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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF MULTIVALUED DIFFERENTIAL SYSTEM[1]

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Abstract. *In this paper we show the relationship between the notions "Asymptotic equilibrium", and "Asymptotic equivalence", and obtain some conditions for asymptotic equivalence of two differential equation systems.*

I. INTRODUCTION

In this paper we consider the following systems

$$\dot{x} \in A(t)\, x + F(t, x)\,, \tag{1}$$

$$\dot{y} = A(t)\, y\,, \tag{2}$$

where $A(t)$ is an $n \times n$ matrix whose elements are integrable on each compact subset of $J = (0, \infty)$; $F(t, x)$ is a nonempty compact convex subset of $J \times R^n$ for each $(t, x) \in J \times R^n$. By solution of (1) we mean an absolutely continuous function $x(t)$ such that

$$\dot{x}(t) \in A(t) \times (t) + F(t, x(t))$$

almost everywhere (a.e.).

Definition 1. The system (1), (2) are said to be asymptotic equivalent if to each solution $x(t)$ of (1) there exists a solution $y(t)$ of (2) such that

$$|x(t) - y(t) = o(1) \qquad \text{as } t \to \infty \tag{3}$$

and conversely, to each solution $y(t)$ of (2) there exists a solution $x(t)$ of (1) satisfying (3).

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Definition 2. The system

$$\dot{x} \in F(t, x) \tag{4}$$

is said to be in asymptotic equilibrium if its any solution converges as $t \to \infty$ and for each $c \in R^n$ there exists a solution $x(t)$ of (4) such that $x(t) \to c$ as $t \to \infty$.

In this paper we show a relationship between these notions and obtain some conditions for asymptotic equilibrium and asymptotic equivalence of system (1), (2). Our results extend and generalize those in [4], [5], [7].

2. PRELIMINARY RESULTS

In this section we give some notations and preliminary results which will be needed in the next section.

We shall write $|\cdot|$ for any convenient vector or matrix norm in R^n. If A is a subset of R^n, we define

$$|A| = \sup\{|a| : a \in A\}.$$

If A and B are two subset of R^n we write

$$|A - B| = \sup\{|a - b| : a \in A,\ b \in B\}.$$

We denote by $\Omega(Y)$ the set of all nonempty compact subset of the topological space Y.

We assume throughout this paper that $F : J \times R^n \to \Omega(R^n)$ satisfies the following hypotheses:

(H1) For each $(t, x) \in J \times R^n$, $F(t, x)$ is convex.

(H2) For each $t \in J$, $F(t, x)$ is upper semicontinuous on R^n.

(H3) For each $x \in R^n$, $F(t, x)$ is measurable on J.

Measurability and upper semicontinuity of a multivalued function are defined in [6], [7]. The proof of the following lemma may be found in [7].

Lemma. *Suppose that $F : T \times R^n \to \Omega(R^n)$ satisfies hypotheses (H1) - (H3) and for each $x \in R^n$, $|F(t, x)| \le g(t)$ a.e. on J, where $g(t)$ is locally integrable on J. Then to each $(t_0, x_0) \in J \times R^n$ the system (4) has a solution $x(t, t_0, x_0)$ on J.*

3. ASYMPTOTIC EQUILIBRIUM

In this section we give some sufficient conditions for the asymptotic equilibrium of the system (4).

Theorem 1. *Let $F(t, x)$ satisfy the following conditions: for each $x \in R^n$*

$$|F(t, x)| \le \varphi(t)\, h(|x|) + \psi(t) \ \text{a.e. on } J,$$

where nonnegative functions $\varphi(t)$, $\psi(t)$ are integrable on J; $h(u)$ is nonnegative, monotone nondecreasing and continuous in u such that

$$\int_0^\infty \frac{du}{h(u)} = \infty .$$

Then

1. For each $(t_0, x_0) \in J \times R^n$ there exists a solution $x(t, t_0, x_0)$ of (4) on J.

2. The system (4) is in asymptotic equilibrium.

Proof. Our proof is modified from [7]. Let

$$d = x_0 + \int_{t_0}^\infty \psi(\tau) d\tau .$$

Choose $r > d$ such that

$$\int_d^r \frac{du}{h(u)} > \int_0^\infty \varphi(\tau) d\tau .$$

Consider the problem

$$\begin{aligned} &x \in \widetilde{F}(t, x) \\ &x(t_0) = x_0 \end{aligned} \qquad (5)$$

where

$$\widetilde{F}(t, x) = \begin{cases} F(t, x) & \text{if } |x| < r \\ F\left(t, \frac{rx}{|x|}\right) & \text{if } |x| \ge r \end{cases}$$

It's easy to verify that $\widetilde{F}(t, x)$ satisfies (H1) - (H3) and for each $x \in R^n$

$$|\widetilde{F}(x, t)| \leq \varphi(t)\, h(r) + \psi(t) \text{ a.e. on } J.$$

Since the function $g(t) = h(r)\,\varphi(t) + \psi(t)$ is integrable on J, according to lemma the problem (5) has a solution $x(t) = x(t, t_0, x_0)$ on J. It remains to show that

$$|x(t)| < r \quad \text{for } t \geq t_0.$$

Suppose contrarily that there exists a first moment $T > t_0$ such that $|x(T)| = r$. Then $|x(t)| < r$ for $t \in (t, t_0)$, therefore $\dot{x}(t) \in F[t, x(t)]$ a.e. on (t_0, T) and

$$\frac{d}{dt}|x(t)| \leq |\dot{x}(t)| \leq |F(t, x(t))| \leq \varphi(t)\, h(|x(t)|) + \psi(t)$$

hence

$$|x(t)| \leq |x_0| + \int_{t_0}^{t} \psi(\tau)d\tau + \int_{t_0}^{t} \varphi(\tau)h(|x(\tau)|)d\tau \leq d + \int_{t_0}^{t} \varphi(\tau)h(|x(\tau)|)d\tau$$

According to the theorem about the integral inequality (see [2])

$$|x(t)| \leq v(t)$$

where $v(t)$ is a solution of the problem

$$\begin{aligned} \dot{\nu} &= \varphi(t)\, h(v) \\ v(t_0) &= d \end{aligned} \tag{6}$$

From (6) we have

$$\int_{d}^{v(t)} \frac{du}{h(u)} = \int_{t_0}^{t} \varphi(\tau)d\tau$$

hence

$$\int_{d}^{r} \frac{du}{h(u)} \leq \int_{d}^{v(T)} \frac{du}{h(u)} \int_{t_0}^{T} \varphi(\tau)d\tau \leq \int_{0}^{\infty} \varphi(\tau)d\tau.$$

This is a contradiction. Thus $x(t, t_0, x_0)$ is a solution of (4) on J and

$$|x(t, t_0, x_0)| \leq r \quad \text{for } t \geq t_0 .$$

Since

$$x(t, t_0, x_0) = x_0 + \int_{t_0}^{t} \dot{x}(\tau) d\tau ,$$

and

$$\int_{t_0}^{t} |\dot{x}(\tau)| d\tau \leq \int_{t_0}^{t} \varphi(\tau) h(r) d\tau + \int_{t_0}^{t} \psi(\tau) d\tau$$

$\lim_{t \to \infty} x(t, t_0, x_0)$ exists for each $(t_0, x_0) \in J \times R^n$.

Let now $c \in R^n$ be fixed and $M > 0$ such that $M > 2|c|$. Let t_0 satisfy the condition

$$h(M) \int_{t_0}^{\infty} \varphi(\tau) d\tau + \int_{t_0}^{\infty} \psi(\tau) d\tau < \frac{M}{2} .$$

Consider

$$\Gamma = \{x \in C((t_0, \infty), R^n) : |x|_c \leq M\} .$$

Clearly, Γ is closed, bounded and convex. For each $x \in C(R^n)$ there exists a measurable on (t_0, ∞) f_x such that $f_x(t) \in R(t, x(t))$ a.e. on (t_0, ∞) (see [7]).

Define now a map

$$S_x = c - \int_{t}^{\infty} f_x(\tau) d\tau ; \ \ x \in \Gamma , \ t \geq t_0 .$$

We have

$$|S_x|_c \leq |c| + \int_{t_0}^{\infty} \varphi(\tau) h(M) + \int_{t_0}^{\infty} \psi(\tau) d\tau \leq M ,$$

$$|(S_x)(t_1) - (S_x)(t_2)| \leq \int_{t_1}^{t_2} \varphi(\tau) h(M) d\tau + \int_{t_1}^{t_2} \psi(\tau) d\tau$$

therefore $S\Gamma \subset \Gamma$ and the functions of set $\{Sx,\ x \in \Gamma\}$ are uniformly bounded and equicontinuous.

According to our assumption, $Sx(t) \to c$ uniformly on Γ as $t \to \infty$ that's why set $S\Gamma$ is relatively compact on J. By the Schauder theorem there exists $x(t) \in \Gamma$ such that $x = S_x$. It's easy to verify that $\dot{x}(t) \in F(t,\ x(t))$ a.e. on J and $\lim\limits_{t\to\infty} x(t) = c$. Theorem is proved.

As a corollary we have:

Theorem 2. *Suppose that the system (2) is in asymptotic equilibrium; $F(t,\ x)$ satisfies the conditions of Theorem 1. Then the system (1) is in asymptotic equilibrium.*

Proof. Let $Y(t)$ be the fundamental matrix of (2) such that $Y(t) = I + o(1)$ as $t \to \infty$, where I is an unit matrix. According to our assumption $Y(t)$ exists. Consider the system

$$\dot{z} \in Y^{-1}(t)\, F(t,\ Y(t)z)\,. \tag{7}$$

It's easy to verify that $\Phi(t,\ z) = Y^{-1}(t)\, F(t,\ Y(t)z)$ satisfies (H1) - (H3) and the conditions of Theorem 1. Therefore the system (7) is in asymptotic equilibrium. It's remain to remark that $x(t) = Y(t)\, z(t)$ is a solution of system (1).

Theorem 3. *Let $F(t,\ x)$ satisfy the condition*

$$|F(t,\ x) - F(t,\ y)| \le \varphi(t)\, h(|x-y|) \quad \textit{a.e. on } J\,,$$

where the positive functions φ, h are the same as in Theorem 1 and

$$\int_0^\infty |F(t,\ 0)|dt < \infty\,.$$

Then the statement of Theorem 1 is true.

The proof is analogous to that of Theorem 1.

4. ASYMPTOTIC EQUIVALENCE

In this section we show a relationship between the asymptotic equilibrium and asymptotic equilibrium and obtain some conditions for the asymptotic equivalence of the systems (1) and (2).

Let $Y(t)$ be the fundamental matrix of (2) such that $Y(0) = I$. It's easy to verify that if $x(t)$ is a solution of (1) then $z(t) = Y^{-1}(t)x(t)$ is a solution of (7) and inversely if $z(t)$ is a solution of (7) then $x(t) = Y(t)\, z(t)$ is a solution of (1).

Theorem 4. *Suppose $|Y(t)| \leq M$ for any $t \in J$. Then, the asymptotic equilibrium of the system (7) is a sufficient condition for the asymptotic equivalence of the system (1) and (2).*

Proof. Let $x(t)$ be an arbitrary solution of (1). Then $z(t) = Y^{-1}(t)\, x(t)$ is a solution of (7). By virtue of the assumption, $z_\infty = \lim_{t\to\infty} z(t)$ exists.

Consider $y(t) = Y(t)\, z_\infty$. It is a solution of (2) and

$$\lim_{t\to\infty} |x(t) - y(t)| \leq M \lim_{t\to\infty} |z(t) - z_\infty| = 0\,.$$

Let now $y(t)$ be an arbitrary solution of (2). Then $y(t) = Y(t)\, y_0$, $y_0 \in R^n$. According to the assumption there exists a solution $z(t)$ of (7) such that $\lim_{t\to\infty} z(t) = y_0$. Consider $x(t) = Y(t)\, z(t)$. It is a solution of (1) and $\lim_{t\to\infty} |x(t) - y(t)| \leq \lim_{t\to\infty} |Y(t)|\, |z(t) - y_0| = 0$.

Thus the system (1) and (2) are asymptotically equivalent.

Remark. The case when (1) is an ordinary differential system, Theorem 1 was proved in [4] and partly was used in [5]. Even in the case when (1) is an ordinary differential system, Theorem 1 isn't reversible, in general. Consider the following example:

$$\dot{x} = Ax + B(t)\, x\,, \tag{1'}$$

$$\dot{y} = Ay\,, \tag{2'}$$

where

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 & e^{-t} \\ e^{-t} & 0 \end{bmatrix}.$$

In this case

$$Y(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}, \quad Y^{-1}(t) = \begin{bmatrix} e^{t} & 0 \\ 0 & e^{2t} \end{bmatrix},$$

$$Y^{-1}(t)\, B(t)\, Y(t) = \begin{bmatrix} 0 & e^{-2t} \\ 1 & 0 \end{bmatrix}.$$

By Levinson's theorem (see [3]) the system (1') and (2') are asymptotically equivalent. However the system

$$\dot{z} = Y^{-1}(t)\, B(t)\, Y(t)\, z$$

isn't in asymptotic equilibrium. In fact, this system can be written in the form

$$\dot{z}_1 = e^{-2t} z_2$$
$$\dot{z}_2 = z_1$$

Suppose that this system was in asymptotic equilibrium. Then for $c = (1, 1)$ there exists a solution $(z_1(t), z_2(t))$ such that $z_1(t) \to 1$, $z_2(t) \to 1$ as $t \to \infty$. From $\dot{z}_2(t) = z_1(t)$ it follows that $\dot{z}_2(t) \to 1$ as $t \to \infty$. Therefore

$$1 - \varepsilon < \dot{z}_2(t) < 1 + \varepsilon \quad \text{for } t \geq T > 0 .$$

Hence
$$z_2(t) > z_2(T) + (1 - \varepsilon)(t - T) .$$

This is a contradiction.

However we have the following.

Theorem 5. *Let $|Y^{-1}(t)| \leq M$ for any $t \in J$. Then, the asymptotic equilibrium of system (7) is a necessary condition for the asymptotic equivalence of the system (1) and (2).*

Proof. Let $c \in R^n$ and $y(t) = Y(t)\, c$ be a solution of (2). According to our assumption there exists a solution $x(t)$ of (1) such that $|x(t) - y(t)| = o(1)$ as $t \to \infty$. Consider $z(t) = Y^{-1}(t)\, x(t)$. It's a solution of (7) and $|z(t) - c| \leq |Y^{-1}(t)|\, |x(t) - y(t)| \leq M |x(t) - y(t)|$.

That mean $z(t) \to c$ as $t \to \infty$.

Let now $z(t)$ be an arbitrary solution of (7). Then $x(t) = Y(t)\, z(t)$ is a solution of (1). By virtue of assumption there exists a solution $y(t) = Y(t)\, y_0$ of (2) such that $|x(t) - y(t)| = O(1)$ as $t \to \infty$.
We have then

$$|z(t) - y_0| \leq |Y^{-1}(t)|\, |x(t) - y(t)| \leq M\, |x(t) - y(t)| = o(1) .$$

This shows that $z(t) \to y_0$ as $t \to \infty$. Thus the system (7) is in asymptotic equilibrium.

As a corollary of the Theorems (4) and (5) we have:

Theorem 6. *Suppose that every solution of (2) is bounded on J and*

$$\inf_{t \geq t_0} \int_0^t \operatorname{tr} A(\tau) d\tau > -\infty . \tag{8}$$

Then, the system (1) and (2) are asymptotically equivalent if and only if the system (7) is in asymptotic equilibrium.

In fact, the inequality (8) and the boundedness of solutions of (2) on J imply the boundedness of $|Y^{-1}(t)|$ on J.

Basing on Theorem 4 we can obtain different conditions for the asymptotic equivalence of (1) and (2).

Theorem 7. *Let the conditions of Theorem 6 be satisfied.* $F(t, x)$ *is the same as in Theorem 1 or in Theorem 2. Then (1) and (2) are asymptotically equivalent.*

Proof. In this case the function $\Phi(t, z) = Y^{-1}(t) F(t, Y(t)z)$ satisfies the conditions of Theorem 1 or Theorem 2, respectively. Therefore the system (7) is in asymptotic equilibrium. By Theorem 4 the system (1) and (2) are asymptotically equivalent.

Remark. The condition (8) in the Theorem 6 may be replaced by the weaker condition:

$$|Y^{-1}(t)| \leq M \text{ for any } t \in J .$$

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REFERENCES

1. F. Brauer, *Global behaviour of solutions of ordinary differential equations*, J. Math. Anal. Appl., **2** (1961), 145 - 158.
2. P. Harrtman, *Ordinary differential equations*, New York - London - Sydney, 1964.
3. N. Levinson, *The asymptotic behaviour of system of linear differential equations*, Amer. J. Math., **68** (1946), 1 - 6.
4. Nguyen The Hoan, *Asymptotic equivalence of the system of differential equations*, Izv. Acad. Nauk ASSR, No. 2 (1975), 35 - 40 (Russian).

5. Nguyen The Hoan, *Some asymptotic behaviours of solutions of non-linear systems of differential equations*, Differentialnye Uravnenija, **12** (4) (1981), 624 - 628 (Russian).
6. S. W. Seah, *Asymptotic equivalence of multivalued differential systems*, Boll. Unione Mat. Ital., **17-B** (1980), 1124 - 1145.
7. S. W. Seah, *Existence of solutions and asymptotic equilibrium of multivalued differential systems*, J. Math. Anal. Appl., **89** (1982), 648 - 663.

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