $\gamma = (r, \beta) \in 2\mathbf{Z}_+^d \times \mathbf{Z}_+^d$ such that*

(i) γ satisfies Condition (R) with respect to A,

(ii) $\rho = \rho(A, \gamma) \ge \theta$,

(iii) $\nu(A, \gamma) = \nu$.

Moreover, if γ is such a pair and $f \in L_p(\mathbf{T}^d)$, then there holds the Jackson type direct inequality of weak form

$$E_{2^n}^A(f)_p \le C\Big(\sum_{m=n+1}^{\infty} \{\Omega_A^\gamma(f, 2^{-m})_p\}^{p^*} \Big)^{1/p^*} \tag{2}$$

for any nonnegative number n whenever the right side is finite. In addition we have the Stechkin - Timan type inverse inequality

$$\Omega_A^\gamma(f, 2^{-n})_p \leq C\Big(\sum_{m=0}^{n} \{2^{-\rho(m-n)}(n-m)^{\nu-1} E_{2^m}^A(f)_p\}^{p^*}\Big)^{1/p^*}$$

for any natural number n.

Theorem 2. *Under the assumptions of Theorem 1, let $\varphi \in \Phi$ and φ satisfy Conditions (BS) and (Z_θ). Then for any pair $\gamma = (r, \beta)$ satisfying Conditions (i) - (ii) in Theorem 1, we have*

$$\mathcal{E}_{p,q}^{A,\varphi} = \mathcal{B}_{p,q}^{A,\gamma,\varphi}.$$

Moreover, for functions $f \in \mathcal{E}_{p,q}^{A,\varphi}$

$$|f|_{\mathcal{E}_{p,q}^{A,\varphi}} \asymp |f|_{\mathcal{B}_{p,q}^{A,\gamma,\varphi}}.$$

From Theorem 1 it follows that we can constructively find a pair $\gamma = (r, \beta)$ (with $\nu(A, \gamma) = 1$) for which there hold the inequality (2) and the inequality

$$\Omega_A^\gamma(f, 2^{-n})_p \leq C\Big(\sum_{m=0}^{n} \{2^{-\rho(m-n)} E_{2^m}^A(f)_p\}^{p^*}\Big)^{1/p^*}.$$

5. The methods employed in the proofs of Theorems 1 - 2 rest on the Littlewood - Paley theorem and Marcinkiewicz multiplier theorem and some generalizations of the discrete Hardy inequalities and, in particular, are a refinement of those in [1], [4].

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