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ON LATTICES L DETERMINED BY Sub(L) UP TO ISOMORPHISM

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Abstract. *Denote by* Sub(L) *the lattice determined by all sublattices of a lattice* L. *The aim of this paper is to give some types of lattices* L *which are determined by* Sub(L) *up to isomorphism.*

1. INTRODUCTION

Let L be a lattice. We say that the condition (G) holds for L if Sub(L) determines L up to isomorphism, that is , if $\mathrm{Sub}(L) \cong \mathrm{Sub}(L')$ for some lattices L' then $L \cong L'$. "Find conditions under which L satisfies (G)" is a problem proposed by G. Grätzer in 1978 [5].

In [1] H.M. Chuong dealt with this problem in the larger sense: he has given one type of lattices L which are determined by Sub(L) up to isomorphism or dual isomorphism. Following H.M. Chuong, in [2], by the contractible sublattice method we have proved the following:

(I) If L has no contractible sublattices then Sub(L) determines L up to isomorphism or dual isomorphism. With the help of this result, in [3] we have considered the lattices L which are determined by Sub(L) up to isomorphism and showed:

(II) If L is totally symmetric and has no contractible sublattices then L satisfies condition (G).

So far, it has still been difficult to indicate the lattices that satisfy (G). In this paper we will present some interesting lattices which satisfy this condition. One of the main results of this paper, Proposition 3.2, asserts that if a lattice L has direct decompositions the L has no contractible sublattices.

This proposition is very useful for our problem, first because it gives a criterion for identifying some lattices satisfying (I).

On the other hand, it allows to construct the lattices satisfying (II). They are:

a) $\Pi(L_i,\ i \in I)$ where L_i with $|L_i| > 1,\ i \in I,\ |I| > 1$, are totally symmetric lattices;

b) $L \times L^*$ for an arbitrary lattice L.

The other result of the paper concerns with the lattices with contractible sublattices. By Theorem 3.5 we will show some lattices, which satisfying condition (G) besides the lattices having no contractible sublattices mentioned in (II).

2. PRELIMINARIES

In this section we recall some concepts and prove three lemmas which will be needed in the sequel.

From now on, for $a, b \in L$ we denote by $a\ S\ b$ and $a \,\|\, b$ when a is comparable and uncomparable with b, respectively.

Definition 2.1. A proper sublattice A of a lattice L with $|A| > 1$, is called a contractible sublattice if the following two conditions hold:

(a) A is convex.

(b) If $\langle a, b; c, d\rangle$ is a square in L then $c \in A \Leftrightarrow d \in A$.

Lemma 2.2. *Let A be a contractible sublattice of L and $a \in A,\ k \in L \setminus A$. Then:*

(P1): *If $k < a$ then $k < x,\ \forall x \in A$.*

(P2): *If $k > a$ then $k > x,\ \forall x \in A$.*

Proof. Let $k < a$. Consider an arbitrary element $x \in A$.

a) If $x < k$ then according to (a) of Definition 2.1 we have $k \in A$, but it is impossible.

b) If $x \,\|\, k$ then $x < x \vee k \leq x \vee a$; by (a) we have $x \vee k \in A$ and by (b) $x \wedge k \in A$. From $x \wedge k < k < a$ it follows $k \in A$, which is also impossible.

Thus, $k < x$ and (P1) is proved.

By duality we have (P2) and the lemma is proved.

We say that a lattice L has a linear decomposition if there exist a chain I with $|I| > 1$ and the sublattices $L_i,\ i \in I$ of L such that $L = \cup(L_i,\ i \in I)$ and for $i, j \in I,\ i < j$, then $a < b$ for all $a \in L_i,\ b \in L_j$. Futher, $\cup(L_i,\ i \in I)$ is a completely linear decomposition of L if every linear member $L_i,\ i \in I$, is not linearly decomposable.

Lemma 2.3. *If a lattice L is not linearly decomposable and $\varphi : L \to L'$ is a square preserving bijection then L' is not linearly decomposable.*

Proof. First, we recall that for a square preserving bijection $\varphi : L \to L'$, there hold two assertions:

a) X is a sublattice in L iff $\varphi(X)$ is a sublattice in L'.

b) $x\,S\,y$ iff $\varphi(x)\,S\,\varphi(y)$ $(\forall x, y \in L)$.

Now, we assume that L' has a linear decomposition $X' \cup Y'$ such that $x' < y'$, $\forall x' \in X', y' \in Y'$. Denote $X = \varphi^{-1}(X')$, $Y = \varphi^{-1}(Y')$ then X, Y are the sublattices in L such that $X \cap Y = \emptyset$ and $x\,S\,y$, $\forall x \in X$, $y \in Y$. Moreover, if A and B are the sublattices in X and Y, respectively, then $A \cup B$ is a sublattice in L, because all elements of A are comparable with ones of B.

Consider $y_0 \in Y$. Without loss of generality we can assume that $\exists x \in X$ such that $x < y_0$. Putting $X_0 = \{x \mid x \in X,\ x < y_0\}$, we show that it is a sublattice in L.

Indeed, take $x_1,\ x_2 \in X_0$, $x_1 \| x_2$, then $x_1 \wedge x_2 < x_1 \vee x_2 \leq y_0$. Since $x_1 \vee x_2 \in X$ and $X \cap Y = \emptyset$, therefore $x_1 \vee x_2 < y_0$. Thus, $x_1 \wedge x_2$, $x_1 \vee x_2 \in X_0$.

For x_0 we have the alternative:

1) If $X_0 = X$, we take $Y_0 = \{y \mid y \in Y,\ y > x, \forall x \in X\}$. For $y_1,\ y_2 \in Y_0$, $y_1 \| y_2$, we have $y_1 \vee_y 2 > y_1 \wedge y_2 \geq x$, $\forall x \in X$, where the equality does not occur as $X \cap Y = \emptyset$. This means that $y_1 \wedge y_2$, $y_1 \vee y_2 \in Y_0$, i.e. Y_0 is a sublattice.

1a) If $Y_0 = Y$ then $X \cup Y$ is a linear decomposition of L.

1b) If $Y_0 \neq Y$, we denote $Y_1 = Y \setminus Y_0$. It is observed that: $y \in Y_1$ iff $y < x$ for some $x \in X$. This permits us to prove that Y_1 is a sublattice in Y, therefore $X \cup Y_1$ is a sublattice in L. Thus, Y_0 and $X \cup Y_1$ form a linear decomposition of L.

2) If $X_0 \neq X$, consider $X_1 = X \setminus X_0 = \{x \mid x \in X,\ x > y_0\}$ and $Y_2 = \{y \mid y \in Y,\ y > x,\ \forall x \in X_0\}$. It is easy to deduce that these sets are the sublattices in L.

2a) If $Y_2 = Y$ then $X_1 \cup Y$ is a sublattice and thus X_0, $X_1 \cup Y$ give a linear deposition of L.

2b) if $Y_2 \neq Y$ we take $Y_3 = Y \setminus Y_2$. Clearly, $y \in Y_3$ iff $y < x$, for some $x \in X_0$, therefore Y_3 also forms a sublattice. Thus, we have the sublattices $X_0 \cup Y_3$, $X \cup Y_2$ which form a linear decomposition of L.

In short, we have showed that L is linearly decomposed, which is the contradiction and the lemma is proved.

Lemma 2.4. *Let $L = \cup(A_i,\ i \in I)$ and $L' = \cup(B_i,\ i \in I)$ be two completely linearly decomposed lattices with a finite chain I. If $g : L \to L'$ is an isomorphism then $g(A_i) = B_i,\ \forall i \in I$.*

Proof. As the chain I is finite (and $|I| > 1$) we can assume that $I = \{1, \dots, n\}$ (with the natural order). From this we have $L = A_1 \cup \cdots \cup A_n$ and $L' = B_1 \cup \cdots \cup B_n$.

1) We prove that $g(A_1) = B_1$. Consider an arbitrary element $a \in A_1$. If $g(a) \notin B_1$ then $g(a) > b, \forall b \in B_1$. Because of the isomorphism of g we have $g^{-1}(b) < a, \forall b \in B_1$. Denote $X = g^{-1}(B_1)$ and $Y = A_1 \setminus X$. Obviously, X is a sublattice in A_1. On the other hand, as $Y = \{y \mid y \in A_1,\ g(y) > b, \forall b \in B_1\}$, we conclude that Y is also a sublattice in A_1. Note that $x < y,\ \forall x \in X,\ y \in Y$. Therefore A_1 is decomposed into X, Y. But this contradicts the assumption that A_1 is not linearly decomposable.

Thus, we have $g(a) \in B_1$ and hence, $g(A_1) \subseteq B_1$.

Conversely, considering the isomorphism $g^{-1} : L' \to L$ we have $g^{-1}(B_1) \subseteq A_1$.

Consequently, we obtain $g(A_1) = B_1$.

2) Denote $L_1 = A_2 \cup \cdots \cup A_n$ and $L_1' = B_2 \cup \cdots \cup B_n$. The restriction of g on L_1 is an isomorphism between L_1 and L_1'. By the similar arguments as in part 1) we have $g(A_2) = B_2$.

Thus, by this way we obtain $g(A_i) = B_i,\ i \in I$. The lemma is proved.

Finally, we recall a result of N. D. Filippov [4], that is:

(F): Let L, L' be the lattices, then $\mathrm{Sub}(L) \cong \mathrm{Sub}(L')$ iff there exists a square preserving bijection $\varphi : L \to L'$.

Let L be a lattice satisfying (G) and $\varphi : L \to L'$, a square preserving bijection. Due to (F), we have $\mathrm{Sub}(L) \cong \mathrm{Sub}(L')$ and by (G) there exists an isomorphism $f : L \to L'$. Thus, we have

Remark. If a lattice L satisfies (G) then every square preserving bijection $\varphi : L \to L'$ for some lattice L', induces an isomorphism $f : L \to L'$.

3. RESULTS

Let $L_i,\ i \in I$, be the lattices, whose direct product is denoted by $\Pi(L_i,\ i \in I)$. The element $f \in \Pi(L_i,\ i \in I)$ is understood as a map $f : I \to \cup(L_i,\ i \in I)$ (disjoint union) such that $f(i) \in L_i,\ \forall i \in I$.

We say that a lattice L has a direct decomposition if there exist the lattices L_i, $i \in I$, with $|I| > 1$, $|L_i| > 1$, $i \in I$, such that $L \cong \Pi(L_i,\ i \in I)$.

Lemma 3.1. *If a lattice L has a direct decomposition then L has no linear decomposition.*

Proof. Let $L = \Pi(L_i,\ i \in I)$. Suppose that L has a linear decomposition. Without loss of generality we can assume that $L = A \cup B$ such that $u < v$ for all $u \in A,\ v \in B$.

Take $f \in A$, $g \in B$, then $f < g$, i.e. $f(i) \leq g(i)$, $\forall i \in I$. suppose that for some fixed index $j \in I$ there is $f(j) = g(j)$. As $|L_j| > 1$ there are two following possibilities:

(1) $\exists a \in L_j : a > g(j)$ or

(2) $\exists b \in L_j : b < g(j)$.

For the first case, we take $g_1 \in B$ as follows: $g_1(j) = a$, $g_1(i) = g(i)$, $\forall i \neq j$. Thus, we have $f \in A$, $g_1 \in B$ such that $f < g_1$ with $f(j) < g_1(j)$.

For the second case, we define $f_1 \in A$ putting $f_1(j) = b$, $f1(i) = f(i)$, $\forall i \neq j$. We obtain $f_1 \in A$, $g \in B$ such that $f_1 < g$ with $f_1(j) < g(j)$.

Consequently, we can always assume that $\exists f \in A$, $g \in B$ such that $f(i) < g(i)$, $\forall i \in I$.

Now, as $|I| > 1$, from f, g, for some index $i \in I$, we construct p, $q \in L$ as follows: $p(i) = f(i)$, $p(j) = g(j)$, $\forall j \neq i$, $q(i) = g(i)$, $q(j) = f(j)$, $\forall j \neq i$. Thus, we have $p \| q$ and $p \wedge q = f$, $p \vee q = g$.

Since $L = A \cup B$, the element p must belong to one of two linear members, hence, we can suppose that $p \in A$. Now, if $q \in B$ then $p < q$, but it is impossible. Thus, it is necessary $q \in A$, which implies that $g = p \vee q \in A$. This contradicts the fact that $g \in B$.

Thus, L is not linearly decomposable and the proof is complete.

Proposition 3.2. *If a lattice L has a direct decomposition then L has no contractible sublattices.*

Proof. Let $L = \Pi(L_i,\ i \in I)$. Arguing by contradiction we suppose that there exists a contractible sublattice A in L.

Take $f \in A$, as $|A| > 1$ we can assume that $\exists a_i \in L_i$ such that $a_i > f(i)$ for some index i (otherwise, instead of f, we take $f_1 \in A$, $f_1 < f$). We define $\bar{a}_i \in L$ as follows: $\bar{a}_i(i) = a_i$, $\bar{a}_i(j) = f(j)$, $\forall j \neq i$. Thus

$\overline{a}_i > f$. From this moment we have to examine two following cases:

Case 1. If besides $\overline{a}_i$ there exists $\overline{a}_j$ for some $j \neq i$, then from the square $\langle \overline{a}_i, \overline{a}_j, f, \overline{a}_i \vee \overline{a}_j \rangle$ it implies that $\overline{a}_i, \overline{a}_j \in A$ (Definition 2.1).

Thus, in general, whenever there exist $n \in I$ and $a_n \in L_n$ such that $a_n > f(n)$ then it always follows that $a_n \in A$. This permits us to prove the following assertion:

(A): If $h > f$ then $h \in A$.

Proof of (A). We take $h > f$. Assume that $h \notin A$, according to (P2) $h > g$, $\forall g \in A$. Consider an arbitrary index $j \in I$. If $\exists b_j > h(j)$ then $\overline{b}_j \in A$ since $h(j) \geq f(j)$. This deduces that $h > \overline{b}_j$, i.e. $h(j) \geq b_j$, but it is impossible. Thus $h(j)$ must be a unit-element in L_j which is denoted by 1_j.

As $h > f$, there exists an index n such that $f(n) < 1_n$. We take g as follows: $g(n) = f(n)$, $g(j) = 1_j$, $\forall j \neq n$. Clearly $g \notin A$ (otherwise, $h = g \vee \overline{1_n} \in A$, but it is impossible). By virtue of (P2) we have $g > \overline{1_n}$, hence $f(n) \geq 1_n$. The obtained contradiction proves the assertion (A).

Now, we continue to study the case 1: Let $g \in B = L \setminus A$ then, due to (A), g can not be greater than f. If $g \| f$ then $g \vee f \in A$ (as A is contractible). This contradicts the inclusion $g \in L \setminus A$. Thus, we have $g < f$ and hence $g < k$, $\forall k \in A$, by (P1). Further, if $g_1, g_2 \in B$ then $g_1 \wedge g_2, g_1 \vee g_2 \in B$. In short, B is a sublattice and for all $g \in B$, $k \in A$ then $g < k$, i.e. B and A form a linear decomposition of L. This contradicts Lemma 3.1.

Case 2. If there exists $\overline{a}_i$ for a unique index i then it is necessary that $f(j) = 1_j \in L_j$, $\forall j \neq i$. Let $c_j \in L_j$ such that $c_j < f(j)$ for some index j. We have $\overline{c}_j$ by defining $\overline{c}_j(j) = c_j, \overline{c}_j(n) = f(n)$, $\forall n \neq j$.

1) If besides $\overline{c}_j$ there exists $\overline{c}_n$, $n \neq j$, then by the same arguments as in case 1, we have L with a linear decomposition, but it is impossible.

2) If there exists $\overline{c}_j$ for a unique index j then $|I| = 2$. Denote $i = 1$, $j = 2$, we have $L = L_1 \times L_2$. Thus, $f(1)$ is a null-element of L_1, which is denoted by 0, and $f(2)$ a unit-element of L_2, denoted by 1.

For the sake of convenience, we shall denote element of $L_1 \times L_2$ as (a_1, a_2), $a_1 \in L_1$, $a_2 \in L_2$ (thus f is equal to $(0, 1) \in A$).

As $|A| > 1$ we consider three following cases:

(i) If there exists $(0, a_2) \in A$, $a_2 \neq 1$, then $\forall a_1 \in L_1$, $a_1 \neq 0$, we have in L: $(a_1, a_2) \| (0, 1)$ and $(a_1, a_2) \wedge (0, 1) = (0, a_2)$, $(a_1, a_2) \vee (0, 1) = (a_1, 1)$. From $(0, a_2) \in A$ it implies that $(a_1, 1), (a_1, a_2) \in A$.

With this we can deduce that $(x, y) \in A$, $\forall(x, y) \in L$, i.e. $A = L$, but it contradicts the assumption that $A \neq L$.

(ii) By the same arguments, if there exists $(a_1, 1) \in A$, $a_1 \neq 0$, then we also have $A = L$, i.e. we come to the same contradiction.

(iii) If there exists $(a_1, a_2) \in A$, $a_1 \neq 0$, $a_2 \neq 1$, then $(0, a_2) = (a_1, a_2) \wedge (0, 1) \in A$ and $(a_1, 1) = (a_1, a_2) \vee (0, 1) \in A$. Thus, it lead us to the case (i) or case (ii).

In conclusion, the lattice L has no contractible sublattices and the proof of (3.2) is complete.

Now we apply Proposition 3.2 for constructing the lattices which satisfy (II) (Section 1).

Proposition 3.3. *Let L_i with $|L_i| > 1$, $i \in I$, $|I| > 1$, be the totally symmetric lattices then $L = \Pi(L_i,\ i \in I)$ is determined by Sub(L) up to isomorphism.*

Proof. Due to (II) we have that L is totally symmetric and has no contractible sublattices. As $L_i \cong L_i^*$, $i \in I$ (totally symmetric) we have $\Pi(L_i,\ i \in I)^* \cong \Pi(L_i^*,\ i \in I) \cong \Pi(L_i,\ i \in I)$. Thus L is totally symmetric and the proposition now follows directly from Proposition 3.2.

Proposition 3.4. *Let L be an arbitrary lattice then $L \times L^*$ is determined by Sub($L \times L^*$) up to isomorphism.*

The proof is trivial due to Proposition 3.2 and the fact that $(L \times L^*)^* \cong L^* \times L^{**} \cong L^* \times L \cong L \times L^*$.

In the sequel, we deal with some lattices with contractible sublattices, which belong to the class of the linearly decomposable lattices.

Theorem 3.5. *Let $L = \cup(A_i,\ i \in I)$ with a finite chain I, be a completely linearly decomposed lattice. Then L satisfies (G) iff $A_i \cong A_j$, $\forall i, j \in I$ and A_i satisfies (G), $\forall i \in I$.*

Proof. Necessity: First, for an arbitrary $i \in I$, we have to prove that A_i satisfies (G). Let $\varphi : A_i \to B$ be a square preserving bijection for some lattice B, we shall show that $A_i \cong B$.

We construct a linearly decomposed lattice $L' = \cup(A'_j,\ j \in I)$, where $A'_i = B$, $A'_j = A_j$, $\forall j \neq i$. This means that L' consists of B and A_j, $j \in I$, $j \neq i$, as the sublattices which form the linear members, these linear members are ordered by the chain I. Define a map $\varphi' : L \to L'$ putting $\varphi'(a) = \varphi(a)$, $\forall a \in A_i$, $\varphi'(a) = a$, $\forall a \in L\backslash A_i$. Clearly, φ' is a square preserving bijection. Due to Filippov's theorem (see (F) and Remark), φ' determines an isomorphism $g : L \to L'$. Notice that B is not linearly decomposable (Lemma 2.3). Therefore, according to Lemma 2.4, it follows that $g(A_j) = A'_j$, $\forall j \in I$. Thus, we have $g(A_i) = B$, that is, $A_i \cong B$.

Now, let $i, j \in I$, $i \neq j$, be two arbitrary and fixed indexes, we have to prove that $A_i \cong A_j$. On the same set I, we establish a new chain denoted by J, which is defined by a lattice isomorphism $f : I \to J$ such that $f(i) = j$, $f(j) = i$, $f(n) = n$, $\forall n \in I$, $n \neq i, j$. By the lattices A_i, $i \in I$ and the chain J, we construct a linearly decomposed lattice $L' = \cup(A_k,\ k \in J)$ (i.e. L' contains A_k, $k \in J$, as the sublattices which are linear members, ordered by the chain J).

Take a map $\varphi : L \to L'$, $\varphi(a) = a$, $\forall a \in L$, then φ is a square preserving bijection. Note that, since L, L' are the lattices, φ is not identity. Indeed, consider $a \in A_i$, $b \in A_j$ then $\varphi(a) \in A_{f(j)}$, $\varphi(b) \in A_{f(i)}$, (since $f(j) = i$, $f(i) = j$). Without loss of generality we can assume that $i < j$ in I which implies that $f(i) < f(j)$ in J. Therefore we have $a < b$ in L and $\varphi(a) > \varphi(b)$ i.e. $a > b$ in L'.

Due to (F), φ induces an isomorphism $g : L \to L'$. Applying Lemma 2.4 we have $g(A_n) = A_{f(n)}$, $\forall n \in I$. Thus, $g(A_i) = A_j$, i.e. $A_i \cong A_j$ which is to be proved.

Sufficiency: Let $\mathrm{Sub}(L) \cong \mathrm{Sub}(L')$ for some lattice L', we have to prove that $L \cong L'$. Due to (F), there exists a square preserving bijection $\varphi : L \to L'$.

If each A_i, $i \in I$, consists of one element, then L is a finite linear lattice. As φ preserves the squares, lattice L' is also linear and $|L| = |L'|$. Therefore $L \cong L'$.

Now, assume that for every index $i \in I$, $|A_i| > 1$, therefore A_i forms a contractible sublattice. Denote $\varphi(A_i) = A'_i$, $i \in I$ and $x = \varphi^{-1}(x')$ for an arbitrary $x' \in L'$.

Let $i \in I$ be an arbitrary index, then clearly A'_i is a sublattice of L'. We shall show that A'_i is contractible.

a) Take $x' \in L'$ such that $a' < x' < b'$ with $a, b' \in A'_i$. We deduce that $x' \in A'_i$. Indeed, if $x' \notin A'_i$ then $x \notin A_i$, this means that $x\,S\,z$, $\forall z \in A_i$, i.e. $x'\,S\,z'$, $\forall z' \in A'_i$. Therefore A'_i is linearly decomposed into $X' = \{z' \mid z' \in A'_i,\ z' < x'\}$ and $Y' = \{z' \mid z' \in A'_i,\ z' > x'\}$ $(a' \in X',\ b' \in Y')$. But A_i is not linearly decomposable. This contradicts Lemma 2.3.

b) Let $\langle u', v'; c', d'\rangle$ be a square in L'. Considering the square $\langle u, v; c, d\rangle$ and the contractible sublattice A_i in L, we have $c' \in A'_i \Leftrightarrow c \in A_i \Leftrightarrow d \in A_i \Leftrightarrow d' \in A'_i$.

In short, A'_i satisfies (a), (b) of Definition 2.1.

Consequently, for every linear member A_i, $i \in I$, we obtain a corresponding contractible sublattice A'_i in L'. By Lemma 2.2 it is easy to deduce that A'_i, $i \in I$, form the linear members in L', i.e. $L' = \cup(A'_j,\ j \in J)$, where J is a chain defines on the same set I $(i, j \in J,\ i < j \Leftrightarrow \exists x' \in A'_i,\ y' \in A'_j : x' < y')$. Since I is finite, there exists a lattice isomorphism $f : I \to J$. Let $f(i) = j$. The restriction of φ on A_j is a square preserving bijection $A_j \to A'_j$. as A_j satisfies (G), it follows that $A_j \cong A'_j$. On the other hand, $A_i \cong A_j$, therefore there exists an isomorphism $g_i : A_i \to A'_j$.

In conclusion, we obtain a family of the isomorphism $g_i : A_i \to A'_{f(i)}$, $i \in I$, which determines an isomorphism $g : L \to L'$ such that the restriction of g on A_i is g_i, $\forall i \in I$. Thus $L \cong L'$ and the proof of Theorem 3.5 is complete.

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