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SYLOW p-SUBGROUPS OF FINITE DIMENSIONAL LOCALLY COMPACT GROUPS WITH A FINITE NUMBER OF CONNECTED COMPONENTS

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Abstract. *Let G be a finite dimesional locally compact group with a finite number of connected components. We prove that all the Sylow p-subgroups of G are conjugate, where p is a fixed prime number and we give also two counter-examples.*

INTRODUCTION

Sylow p-subgroups are interesting in both the abstract and topological group-theories. Many interesting results were obtained, see for examples [1], [2], [3], [4], [10], [11], [12]. One of the most interesting results is that all Sylow p-subgroups of a finite group are conjugate. It has many applications in other problems, say in Brauer theory. One of questions in which we are interested is: what kind of results are available also for topological groups? Our purpose is to study the classification of Sylow p-subgroups under conjugation.

We prove that in an arbitrary finite dimensional locally compact groups with a finite number of connected components, all the Sylow p-subgroups are conjugate one-to-another. This means that there is only one conjugation class of Sylow p-subgroup. We show then that if the conditions are not satisfied there may be counter-examples.

The main part of proofs were verified before for the group case. We use the presentation of an arbitrary locally compact group as a projective limit of Lie groups. Our technique therefore is developed in this context.

1. SYLOW p-SUBGROUPS OF FINITE DIMENSIONAL LOCALLY COMPACT GROUPS WITH FINITE NUMBER OF CONNECTED COMPONENTS

All groups we shall concern are supposed to be locally compact and we shall not repeat this later.

Lemma 1. *Let G be a group with a finite number of connected components and P a p-subgroup of G. Then, the closure $\overline{P}$ of P is compact.*

Proof. The lemma was proved in [7] for Lie group case.

Consider the general case. Following Yamabe's theorem [11], [12] in G there exists a compact normal subgroup H such that G/H is a Lie group. Then G/H is a Lie group satisfying the conditions of the lemma. One deduces that $\overline{PH/H} = P^*$ is a compact subgroup, and hence $\overline{P}$ is a compact subgroup.

Proposition 1. *Let S_p be a Sylow p-subgroup of a Lie group G. Then the connected component $(\overline{S}_p)_0$ of indentity is a torus.*

Proof. Following lemma and the well-known Cartan - Iwasawa - Maltsev theorem, we can suppose G to be a compact Lie group. If G is a linear group, then our proposition is derived from a theorem in [9]: in that case, $\overline{S}_p$ has a normal Abelian subgroup H of finite index. Thus $(\overline{S}_p)_0 \subseteq H$ and $(\overline{S}_p)_0$ is a torus.

In general case, we consider $Ad(G) = G/Z$, which is a linear Lie group, following the Lie theory and Ado theorem. Thus $\overline{S}_pZ/Z$ has a normal Abelian subgroup H^* of finite index. The preimage $H = \varphi^{-1}(H^*)$ of the natural projection $\varphi\colon G \to G/Z$ is a compact solvable normal subgroup and $(\overline{S}_pZ/Z)/H^* \cong \overline{S}_p/H$ is finite.

Therefore, the connected component H_0 of indentify $e \in H$ is a compact solvable connected group and hence H_0 is Abelian and $\overline{S}_p/H_0$ is finite, because H/H_0 is finite. One deduces that $(\overline{S}_p)_0 \subset H_0$ and $(\overline{S}_p)_0$ is a torus.

Proposition 2. *Suppose G be a compact group, P a p-subgroup of G. Then P is contained in the normalizer $N_G(T)$ in G of some maximal torus T of G.*

Remark. To prove the proposition it is enough to suppose that G is locally compact and G/G_0 is compact.

Proof. The proposition is valid for Lie group case, see [7] . In general case

$$G = \varprojlim(\{G_\beta\}, \{\varphi_{\beta\alpha}\}_{\beta>\alpha}),$$

where G_β are Lie groups, see [8].

Denote F_β the set of all the closures $\overline{H}_\beta$ of Sylow p-subgroup H_β of G_β. Because in Lie groups the Sylow p-subgroups are conjugate, so are also their closures.

Suppose $\overline{H}_\beta \in F_\beta$. Then, as said previously, there exists a maximal torus T_β of G_β such that $\overline{H}_\beta \subset N_{G_\beta}(T_\beta)$.

Denote T_β^p the Sylow p-subgroup of the torus. Then $T_\beta^p \subset H_\beta$ and therefore $(\overline{H}_\beta)_0 = T_\beta = \overline{T}_\beta^p$. Clearly

$$F_\beta = \{g_\beta \overline{H}_\beta g_\beta^{-1} \mid g_\beta \in G_\beta\}.$$

We endow F_β with the natural topology of conjugate subgroups. This means that a complete system of neighbourhoods of H_β is the set of all $W_\beta = \langle U_\beta \bar{H}_\beta \mid U_\beta$ is a neighbourhood of $G_\beta\rangle$.

Thus $F_\beta \cong G_\beta / N_{G_\beta}(\overline{H}_\beta)$ and so F_β is compact.

We define $F_\beta \supset F_\alpha$, if $\beta > \alpha$. Because in G_β the Sylow p-subgroup are conjugate, the morphism $\varphi_{\beta\alpha}\colon G_\beta \to G_\alpha$ induces $\pi_{\beta\alpha}\colon F_\beta \to F_\alpha$, i.e $\pi_{\beta\alpha}(\overline{H}_\beta) \to \varphi_{\beta\alpha}(\overline{H}_\beta)$.

It is easy to see that $\pi_{\beta\alpha}$ are continuous maps, having transitive property on indices,

i.e. $\pi_{\beta\alpha}.\pi_{\alpha\gamma} = \pi_{\beta\gamma}$ if $\beta > \alpha > \gamma$.

Thus $\mathcal{F} = (\{F_\beta\}, \{\pi_{\beta\alpha}\}_{\beta>\alpha})$ is a projective system. Let $S \in \varprojlim(\{F_\beta\}, \{\pi_{\beta\alpha}\}_{\beta>\alpha})$, then $S = (\{\overline{H}_\beta\}, \{\varphi_{\beta\alpha}\}_{\beta>\alpha})$. Pose

$$\overline{H} = \varprojlim(\{\overline{H}_\beta\}, \{\varphi_{\beta\alpha}\}_{\beta>\alpha}).$$

From the construction $\overline{H}_0$ is a maximal torus of G.

Suppose Q is a Sylow p-subgroup of G, $\varphi_\beta\colon G \to G_\beta$ the natural projections. Then

$$\overline{Q} = \varprojlim(\{\overline{Q}_\beta\}, \{\varphi_{\beta\alpha}\}_{\beta>\alpha}) \quad \text{with} \quad Q_\beta = \varphi_\beta(Q).$$

Denote by D_β the set of elements $d_\beta \in Q_\beta$ such that $d_\beta \bar{Q}_\beta d_\beta^{-1} \subset \overline{H}_\beta$. Then D_β is a closed subset of G_β. By an analogy, we have a system

$$\omega_{\beta\alpha}\colon D_\beta \to D_\alpha, \quad \forall \beta > \alpha$$

such that $\omega_{\beta\alpha}(d_\beta) = \varphi_{\beta\alpha}(d_\beta)$, $d_\beta \in D_\beta$ and $\mathcal{D} = (\{D_\beta\}, \{\omega_{\beta\alpha}\}_{\{\beta>\alpha\}})$ is a projective system.

Let $d \in \varprojlim(\{D_\beta\}, \{\omega_{\beta\alpha}\}_{\{}\beta > \alpha\})$. Then $d = (\{d_\beta\}, \{\omega_{\beta\alpha}\}_{\beta>\alpha})$ with $d_\beta \in G_\beta$ and $d\bar{Q}d^-1 \subset \overline{H}$. The proposition is proved.

Theorem 1. *Suppose that G is a finite dimensional locally compact group with a finite number of connected components. Then the Sylow p-subgroups of G are conjugate one - to - another.*

Proof. Following the well-known theorem of Cartan - Iwasawa - Maltsev, the maximal compact subgroups of G are conjugate, and therefore by virtue of Lemma 1, we can assume G to be compact.

Following [7], in G the maximal tori are conjugate and following Proposition 2, we need only to prove that in $N_G(T)$, the Sylow p-subgroups are conjugate.

First, we show that there exists in G a normal subgroup H in $N_G(T)$, such that G/H is a Lie group. Really, following [9] there exists in G a totally disconnected normal subgroup K such that G/K is a Lie group. Because G/G_0 is finite, G and G_0 are compact, as assumed , following [9], $G/(G_0 \cap K)$ is also a Lie group. The group $G_0 \cap K$ is compact and totally disconnected and therefore is contained in the center of G_0. The maximal torus of G is also contained in G_0. Then $H = G_0 \cap K \subseteq N_G(T)$ for some maximal torus T of G.

Denote $\varphi: G \to G/H$ the natural epimorphism. If T is a maximal torus of G, then $T' = \varphi(T)$ is a maximal torus of G/H. Because $H \subset N_G(T)$ then $\varphi^-1(N_{G(T)}(T')) = N_G(T)$. We have the isomorphic finite groups

$$N_G(T).H/T.H \cong N_{G/H}(T')/T' \cong N_G(T)/T\,.$$

In Lie group $N_{G/H}(T')$ the Sylow p-subgroup are conjugate, following [7]. Therefore, the same fact is valid in $N_G(T)$. But the Sylow p-subgroups of G are the same of $N_G(T)$. The theorem is proved.

2. COUNTER - EXAMPLES

We construct in this section two counter - examples for Theorem 1 to show that the conditions of the theorem are really needed. The first example gives us a totally disconnected compact group G such that

G/G_0 is infinite and in G the Sylow 2-subgroups are divided into an infinite number of conjugation classes. The second example will show an analogous situation when $\dim G = \infty$.

Theorem 2. *Let $G_{\alpha\beta} := \{a_{\alpha\beta}, b_{\alpha\beta}\}\alpha, \beta = 1, 2, \ldots$ be the order $2^{\alpha+2}$ groups defined by the relation*

$$a^{2^{\alpha+1}} = b_{\alpha\beta}^2 = e\,,$$

$$a_{\alpha\beta} b_{\alpha\beta} = b_{\alpha\beta} a_{\alpha\beta}^{-1}\,.$$

Then, in the topological direct product

$$G = \prod_{\beta=1}^{\infty} F_\beta \text{ where } F_\beta = \prod_{\alpha=1}^{\infty} G_{\alpha\beta}$$

there are infinitely many classes of conjugate Sylow 2-subgroups.

Proof. Les us denote

$$g_\beta^0 = (b_{1\beta} a_{1\beta}^2, b_{2\beta} a_{2\beta}^2, \ldots, b_{m\beta} a_{m\beta}^2, \ldots)$$

for $\beta = 1, 2, \ldots, n$; and $h_\beta = (b_{1\beta} a_{1\beta}, b_{2\beta} a_{2\beta}, \ldots)$. We introduce some symbols

$$g_0 = (g_1^0, g_2^0, \ldots, g_\alpha^0, \ldots)$$

$$g_1 = (b_1^0, g_2^0, \ldots, g_\alpha^0, \ldots)$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$g_i = (h_1, h_2, \ldots h_i, g_{i+1}^0, \ldots, g_\alpha^0, \ldots).$$

From the defining relations between generators, we have

$$g_t^0 = (e) = (1, 1, \ldots), \forall t = 0, 1, 2, \ldots$$

Remark that g_β and h_β are not cojugate in F_β. In fact

$$a_{\alpha\beta}^{-k} b_{\alpha\beta}^{-1} b_{\alpha\beta} a_{\alpha\beta}^2 b_{\alpha\beta} a_{\alpha\beta}^k = a_{\alpha\beta}^{-k} b_{\alpha\beta} a_{\alpha\beta}^k a_{\alpha\beta}^{-2} = b_{\alpha\beta} a_{\alpha\beta}^{2\alpha-2} \neq b_{\alpha\beta} a_{\alpha\beta}\,.$$

Because $a_{\alpha\beta}^{2k-3} \neq 1$

Denote S_t the Sylow 2-subgroup of G, containing g_t. All S_t are different one-from-another. Indeed, if $g_{m_1}, g_{m_2} \in S_m$, for some m_1 and $m_2 > m_1$, then

$$g_{m_2}^{-1} g_{m_1} = (\dots, h_{m_1}^{-1} g_{m_2}^0, e, e, \dots) \in S_m .$$

But

$$h_{m_2}^{-1} g_{m_2}^0 = (a_{1m_1}, a_{2m_2}, \dots, a_{km_k}, \dots)$$

has infinite order. This contradicts the assumption that S_m is a Sylow 2 – subgroup. We show now, that $g^{-1} g_{m_1} g \notin S_{m_2} \forall g \in G$, if $m_1 \neq m_2$. Suppose $g^{-1} g_{m_1} g \in S_{m_2}$. Then we have

$$b_{\alpha\beta} a_{\alpha\beta}^k b_{\alpha\beta} a_{\alpha\beta}^2 a_{\alpha\beta}^{-k} b_{\alpha\beta}^{-1} = a_{\alpha\beta}^{-2k+2} b_{\alpha\beta}^{-1}, a_{\alpha\beta}^k b_{\alpha\beta} a_{\alpha\beta}^2 a_{\alpha\beta}^{-k} = b_{\alpha\beta} a_{\alpha\beta}^{-2k+2} .$$

It follows from the first relation that

$$g g_{m_1} g^{-1} g_{m_2} = (\dots, a_{1m_2}^{3-2k}, a_{2m_2}^{3-2k}, \dots, a_{nm_2}^{3-2k}, \dots) .$$

From the second relation we have

$$g_{m_2}^{-1} g g_{m_1} g^{-1} = (\dots a_{1m_1}^{1-2k}, a_{2m_1}^{1-2k}, \dots, a_{nm_1}^{1-2k}, \dots) .$$

In every case we have an infinite order element. Thus $g g_{m_1} g^{-1} \notin S_{m_2}, \forall g \in G$, if $m_1 \neq m_2$. This means that the Sylow 2-subgroups S_{m_1} and S_{m_2} are not conjugate if $m_1 \neq m_2$. The theorem is proved.

Theorem 3. *Let $G_{\alpha\beta} = U(2)$ be the unitary group and*

$$G = \prod_{\alpha,\beta=1}^{\infty} G_{\alpha\beta} ,$$

the topologycal direct product. Then in G there are infinitely many non-conjugate Sylow 2-subgroups.

Proof. We first remark that U(2) contains the infinite diedral group

$$D_{\alpha\beta} = \left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{bmatrix} \quad \varepsilon^{2^\alpha} = 1, \alpha = 1, 2, \dots \right\} .$$

We choose g_t in the same way as in the proof of Theorem 2, and use the some notion S_t, as the Sylow 2-subgroup containing g_t and the direct product of goups $D_{\alpha\beta}$.

We show now that if $g \in G$, $m_1 \neq m_2$ and $gS_{m_1}g^{-1} = S_{m_2}$ then

$$g \in \prod_{\alpha,\beta=1}^{\infty} G_{\alpha\beta},$$

the topological direct product of $D_{\alpha\beta}$.

Indeed, $g = (g_{\alpha\beta})$ and upto a scalar factor, either

$$g_{\alpha\beta} = \begin{bmatrix} h_{\alpha\beta} & 0 \\ 0 & h_{\alpha\beta}^{-1} \end{bmatrix} \text{ or } g_{\alpha\beta} = \begin{bmatrix} 0 & h_{\alpha\beta} \\ h_{\alpha\beta}^{-1} & 0 \end{bmatrix}.$$

In the first case

$$\begin{bmatrix} h_{\alpha\beta} & 0 \\ 0 & h_{\alpha\beta}^{-1} \end{bmatrix} \begin{bmatrix} 0 & \varepsilon \\ \varepsilon^{-1} & 0 \end{bmatrix} \begin{bmatrix} h_{\alpha\beta}^{-1} & 0 \\ 0 & h_{\alpha\beta} \end{bmatrix} = \begin{bmatrix} 0 & h_{\alpha\beta}^{2}\varepsilon \\ h_{\alpha\beta}^{-2}\varepsilon^{-1} & 0 \end{bmatrix} \in S_{m_2}^{\alpha\beta},$$

where $S_{m_2} = (S_{m_2}^{\alpha\beta})$. From the construction $S_{m_2}^{\alpha\beta} = D_{\alpha\beta}$.
Thus

$$\begin{bmatrix} 0 & h_{\alpha\beta}^{2}\varepsilon \\ h_{\alpha\beta}^{-2}\varepsilon^{-1} & 0 \end{bmatrix} \begin{bmatrix} 0 & \varepsilon \\ \varepsilon^{-1} & 0 \end{bmatrix} = \begin{bmatrix} h_{\alpha\beta}^{2} & 0 \\ 0 & h_{\alpha\beta}^{-2} \end{bmatrix} \in S_{m_2}^{\alpha\beta}.$$

In virtue of the structure of direct product, we have

$$g_{\alpha\beta} = \begin{bmatrix} h_{\alpha\beta} & 0 \\ 0 & h_{\alpha\beta}^{-1} \end{bmatrix} \in S_{m_2}^{\alpha\beta}.$$

By analogy, in the second case we also have

$$g_{\alpha\beta} = \begin{bmatrix} 0 & h_{\alpha\beta} \\ h_{\alpha\beta}^{-1} & 0 \end{bmatrix} \in S_{m_2}^{\alpha\beta}.$$

We have then

$$g \in \prod_{\alpha,\beta=1}^{\infty} D_{\alpha\beta}.$$

One reduces the proof of theorem to the same assertion for

$$G^1 = \prod_{\alpha,\beta=1}^{\infty} D_{\alpha\beta}.$$

The last assertion can be proved in the same way as Theorem 2. The proof is now complete .

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