VIETNAM JOURNAL OF MATHEMATICS
Volume 24, Number 4, 1996

ON SOLVABILITY OF A CLASS OF SINGULAR INTEGRAL EQUATIONS WITH ROTATION [1]

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Abstract. *It is well-known that the complete singular integral equations do not admit solutions in a closed form. However, there exist several special cases of singular integral equations, which can be solved effectively.*

The paper deals with some fundamental properties of integral operators with shifts and applied to obtain all solutions of equations of the type (1). This class contains a lot of the equations in [1, 8].

Consider singular integral equations of the form

$$a(t)\varphi(t) + \sum_{j=0}^{n-1} \frac{1}{\pi i} \int_\Gamma \frac{\tau^{n-1-j} t^j}{\tau^n - t^n} m_j(\tau, t)\varphi(\tau) d\tau = f(t), \tag{1}$$

where

$$a(t), m_j(t,t), \varphi(t), f(t) \in X = H^\mu(\Gamma),$$

$$\Gamma = \{t : |t| = 1\}, \quad D^+ = \{z : |z| < 1\}, \quad D^- = \{z : |z| > 1\}.$$

Define

$$(S\varphi)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau)}{\tau - t} d\tau$$

$$(S_k\varphi)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\tau^{n-1-k} t^k}{\tau^n - t^n} \varphi(\tau) d\tau \tag{2}$$

$$(M_k\varphi)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\tau^{n-1-k} t^k}{\tau^n - t^n} m_k(\tau, t)\varphi(\tau) d\tau$$

$$(W\varphi)(t) = \varphi(\epsilon_1 t), \ \ \epsilon_1 = \exp(2\pi i/n), \ \ \epsilon_k = \epsilon_1^k,$$

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[1] This work was supported in part by the NBRP in NS, Vietnam.

where $m_k(\tau,t)$ are functions satisfying Hölder's condition in both variables $(\tau,t)\in\Gamma\times\Gamma$.

We need the following equalities ([4], [5])

$$SW=WS,\ S_kS=SS_k,\ S_kW=WS_k, \tag{3}$$

$$I=P+Q,\ S=P-Q,\ P^2=P,\ Q^2=Q,\ PQ=QP=0, \tag{4}$$

where

$$P=\frac{1}{2}(I+S),\ Q=\frac{1}{2}(I-S),\ P_iP_j=\delta_{ij}P_j.$$

Denote by $P_j, j=1,2,...,n$ the projectors induced by operator W, then (see [8])

$$\begin{cases} P_j &= \frac{1}{n}\sum\limits_{\nu=1}^{n}\epsilon_j^{n-1-\nu}W^{\nu+1} \\ W^k &= \sum\limits_{j=1}^{n}\epsilon_j^kP_j, \end{cases} \tag{5}$$

$$X=X^+\bigoplus X^-,\ X=\bigoplus_{j=1}^{n}X_j, \tag{6}$$

$$\begin{cases} S_k=SP_k=P_kS \\ M_k=m_kS_k+N_kP_k=(m_kS+N_k)P_k \\ N_kP_j=P_jN_k, \end{cases} \tag{7}$$

where

$$X^+=P(X),\ X^-=Q(X),\ X_j=P_j(X),$$

$$(N_k\varphi)(t)=\frac{1}{\pi i}\int\frac{m_k(\tau,t)-m_k(t,t)}{\tau-t}\varphi(\tau)d\tau,$$

$$(m_k\varphi)(t)=m_k(t,t)\varphi(t),$$

and $m_k(\tau,t)$ are assumed to satisfy the following condition

$$m_k(\tau,t)=m_k(\epsilon_1\tau,t)=m_k(\tau,\epsilon_1t).$$

Note that [4]

$$(S+N_k)^2=I, \tag{8}$$

if the function

$$\frac{m_k(\tau,t)-m_k(t,t)}{\tau-t}$$

admits an analytic continuation in both variables onto D^+.

Lemma 1 ([4]). *Suppose that $K(\tau,t)$ admits an analytic continuation onto D^+ in both variables τ, t and $K(\tau,t) = K(\epsilon_1\tau,t) = K(\tau,\epsilon_1 t)$. Suppose that the function $(\tau-t)^{-1}[K(\tau,t)-K(t,t)]$ is continuous in $(\tau,t)\in\Gamma\times\Gamma$. Then*

1. $\Phi^+ = \frac{1}{\pi i}\int \frac{\tau^{n-1-k}t^k}{\tau^n-t^n}K(\tau,t)\varphi(\tau)d\tau \in X^+$ *for every* $\varphi\in X$.
2. $\Phi^+ = 0$ *for every* $\varphi \in X^+$.

In what follows for every function $a\in X$, we write $(K_a\varphi)(t) = a(t)\varphi(t)$.

Lemma 2. *Let $a\in X$ be fixed. Then for any $k,j\in\{1,2,...,n\}$ there exists $b\in X$ such that $K_bX\subset X_k$ and $P_kK_aP_j = K_bP_j$.*

Proof. By equality (5), we obtain

$$\begin{aligned} P_kK_aP_j &= \frac{1}{n}\sum_{\nu=1}^{n}\epsilon_k^{n-1-\nu}W^{\nu+1}K_aP_j = \frac{1}{n}\sum_{\nu=1}^{n}\epsilon_k^{n-1-\nu}a(\epsilon_{\nu+1}t)W^{\nu+1}P_j \\ &= \frac{1}{n}\sum_{\nu=1}^{n}\epsilon_k^{n-1-\nu}a(\epsilon_{\nu+1}t)\sum_{\mu=1}^{n}\epsilon_\mu^{\nu+1}P_\mu P_j \\ &= \Big(\frac{1}{n}\sum_{\nu=1}^{n}\epsilon_{\nu+1}^{j-k}a(\epsilon_{\nu+1}t)\Big)P_j = a_{kj}(t)P_j, \end{aligned}$$

where

$$a_{kj}(t) = \frac{1}{n}\sum_{\nu=1}^{n}\epsilon_{\nu+1}^{j-k}a(\epsilon_{\nu+1}t). \tag{9}$$

Put $a_{kj}(t) = b(t)$ then $P_kK_aP_j = K_bP_j$. It implies $K_bP_j\subset X_k$. Lemma is proved.

Corollary 1. *Let $a\in X$. Then for every $k,j\in\{1,2,...,n\}$ the following identity yields*

$$P_kK_{a_{kj}} = K_{a_{kj}}P_j,$$

where $a_{kj}(t)$ is defined by (9).

Proof. For an arbitrary $\varphi\in X$, we have

$$P_kK_{a_{kj}}\varphi = \frac{1}{n}\sum_{\nu=1}^{n}\epsilon_k^{n-1-\nu}W^{\nu+1}a_{kj}(t)\varphi(t)$$
$$= \Big(\frac{1}{n}\sum_{\nu=1}^{n}\epsilon_k^{n-1-\nu}W^{\nu+1}\Big)\Big(\frac{1}{n}\sum_{\mu=1}^{n}\epsilon_{\mu+1}^{j-k}a(\epsilon_{\mu+1}t)\Big)\varphi(t)$$
$$= \frac{1}{n}\sum_{\nu=1}^{n}\Big[\frac{1}{n}\sum_{\mu=1}^{n}\epsilon_{\mu+1}^{j-k}\epsilon_{\nu+1}^{j-k}a(\epsilon_{\nu+1}\epsilon_{\mu+1}t)\Big]\epsilon_{\nu+1}^{k-j}\epsilon_k^{n-1-\nu}W^{\nu+1}\varphi(t)$$
$$= \frac{1}{n}\sum_{\nu=1}^{n}a_{kj}(t)\epsilon_{\nu+1}^{k-j}\epsilon_k^{n-1-\nu}W^{\nu+1}\varphi(t)$$
$$= a_{kj}(t)\Big[\frac{1}{n}\sum_{\nu=1}^{n}\epsilon_j^{n-1-\nu}W^{\nu+1}\Big]\varphi(t) = a_kj(t)P_j\varphi = K_{a_{kj}}P_j\varphi,$$

which proves the corollary.

Now we deal with the following equation (in X)

$$(M\varphi)(t) = \sum_{j=1}^{n}a_j(t)(S+N_j)P_j\varphi = f(t), \tag{10}$$

where

$$a_j \in X,\ \ (N_j\varphi)(t) = \int_\Gamma n_j(\tau,t)\varphi(\tau)d\tau,\quad j=1,2,...,n.$$

Lemma 3. *Suppose that $n_j(\tau,t), j=1,2,...,n$ admit an analytic continuation onto D^+ in both variables τ,t and $n_j(\tau,t) = n_j(\epsilon_1\tau,t) = n_j(\tau,\epsilon_1 t)$. Then $\varphi(t)\in X$ is a solution of (10) if and only if $\varphi_j = P_j\varphi,\ j=1,2,...,n$ is a solution of the following system*

$$\sum_{j=1}^{n}a_{kj}(t)(S+N_j)\varphi_j = P_kf,\ \ k=1,2,...,n \tag{11}$$

where $a_{kj}(t)$ is defined by (9).

Proof. Let $\varphi(t)$ be a solution of (10). Acting to both sides of equation (10) by operators P_k, respectively, we get

$$\sum_{j=1}^{n}P_ka_j(t)(S+N_j)P_j\varphi = P_kf,\quad k=1,2,...,n.$$

According to Lemma 2, the last system can be written as follows

$$\sum_{j=1}^{n} a_{kj}(t)(S+N_j)P_j\varphi = P_k f, \quad k = 1,2,...,n.$$

Put $\varphi_j = P_j\varphi,\ j = 1,2,...,n$, it follows that $\{\varphi\}_{i=\overline{1,n}}$ is a solution of (11).

Conversely, suppose that there exists a $\varphi \in X$ such that $\varphi_j = P_j\varphi, j = 1,2,...,n$ is a solution of (11). We have

$$f = \sum_{k=1}^{n} P_k f = \sum_{k=1}^{n}\sum_{j=1}^{n} a_{kj}(t)(S+N_j)P_j\varphi = \sum_{k=1}^{n}\sum_{j=1}^{n} P_k a_j(t)(S+N_j)P_j\varphi$$

$$= \sum_{j=1}^{n}\Big(\sum_{k=1}^{n} P_k\Big)a_j(t)(S+N_j)P_j\varphi = \sum_{j=1}^{n} a_j(t)(S+N_j)P_j\varphi = M\varphi .$$

Thus, $\varphi(t)$ is a solution of (10). The lemma is proved.

Corollary 2. *If* $\{\varphi_i\}_{i=\overline{1,n}} \in X$ *is a solution of (11), then* $\{P_i\varphi_i\}_{i=\overline{1,n}} \in X_i$*, respectively, is a solution of (11).*

Proof. Suppose that $\{\varphi_i\}_{i=\overline{1,n}} \in X$ is a solution of (11). Acting to both sides of equations (11) by operators P_k, respectively, and using the result of Corollary 1, we get

$$\sum_{j=1}^{n} a_{kj}(t)(S+N_j)P_j\varphi_j = P_k f, \quad k = 1,2,...,n.$$

Hence $\{P_i\varphi_i\}_{i=\overline{1,n}} \in X_i$ is a solution of (11).

Corollary 3. *If* $\{\varphi_i\}_{i=\overline{1,n}} \in X$ *is a solution of (11), then* $\varphi = P_1\varphi_1 + P_2\varphi_2 + ... + P_n\varphi_n$ *is a solution of (10).*

Due to results of Lemma 3 and Corollaries 2, 3, it is enough to solve system (11) in X instead of solving (10) in X.

Theorem 1. *Suppose that conditions of Lemma 3 are satisfied and that*

$$det A(t) = det\Big[a_{kj}(t)\Big]_{k,j=\overline{1,n}} \neq 0.$$

Then, for every $f \in X$*, the equation (10) has an unique solution of the form*

$$\varphi(t) = \sum_{k=1}^{n}\sum_{j=1}^{n}(S+N_k)P_k b_{kj}(t)P_j f \tag{12}$$

where $b_{kj}(t)$, $k,j = 1,2,...,n$ *are defined from the matrix* $\Big[A(t)\Big]^{-1}$

$$\Big[A(t)\Big]^{-1} = \Big[b_{kj}(t)\Big].$$

Proof. We rewrite system (11) in the following form

$$A\mathbf{K}\Phi = F,$$

where

$$\mathbf{K} = \Big[\delta_{kj}(S+N_j)\Big]_{k,j=\overline{1,n}},$$

$$\Phi = (\varphi_1, \varphi_2, ..., \varphi_n); \quad F = (P_1 f, P_2 f, ..., P_n f).$$

The equality (8) follows that $\mathbf{K}$ is invertible and $\mathbf{K}^{-1} = \mathbf{K}$. Hence we obtain $\Phi = \mathbf{K}A^{-1}F$, i.e.

$$\varphi_k(t) = \sum_{j=i}^{n}(S+N_k)b_{kj}(t)P_j f. .$$

Using Corollary 2, we conclude that every solution of (10) is of the form (12). The proof is complete.

Now we deal with equation (1) on Γ

$$\sum_{j=1}^{n}\Big\{a(t) + b_j(t)(S+N_j)P_j\Big\}\varphi = f(t), \tag{13}$$

where

$$b_j(t) = m_j(t,t), \; b_n(t) = b_0(t), \; N_n = N_0,$$

$$(N_j\varphi)(t) = \frac{1}{\pi i}\int_{\Gamma}\frac{m_j(\tau,t) - m_j(t,t)}{\tau - t}\varphi(\tau)d\tau$$

and $m_j(\tau,t)$ satisfy the condition

$$m_j(\epsilon_1\tau, t) = m_j(\tau, \epsilon_1 t) = m_j(\tau, t).$$

As Lemma 3 and Corollaries 2 and 3, one can prove the following results

Lemma 4. *Suppose that* $n_j(\tau,t) = (\tau - t)^{-1}[m_j(\tau,t) - m_j(t,t)]$ $j = 1,2,...,n$ *admit an analytic continuation onto* D^+ *in both variables and suppose that*

$$n_j(\epsilon_1\tau,t) = n_j(\tau,\epsilon_1 t) = n_j(\tau,t).$$

Then $\varphi(t) \in X$ *is a solution of (13) if and only if* $\varphi_j = P_j\varphi$, $(j = 1,2,...,n)$ *is a solution of the following system*

$$\sum_{j=1}^{n}\Big[a_{kj}(t) + b_{kj}(t)(S + N_j)\Big]P_j\varphi = P_k f, \quad k = 1,2,...,n. \tag{14}$$

Corollary 4. *Suppose that condition of Lemma 4 are satisfied. If* $\{\varphi_i\}_{i=\overline{1,n}}$ *is a solution of (14) in X, then* $\psi_i = \{P_i\varphi_i\}_{i=\overline{1,n}}$ *is a solution of (14) in* X_i.

Corollary 5. *If* $\{\varphi_i\}_{i=\overline{1,n}} \in X$ *is a solution of (14), then* $\varphi = P_1\varphi_1 + P_2\varphi_2 + ... + P_n\varphi_n$ *is a solution of (13).*

Write

$$\Phi = (P_1\varphi, P_2\varphi, ..., P_n\varphi);$$

$$F = (P_1 f, P_2 f, ..., P_n f);$$

$$A(t) = \Big[a_{kj}(t)\Big]_{k,j=\overline{1,n}};$$

$$B(t) = \Big[b_{kj}(t)\Big]_{k,j=\overline{1,n}};$$

$$L(\tau,t) = \Big[\frac{b_{kj}(t)(m_j(\tau,t) - m_j(t,t))}{\tau - t}\Big]_{k,j=\overline{1,n}};$$

and

$$L = \Big[b_{kj}(t)N_j\Big]_{k,j=\overline{1,n}};$$

$$\mathbf{S} = \Big[\delta_{kj}S\Big]_{k,j=\overline{1,n}}$$

are function-matrices and operator-matrices. Then the system (14) can be written in the form

$$A(t)\Phi + B(t)(\mathbf{S}\Phi) + L\Phi = F. \tag{15}$$

Denote by $H(D^+ \times D^+)$ the set of all two-variable functions $l(z,w)$ admiting an analytic continuation onto $D^+ \times D^+$ in both variables and satisfying the following condition

$$l(\epsilon_1\tau,t)=l(\tau,\epsilon_1 t)=l(\tau,t) \quad \text{on } \Gamma\,.$$

Denote by $H_{n\times n}(D^+\times D^+)$ the set of all function-matrices of order $n\times n$ their elements belonging to $H(D^+\times D^+)$.

Lemm 5. *Suppose that the matrices* $D_\pm(t)=\Big[A(t)\pm B(t)\Big]$ *are invertible and*

$$\Big[A(t)\pm B(t)\Big]^{-1}L(\tau,t)\in H_{n\times n}(D^+\times D^+)\,. \tag{16}$$

Then the system (15) can be represented in the following form

$$\big(A(t)I+B(t)\mathrm{S}\big)\big(I+M\big)=F,$$

where M is an integral operator with the kernel

$$\frac{1}{2\pi i}M(\tau,t)=\big[A(t)+B(t)\big]^{-1}L(\tau,t).$$

Proof. According to (16) and Lemma 1, we get $\mathrm{S}M=M$. Hence $\big(AI+B\mathrm{S}\big)\big(I+M\big)=AI+B\mathrm{S}+L$, which proves the lemma.

Denote by $\Gamma(z), z\in D^+$ the canonical matrix of the following system (see [6])

$$A\Psi+B\mathrm{S}\Psi=0\,. \tag{17}$$

Denote by $\alpha_i, i=1,2,...,n$ the partial indexes of (17). One can assume that $\alpha_1\ge\alpha_2\ge...\ge\alpha_m\ge 0>\alpha_{m+1}\ge...\ge\alpha_n$. Let

$$m^+=\alpha_1+\alpha_2+\cdots+\alpha_m,\quad m^-=|\alpha_{m+1}|+\cdots+|\alpha_n|\,.$$

Lemma 5 and [6] together imply the following theorem

Theorem 2. *Suppose that the assumptions of Lemma 5 are satisfied. Then the system (15) is solvable if and only if*

$$\int_\Gamma F(\tau)T^t(\tau)Q\Big(\frac{1}{\tau+i}\Big)d\tau=0\,, \tag{18}$$

where $T^t(t)$ is the transposed matrix of $T(t)$,

$$T(t)=\Big[\Gamma^+(t)\Big]^{-1}D_+^{-1}(t)=\Big[\Gamma^-(t)\Big]^{-1}D_-^{-1}(t),$$

$$Q(t)=\Big(Q_1(t),Q_2(t),...,Q_n(t)\Big),$$

$$Q_j\left(\frac{1}{t+i}\right) = \begin{cases} 0 & if\ j = 1, 2, ..., m \\ \sum_{k=1}^{|\alpha_j|} c_{kj}(t+i)^{-k} & if\ j = m+1, ..., n. \end{cases}$$

If the conditions (18) are satisfied, every solution of system (15) is of the form

$$\Phi = \big(I - M\big)\big(A_0F + B_0STF + B_0P\big),$$

where

$$A_0(t) = \frac{1}{2}\big[D_+^{-1}(t) + D_-^{-1}(t)\big],$$

$$B_0(t) = \frac{1}{2}\big[\Gamma^+(t) - \Gamma^-(t)\big],$$

$$P(t) = \big(P_1(t), P_2(t), ..., P_n(t)\big),$$

$$P_j\left(\frac{1}{i+t}\right) = \begin{cases} 0, & if\ j = 1, 2, ..., n \\ \sum_{k=1}^{\alpha_j} c_{kj}(t+i)^{-k}, & if\ j = 1, 2, ..., m, \end{cases}$$

c_{kj} are arbitrary complex numbers.

Acknowledgement. The author is greatly indebted to Professor Nguyen Van Mau for valuable advice and various suggestions that led to improvement of this work. The author also wishes to express his deep gratitude to Professors Dang Huy Ruan and Tran Huy Ho for their encouragement and attention to this paper.

REFERENCES

1. F. D. Gakhov, *Boundary value problems,* Oxford 1966 (3 rd Russian complemented and corrected edition, Moscow, 1977).
2. G. S. Litvinchuk, *Boundary value problems and singular integral equations with shift,* Moscow, 1977 (Russian).
3. Ng. V. Mau, *On solution of system of singular integral equations with analytic kernels and reflection,* Annales Polonici Mathematici, LII, 1990.
4. Ng. V. Mau, *On solvability in closed form of the class of singular integral equations,* Diff. equations, USSR, **25** (2) (1989) (Russian).
5. Ng. V. Mau, *Generalized algebraic elements and linear singular integral equations with transformed arguments,* WPW, Warszawa, 1989.

6. N. P. Vekya, *System of singular integral equations*, Moscow, 1970 (Russian).

7. N. I. Muskhelisvili, *Singular integral equations*, Moscow, 1968 (Russian).

8. D. Przeworska-Rolewicz, *Equations with transformed arguments, An algebraic approach*, Amsterdam, Warszawa, 1973.

Received November 27, 1995

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