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THE POPULATION AND ITS RENEWAL FUNCTION[1]

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Abstract. *A concept of "the population" and of corresponding renewal function are constructed in the work as a generalizing renewal processes, non-homogeneous renewal models and ones of renewals by partly failed objects in the discrete case.*

1. INTRODUCTION

Analysing a process of replacements (renewals) of objects in a set A in the time $t \in (0, \infty)$, we can describe it by three sequences of nonnegative random variables:

$$\{\xi_n : n \geq 0\}, \quad \{\nu_n : n \geq 0\}, \quad \{\tau_n^i : n \geq 0\} \quad (i = 1, 2, \dots), \tag{1.1}$$

where

$$\xi_n := \xi_{n-1} + \eta_n \quad (n \geq 1); \quad \xi_o = 0 \quad (\eta_n \geq 0, \text{ for all } n \geq 1) \tag{1.2}$$

and ξ_n is *the n-th renewal moment.* At the moment ξ_n a number ν_n of objects is added to the set A. The random variable ν_n is *the n-th number of renewals* and:

$$\nu_n \in \{1, 2, 3, \dots\} \subset [1, \infty) \quad \text{for } n \geq 0. \tag{1.3}$$

Here ν_0 (the 0-th number of renewal) is *the initial number* of objects belonging to A at the moment $\xi_0 = 0$.

Fixing n $(n = 0, 1, 2, \dots)$ the random variable τ_n^i with

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$$\tau_n^i \in [0, \infty) \quad (1 \le i \le \nu_n) \ ^2 \tag{1.4}$$

is *the residual life-time* of the i-th object in ν_n objects entering into A at the moment ξ_n. It means that the i-th object exists (lives) in A in the time τ_n^i and will be eliminated from A at *the eliminated moment*:

$$\zeta_n^i := \xi_n + \tau_n^i. \tag{1.5}$$

In brief, when we study the evolution of a set A, each object in A can be characterized by a random vector (ξ_n, τ_n^i) which is denoted by (i, n) for simplicity:

$$(i, \xi_n) \equiv (i, n) := (\xi_n, \tau_n^i). \tag{1.6}$$

By this reason, the vector (i, n) in this work represents the corresponding object of A and the set of these vectors: $\{(i, n) : 1 \le i \le \nu_n,\ n \ge 0\}$ represents the set A. Under some assumptions for sequences (1.1), it shows in Section 2 that this set is a generalization of renewal processes (see [1], [2], [6]) of non-homogeneous renewal models (see [4], [5]) and of ones of renewals by partly failed objects (see [3]). A concept of the renewal function will be constructed in Section 3 for the set ("population") A as a generalization of this concept for renewal processes. Using these notions, we will establish an equation of the renewal function in the next work to predict the number of "the renewal individual" in a time $(0, t)$.

2. CONCEPT OF POPULATION

Suppose that the sequences (1.1) fulfill the following assumptions (A) - (D):

(A) $\{\eta_n : n \ge 1\}$ is a sequence of the mutually independent non-negative random variables $\eta_n := \xi_n - \xi_{n-1}$, such that $P\{\eta_1 = 0\} = 0$ and for every $n \ge 2$ the random variable η_n is non-trivial, i.e. we shall exclude the trivial case:

$$\exists n_0 \ge 2 : P\{\eta_{n_0} = 0\} = 1. \tag{2.1}$$

(B) For every $n \ge 0$, the random variables ν_n and ξ_n are independent and

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[2] i.e. $P\{\tau_n^i \le x\} = 0$ for all $x < 0$

$$u_n := E(\nu_n) \leq \bar{u} < \infty \quad \text{(for all } n \geq 0). \tag{2.2}$$

(C) For every $n \geq 0$, the random variables $\tau_n^i (i \geq 1)$ are independent of ξ_n and ν_n.

(D) For every $n \geq 0$, assume that the random variable $\tau_n^i (i \geq 1)$ are nonnegative and identically distributed:

$$F_n(x) := P\{\tau_n^i \leq x\} \quad \text{(for all } i \geq 1). \tag{2.3}$$

Here we exclude also the trivial case, where

$$\exists n_1 \geq 0 : P\{\tau_{n_1}^i = 0\} = 1 \quad \text{(for all } i \geq 1). \tag{2.4}$$

Lemma 2.1. *If $P\{\eta_1 = 0\} = 0$ and the trivial case (2.1) is excluded, then we have:*

$$\forall n \geq 1, \quad \exists \alpha_n > 0 : p_n := P\{\eta_n > \alpha_n\} > 0. \tag{2.5}$$

Moreover, the exclusion of trivial case (2.4) is equivalent to the following condition:

$$\forall n \geq 0, \quad \exists \beta_n > 0 : r_n := P\{\tau_n^i > \beta_n\} > 0 \quad (\forall i \geq 1). \tag{2.6}$$

Proof. We can show (2.5) by contradiction. From the assumption (D) about the exclusion of trivial case (2.4) we obtain (2.6). On the other hand, if the condition (2.6) holds then

$$P\{\tau_n^i = 0\} \leq P\{\tau_n^i \leq \beta_n\} = 1 - r_n < 1 \quad \text{(for all } n \geq 0).$$

It excludes the trivial case (2.4). Hence the condition (2.6) is equivalent to the exclusion of the case (2.4). □

Definition 2.1. Under the condition (A) - (D), the random vector (i, n) is called *the i-th individual* in the n-th renewal. The set of all the individuals:

$$A := \{(i, n) : n \geq 0, \ 1 \leq i \leq \nu_n\} \tag{2.7}$$

is called *the population with residual life-times* τ_n^i. This population is characterized by the sequences of random variables (1.1)[3].

Now we consider a more general renewal model, in which besides the random variable τ_n^i representing the time of existence in A of the individual (i,n), there are two other nonnegative random variables τ_{0n}^i and τ_{1n}^i, where τ_{1n}^i is *the age of the individual* (i,n) *at renewal moment* and it represents the time of existence of (i,n) before entering in A; τ_{0n}^i is *the life-time* of the individual (i,n) and it represents the total time of existence of (i,n).

Then we have

$$\tau_n^i = \begin{cases} \tau_{0n}^i - \tau_{1n}^i & (\text{if } \tau_{0n}^i > \tau_{1n}^i), \\ 0 & (\text{if } \tau_{0n}^i \le \tau_{1n}^i). \end{cases} \tag{2.8}$$

Moreover, for this model the conditions (C), (D) are replaced by the following conditions (C_1), (D_1):

(C_1) For every $n \ge 0$ and $i \ge 1$, the random variables τ_{0n}^i and τ_{1n}^i are independent: τ_{0n}^i and τ_{1n}^i are independent of ξ_n, ν_n.

(D_1) For every $n \ge 0$, the random variables in each sequence $\{\tau_{0n}^i : i \ge 1\}$, $\{\tau_{1n}^i : i \ge 1\}$ are nonnegative and identically distributed:

$$F_{0n}(x) := P\{\tau_{0n}^i \le x\}; \quad F_{1n} := P\{\tau_{1n}^i \le x\} \quad (\text{for all } i \ge 1), \tag{2.9}$$

such that

$$\Pi_n := P\{\tau_{0n}^i > \tau_{1n}^i\} > 0 \quad (\text{for all } n \ge 0). \tag{2.10}$$

Lemma 2.2. *Under the condition (D_1), suppose that for every $n \ge 0$ and $i \ge 1$, the random variables τ_{0n}^i and τ_{1n}^i are independent. Then the condition (2.10) has the following equivalent form:*

$$1 - \Pi_n = \int_0^\infty F_{0n}(x)\,dF_{1n}(x) < 1 \quad (\textit{for all } n \ge 0). \tag{2.11}$$

Moreover, for the random variable τ_n^i defined by (2.8) the trivial case (2.4) does not occur.

[3] i.e. by the renewal moments, the renewal numbers and by the residual life-times of individuals

Proof. Since the random variables τ_{0n}^i and τ_{1n}^i are nonnegative, identically distributed for all $i \geq 1$ (by the assumption (D_1)) and independent, we have (see (2.9))

$$P\{\tau_{0n}^i \leq \tau_{in}^i\} = \int_0^\infty P\{\tau_{0n}^i \leq x\} dF_{in}(x) = \int_0^\infty F_{0n}(x) dF_{1n}(x).$$

Then, it implies that

$$\Pi_n := P\{\tau_{0n}^i > \tau_{1n}^i\} = 1 - \int_0^\infty F_{0n}(x) dF_{1n}(x) \quad \text{(for every } n \geq 0). \tag{2.12}$$

It is easy now to see that (2.10) and (2.11) are equivalent.

On the other hand, we get from (2.8), (2.11), (2.12) that

$$P\{\tau_n^i = 0\} = P\{\tau_{0n}^i \leq \tau_{in}^i\} = \int_0^\infty F_{0n}(x) dF_{1n}(x) = 1 - \Pi_n < 1, \tag{2.13}$$

which excludes the trivial case (2.4). □

Theorem 2.1. *Assume that the conditions (A), (B), (C_1) and (D_1) are satisfied. Then, the set of individuals (2.7) is a population with residual life-times τ_n^i defined by (2.8), where the random variables τ_n^i ($i \geq 1$) have identical distribution:*

$$F_n(x) := P\{\tau_n^i \leq x\} = 1 - \Pi_n \int_0^\infty [1 - F_{0n}(x + y)] dF_{1n}(y) \quad \text{(if } x \geq 0), \tag{2.14}$$

$$F_n(x) := P\{\tau_n^i \leq x\} = 0 \quad \text{(if } x < 0). \tag{2.15}$$

Proof. From (2.8) and (2.10) it is easy to deduce that

$$\begin{aligned} P\{\tau_n^i \leq x\} &= P\{\tau_{0n}^i > \tau_{1n}^i\} P\{\tau_n^i \leq x \,/\, \tau_{0n}^i > \tau_{1n}^i\} \\ &\quad + P\{\tau_{0n}^i \leq \tau_{1n}^i\} P\{\tau_n^i \leq x \,/\, \tau_{0n}^i \leq \tau_{1n}^i\} \\ &= \Pi_n P\{\tau_{0n}^i - \tau_{1n}^i \leq x\} + (1 - \Pi_n) P\{0 \leq x\} \\ &= \Pi_n P\{\tau_{0n}^i \leq x + \tau_{1n}^i\} + (1 - \Pi_n) \quad \text{(for all } x \geq 0). \end{aligned} \tag{2.16}$$

On the other hand, since the random variables τ^i_{0n}, τ^i_{1n} are independent for fixed $n \geq 0, i \geq 1$ (by the assumption (C_1)), from (2.9) we obtain:

$$P\{\tau^i_{0n} \leq x + \tau^i_{1n}\} = \int_0^\infty F_{0n}(x+y)dF_{1n}(y). \tag{2.17}$$

By (2.16) and (2.17) we get

$$F_n(x) = P\{\tau^i_n \leq x\} = \Pi_n \int_0^\infty F_{0n}(x+y)dF_{1n}(y) + (1 - \Pi_n)$$

(for all $x \geq 0$),

i.e. (2.14) is true. Beside, because $\tau^i_n \geq 0$ (see (2.8)), $P\{\tau^i_n < 0\} = 0$. Therefore, we obtain (2.15).

We have thus proved that the random variables τ^i_n ($i \geq 1$) are identically distributed with the distribution function (2.14), (2.15). Using lemma (2.2), from the assumptions (C_1), (D_1), it is easy to see that the conditions (C), (D) are satisfied, in which the distribution (2.3) is defined by (2.14), (2.15). Combining these conditions with the assumptions (A), (B) we get the conditions of Definition 2.1. Hence, the set (2.7) forms a population with residual life-times τ^i_n as required. □

In this case, the set (2.7) is called *the population with life-times τ^i_{0n} and with ages at renewal moments τ^i_{1n}*. This population is characterized by the sequences of random variables below:

$$\{\xi_n : n \geq 0\}, \ \{\nu_n : n \geq 0\}, \ \{(\tau^i_{0n}, \tau^i_{1n}) : n \geq 0\} \ (\text{for } i \geq 1). \tag{2.18}$$ [4]

We now shall study, as a particular case of Theorem 2.1, a population with the life-times and the ages at renewal moments being discrete random variables. In this case, suppose that the conditions (C_1) and (D_1) are replaced by the following assumptions (C_2) and (D_2):

(C_2) For every $n \geq 0$, $i \geq 1$, the random variables τ^i_{0n} and τ^i_{1n} are independent; τ^i_{0n} and τ^i_{1n} are independent on ν_n.

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[4] i.e. by the renewal moments, the renewal numbers, the life-times and by the ages at renewal moments of its individuals

(D_2) For every $n \geq 0$, the discrete random variables in each sequence $\{\tau^i_{0n} : i \geq 1\}$, $\{\tau^i_{1n} : i \geq 1\}$ are identically distributed:

$$P\{\tau^i_{0n} = k\} = p_{0n}(k) \ (1 \leq k \leq T); \quad \sum_{k=1}^{T} p_{0n}(k) = 1; \ p_{0n}(T) > 0, \tag{2.19}$$

$$P\{\tau^i_{1n} = k\} = p_{1n}(k) \ (0 \leq k < T); \quad \sum_{k=0}^{T-1} p_{1n}(k) = 1 \quad (\text{for all } n \geq 0). \tag{2.20}$$

The natural number $T > 1$ satisfying $p_{0n}(T) > 0$ (see (2.19)) is called the *maximal life-time* of all individuals in the population.

Corollary. *Under the conditions (C_2), (D_2), suppose that $\eta_n = 1$ (for all $n \geq 1$) and the discrete random variables ν_n satisfy the assumptions (1.3) and (2.2). Then, the set (2.7) forms a population with the life-times and the ages at renewal moments having the discrete distributions (2.19), (2.20).*

Proof. Since $\eta_n \equiv 1$ for all $n \geq 1$, the condition (A) is evidently satisfied. Moreover from (1.2) we have $\xi_n \equiv n$ for all $n \geq 0$. Now it is easy to see that the condition (B) is verified, because (2.2) holds. Further, we get (C_1) as a straightforward consequence of (C_2).

We deduce from (2.19) and (2.20) that

$$F_{0n}(y) = \begin{cases} 0, & (y < 1), \\ \sum_{k=1}^{m-1} p_{0n}(k), & (m-1 \leq y < m;\ m = 1, \ldots, T), \\ 1, & (y \geq T). \end{cases} \tag{2.21}$$

$$\Delta F_{1n}(m) := F_{1n}(m) - F_{1n}(m-1) = \begin{cases} p_{1n}(m) & (1 \leq m < T), \\ 0 & (m \geq T). \end{cases} \tag{2.22}$$

From (2.19) - (2.22) we obtain $\int_0^\infty F_{0n}(y) dF_{1n}(y) \leq 1 - p_{0n}(T)$. Since $p_{0n}(T) > 0$ (see (2.19)), we have $\int_0^\infty F_{0n}(y) dF_{1n}(y) < 1$. Therefore, by the condition (D_2) and Lemma 2.2 it deduces that the condition (D_1) is satisfied. We have thus verified all conditions of theorem 2.1. □

The population defined in this Corollary is called the *population of discrete model* (2.19), (2.20).

We now consider some particular cases of the populations considered in Definition 2.1 and Theorem 2.1.

Definition 2.2. A population A with the lifetime τ^i_{0n} and the ages at renewal moments $\tau^i_{1n} \equiv 0$ for all $n \geq 1$, is called *a population with renewals by newly-born individuals.*

In this case we have for every $n \geq 1$:

$$\tau^i_n = \tau^i_{0n}; \quad F_{0n}(x) = F_n(x); \quad F_{1n}(x) = \mathbf{1}_{[0,\infty)}(x),$$

where the symbol $\mathbf{1}_X(x)$ denotes the characteristic function of X:

$$\mathbf{1}_X(x) = \begin{cases} 1 & \text{if } x \in X, \\ 0 & \text{if } x \notin X. \end{cases}$$

A population A with the residual life-times τ^i_n identically distributed for all $n \geq 1$, $i \geq 1$:

$$F_n(x) := P\{\tau^i_n \leq x\} \equiv F(x)$$

is called *a homogeneous population.*

Remark 2.1. If in a homogeneous population A we have

$$\nu_n \equiv 1; \quad \tau^i_n = \eta_{n+1} \quad (n \geq 0),$$

then A becomes a renewal process (see [1], [6]).

Remark 2.2. A population A of discrete model (2.19), (2.20) with renewals by newly-born individuals is a non-homogeneous renewal model (see [3]) or a discrete renewal population (see [4], [5]).

Remark 2.3. If A is a homogeneous population of discrete model with the distributions (2.19) of the particular form below:

$$P\{\tau^i_{0n} = k\} = p(k) \quad (k = 1, 2, \ldots, T;\ n = 0, 1, 2, \ldots)$$
$$\sum_{k=1}^{T} p(k) = 1; \quad p(T) > 0 \tag{2.23}$$

then A becomes a homogeneous renewal model (see [3]).

Remark 2.4. A population of discrete model (2.23), (2.20) is exactly a discrete model of renewal by partly failed objects (see [3]).

Using Theorem 2.1 and its Corollary it is easy to see that all different variations of these populations can be transformed to the form in Definition 2.1. So, from now on, saying "population" we have in mind "population with residual life-times" as defined in this definition, unless specified otherwise.

3. RENEWAL FUNCTION OF A POPULATION

Let us extend the concept of renewal function for a renewal process (see [1-3], [6]) to the case of population.

Definition 3.1. Let A be a population, We call the random variable

$$N(t) := \max\{n : \xi_n \leq t\} \quad (\text{for } t \geq 0) \tag{3.1}$$

number of renewal times and the random variable

$$\nu(t) := \sum_{N=0}^{N(t)} \nu_n - \nu_0 \quad (\text{for } t \geq 0) \tag{3.2}$$

number of renewals of the population A in the time $(0, t]$[5].

The expected value of $\nu(t)$:

$$U(t) := E\{\nu(t)\} \quad (\text{for all } t \geq 0) \tag{3.3}$$

is called *the renewal function of population A*.

In order to show that the renewal function is finitely defined, we have to establish some lemmas.

As we have noted in Lemma 2.1 that if the condition (A) is fulfilled, then for each $n \geq 1$, there exists a number $\alpha_n > 0$ such that: $p_n := P\{\eta_n > \alpha_n\} > 0$. Now we suppose, in addition, that

$$\inf_{n \geq 1}\{\alpha_n\} := \alpha > 0, \tag{3.4}$$

$$\inf_{n \geq 1}\{p_n\} := p > 0. \quad \text{Put } : q := 1 - p \tag{3.5}$$

and consider the following random variables:

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[5] with the exclusion of the initial moment $\xi_0 = 0$

$$\overline{\eta}_n := \mathbf{1}_{(\alpha,\infty)}(\eta_n) \quad (n \geq 1);\ \overline{\eta}_0 = 0, \tag{3.6}$$

$$\overline{\xi}_n := \sum_{i=0}^{n} \overline{\eta}_i \quad (n \geq 1);\ \overline{\xi}_0 = 0, \tag{3.7}$$

$$\overline{N}(t) := \max\{n : \overline{\xi}_n \leq t\} \quad (t \geq 0). \tag{3.8}$$

Lemma 3.1. *Assume that the condition (A) and (3.4), (3.5) hold. Then, we have*

$$0 \leq q < 1, \tag{3.9}$$

$$P\{\overline{N}(t) < \infty\} = 1 \quad (t \geq 0), \tag{3.10}$$

$$E\{[\overline{N}(t)]^k\} < \infty \qquad (t \geq 0,\ k = 1, 2, \dots). \tag{3.11}$$

Proof. Setting

$$\overline{p}_n := P\{\eta_n > \alpha\};\quad \overline{q}_n := 1 - p_n = P\{\eta_n \leq \alpha\} \quad (n \geq 1) \tag{3.12}$$

we have

$$\overline{p}_n \geq P\{\eta_n > \alpha_n\} = p_n > 0 \quad (n \geq 1). \tag{3.13}$$

Hence, it implies that

$$0 < p \leq \overline{p}_n \leq 1;\quad 0 \leq \overline{q}_n = 1 - \overline{p}_n \leq 1 - p = q < 1 \tag{3.14}$$

and we have thus proved (3.9).

To prove (3.10) we note first that $\overline{\eta}_n \in \{0, 1\}$ and $\overline{\xi}_n \in \{0, 1, \dots\}$ for all $n \geq 0$. On the other hand, by the definition (3.7), (3.8) we have

$$\overline{\xi}_{\overline{N}(j)} \leq j;\quad \overline{\xi}_{\overline{N}(j)+1} = \overline{\xi}_{\overline{N}(j)} + \overline{\eta}_{\overline{N}(j)+1} > j \quad (j = 0, 1, 2, \dots).$$

We can deduce by contradiction that

$$\overline{\xi}_{\overline{N}(j)} = \overline{\eta}_0 + \overline{\eta}_1 + \cdots + \overline{\eta}_{\overline{N}(j)} = j, \qquad \overline{\eta}_{\overline{N}(j+1)} = 1. \tag{3.15}$$

For any nonnegative integers j and m, we have:

$$P\{\overline{N}(j) = m\} \le P\{\overline{\eta}_0 + \cdots + \overline{\eta}_m = j\}. \tag{3.16}$$

Let us denote by I_m^j, with $1 \le j \le m$, the set of all combinations of j from m elements of the set below:

$$I_m^m := \{1, 2, \ldots, m\}. \tag{3.17}$$

For a combination $\{i_1, i_2, \ldots, i_j\} \in I_m^j$, we consider respectively the random events below:

$$A_1\{i_1, \ldots, i_j\} := \{\overline{\eta}_{i_1} = \cdots = \overline{\eta}_{i_j} = 1\}, \tag{3.18}$$

$$A_0\{i_1, \ldots, i_j\} := \{\overline{\eta}_{i(1)} = \cdots = \overline{\eta}_{i(m-j)} = 0\}, \tag{3.19}$$

where $\{i(1), \ldots, i(m-j)\}$ is the complementary combination of $\{i_1, \ldots, i_j\}$:

$$\{i(1), \ldots, i(m-j)\} = I_m^m \setminus \{i_1, \ldots, i_j\}.$$

Since $\overline{\eta}_0 = 0$, we obtain the following relations between events:

$$\begin{aligned} \{\overline{\eta}_0 + \cdots + \overline{\eta}_m = j\} &= \{\overline{\eta}_1 + \cdots + \overline{\eta}_m = j\} = \\ &= \bigcup_{\{i_1, \ldots, i_j\} \in I_m^j} (A_1\{i_1, \ldots, i_j\} \cap A_0\{i_1, \ldots, i_j\}) \subset \\ &\subset \bigcup_{\{i_1, \ldots i_j\} \in I_m^j} A_0\{i_1, \ldots, i_j\}, \quad (1 \le j \le m). \end{aligned} \tag{3.20}$$

We deduce from (3.16) and (3.20) that

$$P\{\overline{N}(j) = m\} \le \sum_{\{i_1, \ldots, i_j\} \in I_m^j} P(A_0\{i_1, \ldots, i_j\}), \quad (1 \le j \le m). \tag{3.21}$$

Since the random variables $\{\eta_n,\ n \ge 1\}$ are mutually independent by the condition (A), it follows from (3.12), (3.6), (3.14), (3.19) that

$$\begin{aligned} P(A_0\{i_1, \ldots, i_j\}) &= \prod_{n=1}^{m-j} P\{\overline{\eta}_{i(n)} = 0\} = \prod_{n=1}^{m-j} P\{\eta_{i(n)} \le \alpha\} \\ &= \prod_{n=1}^{m-j} \overline{q}_{i(n)} \le q^{m-j}, \quad (i \le j \le m). \end{aligned}$$

Hence, using (3.21) we get

$$P\{\overline{N}(j) = m\} \le C_m^j q^{m-j}, \quad (1 \le j \le m), \tag{3.22}$$

where C_m^j is the number of combinations of j from m elements:

$$C_m^j = \frac{m!}{(m-j)!j!}, \quad (1 \le j \le m). \tag{3.23}$$

The estimation (3.22) is also true for $j = 0$. Indeed, from (3.16), (3.14), (3.12) and (3.6) we have

$$\begin{aligned} P\{\overline{N}(0) = m\} &\le P\{\overline{\eta}_0 + \cdots + \overline{\eta}_m = 0\} = P\{\overline{\eta}_0 = \cdots = \overline{\eta}_m = 0\} \\ &= P\{\overline{\eta}_0 = 0\}\, P\{\eta_1 \le \alpha\} \dots P\{\eta_m \le \alpha\} \\ &= 1.\overline{q}_1 \dots \overline{q}_m \le q^m = C_m^0 q^m. \end{aligned} \tag{3.24}$$

Let $T := [t]$ be the largest integer which is smaller than t. Then, we have: $T \le t < T+1$. Replacing $j = T$ in (3.15), it yields

$$\overline{\xi}_{\overline{N}(T)} = T \le t, \quad \overline{\xi}_{\overline{N}(T)+1} = \overline{\xi}_{\overline{N}(T)} + \overline{\eta}_{\overline{N}(T)+1} = T + 1 > t.$$

It implies that

$$\overline{N}(t) = \overline{N}(T) \quad (\text{for all } t \ge 0). \tag{3.25}$$

On the other hand, we have

$$\{\overline{N}(T) = \infty\} = \bigcap_{n=0}^{\infty} \{\overline{N}(T) \ge n\},$$
$$\{\overline{N}(T) \ge n\} \supset \{\overline{N}(T) \ge n+1\} \quad (\text{for all } n \ge 0),$$

we obtain, therefore, that

$$P\{\overline{N}(T) = \infty\} = \lim_{n\to\infty} P\{\overline{N}(T) \ge n\}. \tag{3.26}$$

Writing:

$$P\{\overline{N}(T) \ge n\} = \sum_{m=n}^{\infty} P\{\overline{N}(T) = m\}.$$

We deduce from (3.22), (3.24) that

$$P\{\overline{N}(T) \geq n\} \leq \sum_{m=n}^{\infty} C_m^T q^{m-T} \quad (0 \leq T \leq n). \tag{3.27}$$

The series of nonnegative terms in the right-hand side of (3.27) is convergent by the D'Alambert's criterion (see (3.24), (3.23)). Hence, by (3.26), (3.27) it has

$$P\{\overline{N}(T) = \infty\} = 0. \tag{3.28}$$

Combining (3.25) and (3.28) we get

$$P\{\overline{N}(t) < \infty\} = P\{\overline{N}(T) < \infty\} = 1 - P\Big\{\overline{N}(T) = \infty\Big\} = 1$$

and (3.10) is proved.

In order to establish the inequality (3.11) we use an argument identical to that above, concretely,

$$\begin{aligned} E\{[\overline{N}(t)]^k\} =& E\{[\overline{N}(T)]^k\} = \sum_{m=0}^{\infty} m^k P\{\overline{N}(T) = m\} \\ &\leq \sum_{m=1}^{\infty} m^k C_m^T q^{m-T} \qquad (\text{for all } t \geq 0,\ k = 1, 2, \ldots). \end{aligned}$$

The convergence of the series in the right-hand side implies (3.11).

The lemma is completely proved. □

Now we proceed to the properties of $N(t)$. Let G_n and H_n be distribution function of ξ_n and η_n, respectively:

$$\begin{aligned} G_n(t) &:= P\{\xi_n \leq t\} \qquad (n = 0, 1, 2, \ldots), \\ H_n(t) &;= P\{\eta_n \leq t\} \qquad (n = 1, 2, \ldots). \end{aligned} \tag{3.29}$$

Lemma 3.2. *With the same assumptions as in the lemma 3.1, we have*

$$P\{N(t) < \infty\} = 1, \tag{3.30}$$

$$E\{[N(t)]^k\} < \infty \quad (\text{for all } k \geq 1), \tag{3.31}$$

$$E\{N(t)\} = \sum_{n=1}^{\infty} G_n(t) < \infty, \tag{3.32}$$

$$G_{n+1}(t) = \int_0^t G_n(t-x)dH_{n+1}(x) \quad \text{(for all } n \geq 0), \tag{3.33}$$

$$G_0(t) = \mathbf{1}_{[0,\infty)}(t).$$

Proof. It is clear from (3.4), (3.6) that

$$\overline{\eta}_n \leq \eta_n/\alpha \quad \text{(for all } n \geq 1),$$

therefore by (3.7) we have

$$\overline{\xi}_n \leq \xi_n/\alpha \quad \text{(for all } n \geq 1)$$

with $\overline{\xi}_0 = \xi_0 \equiv 0$. It implies that for all $t' \geq 0$,

$$\{n : \xi_n \leq \alpha t'\} = \{n : \xi_n/\alpha \leq t'\} \subset \{n : \overline{\xi}_n \leq t'\}.$$

Then, we obtain

$$0 \leq N(\alpha t') \leq \overline{N}(t') \quad \text{(for all } t' \geq 0).$$

A simple replacing t by $t' = t/\alpha$ in (3.10), (3.11) yields (3.30), (3.31).

Now from (3.31) with $k = 1$ we get $E\{N(t)\} < \infty$ for all $t \geq 0$. On the other hand, we have

$$E\{N(t)\} = \sum_{n=0}^{\infty} nP\{N(t) = n\}$$

$$= \sum_{n=1}^{\infty} P\{N(t) = n\} \sum_{m=1}^{n} 1 \quad \text{(for all } t \geq 0).$$

Since the series of nonnegative terms in the right-hand side is convergent, we can write

$$E\{N(t)\} = \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} P\{N(t) = n\} \cdot 1 \quad \text{(for all } t \geq 0).$$

Moreover, it implies from (3.1) that

$$P\{N(t) \geq m\} = P\{\xi_m \leq t\} = G_m(t) \quad \text{for all } m \geq 0.$$

Replacing these probabilities in the series by corresponding values yields

$$E\{N(t)\} = \sum_{m=1}^{\infty} P\{N(t) \geq m\} = \sum_{m=1}^{\infty} G_m(t) < \infty \quad (t \geq 0)$$

and (3.32) is proved.

Finally, since the random variables η_n are mutually independent, we can say that η_{n+1} is independent of $\xi_n = \eta_1 + \cdots + \eta_n$ for every $n \geq 1$. Then, from (1.2) and (3.29) we get

$$\begin{aligned} G_{n+1}(t) &= P\{\xi_n + \eta_{n+1} \leq t\} \\ &= \int_0^t G_n(t-x)dH_{n+1}(x) \quad (\text{for every } n \geq 1). \end{aligned}$$

This implies (3.33) for all $n \geq 1$. In the case of $n = 0$, the relation (3.33) is obviously true, because $H_1(0) = 0$ by assumption (A); therefore, we have

$$\begin{aligned} \int_0^t G_0(t-x)dH_1(x) &= \int_0^t \mathbf{1}_{[0,\infty)}(t-x)dH_1(x) \\ &= H_1(t) - H_1(0) \\ &= P\{\eta_1 \leq t\} = P\{\xi_1 \leq t\} = G_1(t). \ \square \end{aligned}$$

Remark 3.1. In proving the Lemmas 3.1 and 3.2 we have supposed, besides of the condition (A), that (3.4) and (3.5) hold. These conditions say that the chance for two successive renewal moments being very close one to another is rather rare. That is

$$\exists \alpha, p > 0, \quad \text{such that} \quad P\{\eta_n \leq \alpha\} \leq 1 - p \quad (\text{for all } n \geq 1).$$

In the particular case, when the random variables η_n, $n \geq 2$ are identically distributed:

$$P\{\eta_n \leq x\} = H_n(x) \equiv H(x) \quad (\text{for all } n \geq 2)$$

the conditions (3.4), (3.5) are automatically satisfied. In this case, a similar result have been established for renewal processes $\{\eta_n : n \geq 1\}$ (see [6]).

Theorem 3.1. *Assume that the condition (A), (B) and (3.4), (3.5) hold. Then we have*

$$P\{\nu(t) < \infty\} = 1 \quad (\text{for all } t \geq 0); \tag{3.34}$$

$$0 \leq E\{\nu(t)\} = U(t) = \sum_{n=1}^{\infty} u_n G_n(t) < \infty \quad (\text{for all } t \geq 0). \tag{3.35}$$

Proof. Since $1 \leq \nu_n < \infty$, from (3.2) we get

$$P\{N(t) < \infty\} \leq P\Big\{\sum_{n=0}^{N(t)} \nu_n < \infty\Big\} \leq P\{\nu(t) < \infty\} \leq 1 \ (\text{for all } t \geq 0).$$

On the other hand, we have (3.30):

$$P\{N(t) < \infty\} = 1.$$

Combining two relations above, we get (3.34).

To prove (3.35) we note that two events $\{\xi_n \leq t\}$ and $\{n \leq N(t)\}$ are identical. We obtain, therefore

$$\begin{aligned} \mathbf{1}_{[0,t]}(\xi_n) &= \begin{cases} 1 & (\xi_n \leq t), \\ 0 & (\xi_n > t), \end{cases} \\ &= \begin{cases} 1 & (n \leq N(t)), \\ 0 & (n > N(t)), \end{cases} \end{aligned} \tag{3.36}$$

and we can rewrite (3.2) in the form:

$$\nu(t) = \sum_{n=0}^{\infty} \nu_n \mathbf{1}_{[0,t]}(\xi_n) - \nu_0 \quad (\text{for } t \geq 0).$$

Then, since ν_n is independent of ξ_n by assumption (B), we get

$$\begin{aligned} 0 \leq U(t) &= \sum_{n=0}^{\infty} E\{\nu_n \mathbf{1}_{[0,t]}(\xi_n)\} - E(\nu_0) \\ &= \sum_{n=0}^{\infty} u_n E\{\mathbf{1}_{[0,t]}(\xi_n)\}. \end{aligned}$$

It implies that

$$0 \le U(t) = \sum_{n=1}^{\infty} u_n P\{\xi_n \le t\}$$
$$= \sum_{n=1}^{\infty} u_n G_n(t) \le \overline{u} \sum_{n=1}^{\infty} G_n(t) = \overline{u} E\{N(t)\} < \infty$$

(see (2.2) and (3.32)). We have thus proved (3.35). □

Remark 3.2. If in the Theorem 3.1 we replace the condition (B) by the stronger one (B*) below:

(B*) For every $n \ge 0$, ν_n is independent of all ξ_k, $k \ge 0$ and

$$E\{\nu_n^2\} \le \overline{u}^2 < \infty \quad \text{(for all } n \ge 0) \tag{3.37}$$

then, we can prove, beside the relations (3.34), (3.35), that $\nu(t)$ is a Hilbert valued process (see, for example [2]):

$$E\{\nu^2(t)\} < \infty \qquad (t \ge 0). \tag{3.38}$$

Indeed, we know that $[E\{\nu_n\}]^2 \le E\{\nu_n^2\}$. Hence, it follows from (3.37) that (2.2) is satisfied. The condition (B*) is thus really stronger than (B).

To prove (3.38) we shall use (3.36) and the Holder inequality:

$$E\Big\{\Big(\sum_{n=0}^{N(t)} \nu_n\Big)^2\Big\} \le E\Big\{\sum_{n=0}^{N(t)} \nu_n^2 \sum_{m=0}^{N(t)} 1^2\Big\} = E\Big\{\sum_{n=0}^{\infty} \nu_n^2 \mathbf{1}_{[0,t]}(\xi_n)[N(t)+1]\Big\}.$$

Since ν_n is independent of all ξ_k, $k \ge 0$, we get

$$E\Big\{\Big(\sum_{n=0}^{N(t)} \nu_n\Big)^2\Big\} \le \sum_{n=0}^{\infty} E(\nu_n^2) \cdot E\{\mathbf{1}_{[0,t]}(\xi_n)[N(t)+1]\}$$
$$\le \overline{u}^2 \sum_{n=0}^{\infty} E\{\mathbf{1}_{[0,t]}(\xi_n)[N(t)+1]\}$$

$$\le \overline{u}^2 E\Big\{\sum_{n=0}^{N(t)} 1 \cdot [N(t)+1]\Big\} = u^2 E\{[N(t)+1]^2\} < \infty \tag{3.39}$$

(see (3.31)). As $\nu_n \geq 1$ for all $n \geq 0$, we obtain from (3.2) that

$$0 \leq E\{\nu^2(t)\} \leq E\Big\{\Big(\sum_{n=0}^{N(t)} \nu_n\Big)^2\Big\} + E\{\nu_0^2\}.$$

By (3.32) and (3.39) it is easy to see that (3.38) holds. □

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