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Short Communication

MEROMORPHIC FUNCTIONS AND THE TOPOLOGICAL LINEAR INVARIANTS $\overline{DN}$ AND Ω

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1. Main results. Let E and F be locally convex spaces and D an open subset of E. A holomorphic function f defined on a dense open subset D_0 of D with values in F is called meromorphic on D if for every $z \in D$ there exists a neighbourhood U of z in D and two holomorphic functions $h : U \to F$ and $\sigma : U \to \mathbb{C}$ such that

$$f\big|_{U \cap D_0} = \frac{h}{\sigma}\Big|_{U \cap D_0} \text{ with } \sigma \neq 0 .$$

For each meromorphic function f, we put

$$P(f) = \{x \in D : f \text{ is not holomorphic at } x\} .$$

$P(f)$ is called the set of poles of f. By [5] either $P(f) = \emptyset$, or if $P(f) \neq \emptyset$ then $P(f)$ is an analytic subset of codimension 1 in D.

Given $f : E \to F$ a meromorphic function. We say that f is of uniform type if there exists a continuous semi-norm ρ on E and a meromorphic function $g : E_\rho \to F$ such that

$$f = g \circ \omega_\rho ,$$

where E_ρ denotes the Banach space associated to ρ and $\omega_\rho : E \to E_\rho$ is the canonical map.

The uniformity of holomorphic functions between locally convex spaces is defined similarly. In 1982, Colombeau - Mujica [2] established the uniformity of Frechet-valued holomorphic functions defined on dual spaces of Frechet - Montel spaces. Late Meise - Vogt [6] have obtained an important result on the connection between the uniformity of scalar holomorphic functions defined on Frechet nuclear spaces and linear topological invariants on these spaces.

The linear topological invariants which we use in this note have been introduced and investigated by Vogt [10], [11].

For the uniformity of meromorphic functions, recently in [3] it has been shown that the equality

$$\mathcal{M}(E^*, F^*) = \mathcal{M}_u(E^*, F^*), \tag{1}$$

where $\mathcal{M}(E^*, F^*)$ denotes the set of meromorphic functions from E^* to F^* and $\mathcal{M}_u(E^*, F^*)$ the set of meromorphic functions of uniform type from E^* to F^*, is carried out if E is a Schwartz - Frechet space with an absolutely Schauder basis, $E \in (\overline{DN})$ and F is a Frechet space, $F \in (\tilde{\Omega})$.

Using the method of [3] and by improving estimating inequalities, in the first part of this note we prove that (1) holds if E is a Schwartz - Frechet space with an absolutely Schauder basis, $E \in (\overline{DN})$ and $F \in (\Omega)$. Note that we always have the implication $(\tilde{\Omega}) \to (\Omega)$. However, there exist spaces $F \in (\Omega)$ but $F \notin (\tilde{\Omega})$. Next we investigate the problem on extending meromorphic functions in the mean of Silva through a hypersurface on dual of a reflexive Frechet space.

For the formulation of the main result of this note, first we recall some definitions of linear topological invariants.

Let E be a Frechet space having an increasing fundamental system of semi-norms $\{\|\cdot\|_k\}$. For each subset B of E, we define

$$\|\cdot\|_B^* : E^* \to [0, +\infty)$$

by
$$\|u\|_B^* = \sup\{|u(x)| : x \in B\},$$

where E^* denotes the topological dual of E. Instead of $\|\ \|_{U_q}^*$ we write $\|\ \|_q^*$, where

$$U_q = \{x \in E : \|x\|_q \le 1\}.$$

We say that E has the linear topological invariants

$$(\overline{DN}) \quad \exists p\, \forall q\, \exists k\, \forall d > 0\, \exists C > 0 : \|\ \|_q^{1+d} \le C \|\ \|_k \|\ \|_p^d,$$

$$(\Omega) \quad \forall p\, \exists q\, \forall k\, \exists d > 0,\ C > 0 : \|\ \|_q^{*1+d} \le C \|\ \|_k^* \|\ \|_p^{*d}.$$

Now we formulate the main results of the note.

Theorem 1. *Let E, F be Frechet spaces. If E is a Schwartz space having an absolutely Schauder basis and $E \in (\overline{DN})$, $F \in (\Omega)$, then*

$$\mathcal{M}_u(E^*, F^*) = \mathcal{M}(E^*, F^*)\,.$$

Theorem 2. *Let F be a reflexive Frechet space. Then F has a continuous norm if and only if every holomorphic function f on $D \setminus H$, where D is an open set in F^* and H is a hypersurface in D, which can be extended meromorphically in the mean of Silva to H, is meromorphic on D.*

2. Proof of main results. In order to prove Theorem 1 we need the following lemma which is proved by a suitable improvement of the case $(\overline{DN}, \widetilde{\Omega})$ in [3].

Lemma 1. *Let E be a Schwartz - Frechet space having an absolutely Schauder basis and the linear topological invariant $(\overline{DN})$ and F be a Frechet space with the linear totological invariant (Ω). Then every F^* - valued holomorphic function on an open set D in E^* is locally bounded.*

Since this lemma, by an argument analogous to that used for the proof of the Lemma 2.2 in [3], we have

Lemma 2. *Let $f : D \to F^*$ be a meromorphic function, where D is an open subset of a DFS - space E^* with $E \in (\overline{DN})$ and E has an absolutely Schauder basis and F is a Frechet space having the linear topological invariant (Ω). Then there exists a continuous semi-norm ρ on E^* and a meromorphic function $g : D_\rho \to F^*$ where D_ρ is a neighbourhood of $\omega_\rho(D)$ in F^*_ρ such that*

$$f = g \circ \omega_\rho\,.$$

Now Theorem 1 can be proved as follows.

Proof of Theorem 1. Given $f : E^* \to F^*$ a meromorphic function, where E and F are as in Theorem 1. By Lemma 2 there exist a continuous semi-norm ρ on E^* and meromorphic function $g : D_\rho \to F^*$, where D_ρ is a neighbourhood of $E^*/\ker\rho =$n E^*_ρ such that $f = g \circ w_\rho$. Consider the domain of existence D^m_g of g over E^*_ρ.

Since E^*_ρ is a separable Banach space, by [8] D^m_g is pseudoconvex. Hence, the function $\varphi(z) = -\log d(z, \partial D^m_g)$ is plurisubharmonic on D^m_g.

By [9] there exist a continuous semi norm $\rho_1 \geq \rho$ on E^* and a plurisubharmonic function ψ on $E^*_{\rho_1}$ such that $\varphi w_\rho = \psi w_{\rho_1}$. It suffices to show that $\operatorname{Im} w_{\rho_1 \rho} \subseteq D^m_g$ where $w_{\rho_1 \rho} : E^*_{\rho_1} \to E^*_\rho$ is the canonical map.

In the converse case we can find $z \in E^*_{\rho_1}$ such that $w_{\rho_1 \rho}(z) \in \partial D^m_g$. Take a sequence $\{z_n\} \subset E^*/\ker\rho_1$ which converges to z. Then

$$+\infty = \lim_{n\to\infty} \varphi(w_{\rho_1 \rho}(z_n)) = \lim_{n\to\infty} \psi(z_n) < \infty .$$

It is impossible. Hence $\operatorname{Im} w_{\rho_1 \rho} \subset D^m_g$ and $\hat{g} w_{\rho_1} = f$ where $\hat{g} : D^m_g \to F^*$ is naturally meromorphic extension of g.

Theorem 1 is proved.

In the remaining of this note we shall prove Theorem 2.

Proof of Theorem 2. Assume that F has a continuous norm ρ. Then it is easy to see that F^*_ρ is dense in F^*. We prove that f is extended meromorphically on D.

Take $z_0 \in R(H)$, the regular locus of H. We can assume $z_0 = 0$. There exists a neighbourhood of z_0 of the form

$$W = U \times \Delta e\,, \; e \in F^* \text{ such that } H \cap W = U \times 0\,.$$

Since f is holomorphic on $U \times \Delta e$, we can consider the Laurent expansion of f at $z_0 = (0, 0)$:

$$f(z, \lambda) = \sum_{j=-\infty}^{+\infty} a_j(z)\, \lambda^j\,, \; \forall (z, \lambda) \in U \times \Delta^* e\,,$$

where $a_j(z) = \dfrac{1}{2\pi i} \displaystyle\int_{|\lambda|=1} \dfrac{f(\lambda, z)}{\lambda^{j+1}}\, d\lambda$ are holomorphic on U.

By the hypothesis, f can be extended meromorphically in the mean of Silva to H, and it follows that $f\big|_{D \cap F^*_\rho}$ is meromorphic. Hence, we can find $n_0 \in \mathbf{Z}$ such that $a_j(z) = 0$ for $\forall j < n_0$, $z \in U \cap F^*_\rho$. Since $a_j(z)$ are holomorphic on U, $U \cap F^*_\rho$ is dense in U and $a_j(z) = 0$ for $\forall j < n_0$, $z \in U \cap F^*_\rho$, we have $a_j(z) = 0$ for $\forall j < n_0$, $z \in U$. Hence

$$f(z, \lambda) = \sum_{j \geq n_0} a_j(z)\, \lambda^j\,, \; \forall (z, \lambda) \in U \times \Delta^* e\,.$$

It means that f is meromorphic at $z_0 \in R(H)$. Thus, f is meromorphic on $D \setminus S(H)$, where $S(H)$ denotes the singular locus of H. Since $\operatorname{codim} S(H) \geq 2$ it follows that f is extended meromorphically to D.

Conversely, assume that F does not have a continuous norm. By Bessaga - Pelczynski [1] F contains a subspace, which is isomorphic to the space of all number sequences ω. Then we can define a holomorphic function $f : F^* \setminus (0 \times \omega^*) \to \mathbf{C}$ by

$$f(z, z_1, \ldots, z_n) = \sum_{j=1}^{n} \frac{z_j}{z^j}.$$

Obviously, for each $n \geq 1$ f is meromorphic on F_n^*, where

$$F = \lim_n \operatorname{Proj} F_n .$$

However f is not meromorphic on F^*. Theorem 2 is proved.

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