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A Cauchy Like Problem in Plane Elasticity: Regularization by Quasi-reversibility with Error Estimates*

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Abstract. We consider the problem of finding the displacement field in an elastic body from displacements and stresses on a part of boundary of the elastic body. This is an ill-posed problem. We use the method of quasi-reversibility to regularize the problem. An estimate of the error is given.

1. Introduction

Let Ω be a plane elastic body and let Γ_0 be an open subset of $\partial\Omega$. In the present paper, we consider the problem of finding the displacement field on Ω. In fact, let u, v be the displacements in the $x-$ and $y-$directions respectively and let the stress field $\sigma_x, \sigma_y, \tau_{xy}$ satisfy the following system of equations

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + X = 0, \tag{1.1}$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + Y = 0, \tag{1.2}$$

where X, Y, the given body forces (in the x-, y-directions respectively), are assumed to be in $H^1(\Omega)$.

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Assuming plane stress, we have the following relations

$$\tau_{xy} = G\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big) \equiv G\gamma_{xy}, \tag{1.3}$$

$$\sigma_x - \nu\sigma_y = E\frac{\partial u}{\partial x}, \quad \sigma_y - \nu\sigma_x = E\frac{\partial v}{\partial y}, \tag{1.4}$$

where E, G, ν can be calculated from the Lamé coefficients λ, μ as follows (cf. [10])

$$G = \frac{\mu}{2}, \quad \nu = \frac{\lambda}{\lambda+\mu}, \quad E = \frac{\mu(2\lambda+\mu)}{\lambda+\mu}. \tag{1.5}$$

Let the displacements and the surface stresses be given on the portion Γ_0 of $\partial\Omega$, i.e.,

$$(u,v)|_{\Gamma_0} = (f_0, g_0) \tag{1.6}$$

and

$$\ell\sigma_x + m\tau_{xy} = \overline{X} \quad \text{on} \quad \Gamma_0, \tag{1.7}$$

$$m\sigma_y + \ell\tau_{xy} = \overline{Y} \quad \text{on} \quad \Gamma_0, \tag{1.8}$$

where (ℓ, m) is the exterior unit normal to $\partial\Omega$. Here (f_0, g_0), $(\overline{X}, \overline{Y})$ are the surface displacements and surface stresses respectively.

Proceeding as in [1], we get after some rearrangements the system

$$\Delta U = -R(U) + \chi, \tag{1.9}$$

where $U = (u, v, e)$, $e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$, $R = (R_1, R_2, R_3)$, $\chi = (\chi_1, \chi_2, \chi_3)$ with

$$R_1(U) = \frac{1+\nu}{1-\nu}\frac{\partial e}{\partial x} + \frac{2}{G}\frac{\partial G}{\partial x}\frac{\partial u}{\partial x} + \frac{1}{G}\frac{\partial G}{\partial y}\gamma_{xy} + \frac{e}{G}\frac{\partial}{\partial x}\Big(\frac{2G\nu}{1-\nu}\Big), \tag{1.10}$$

$$R_2(U) = \frac{1+\nu}{1-\nu}\frac{\partial e}{\partial y} + \frac{2}{G}\frac{\partial G}{\partial y}\frac{\partial v}{\partial y} + \frac{1}{G}\frac{\partial G}{\partial x}\gamma_{xy} + \frac{e}{G}\frac{\partial}{\partial y}\Big(\frac{2G\nu}{1-\nu}\Big), \tag{1.11}$$

$$\begin{aligned} R_3(U) = \frac{1-\nu}{G}\Big\{ & \frac{\partial e}{\partial x}\frac{\partial}{\partial x}\left(\frac{G(1+\nu)}{1-\nu}\right) + \frac{\partial e}{\partial y}\frac{\partial}{\partial y}\Big(\frac{G(1+\nu)}{1-\nu}\Big) \\ & - \frac{\partial G}{\partial x}R_1(U) - \frac{\partial G}{\partial y}R_2(U) + \frac{\partial^2 G}{\partial x^2}\frac{\partial u}{\partial x} \\ & + \frac{\partial^2 G}{\partial y^2}\frac{\partial u}{\partial y} + \frac{\partial^2 G}{\partial x\partial y}\gamma_{xy} + e\Delta\Big(\frac{G\nu}{1-\nu}\Big)\Big\}, \end{aligned} \tag{1.12}$$

and

$$\chi_1 = -X/G, \quad \chi_2 = -Y/G, \tag{1.13}$$

$$\chi_3 = -\frac{1-\nu}{G}\Big(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\Big) - \frac{1-\nu}{G^2}\Big(X\frac{\partial G}{\partial x} + Y\frac{\partial G}{\partial y}\Big). \tag{1.14}$$

From now on, we shall consider the portion Γ_0 as a subset of the segment $\{(x,0) : 0 < x < \pi\}$. In this case $(\ell, m) = (0, -1)$. Hence (1.7), (1.8) can be rewritten as

$$\tau_{xy} = -\overline{X}, \quad \sigma_y = -\overline{Y}. \tag{1.15}$$

By direct computation, one has

$$U|_{\Gamma_0} = (f_0, g_0, B_0(F_0)), \tag{1.16}$$

$$\frac{\partial U}{\partial y}|_{\Gamma_0} = B(F_0) \equiv (B_1(F_0), B_2(F_0), B_3(F_0)), \tag{1.17}$$

where $F_0 = (f_0, g_0, \overline{X}, \overline{Y})$ and

$$B_0(F_0) = (1-\nu)\frac{\partial f_0}{\partial x} - \frac{(1-\nu)\overline{Y}}{2G}, \tag{1.18}$$

$$B_1(F_0) = -\overline{X}/G - \frac{\partial g_0}{\partial x}, \quad B_2(F_0) = -(1-\nu)\overline{Y}/G - \nu\frac{\partial f_0}{\partial x}, \tag{1.19}$$

$$\begin{aligned} B_3(F_0) = &-(1-\nu)\frac{\partial^2 g_0}{\partial x^2} - \frac{1}{G}\frac{\partial}{\partial y}\Big(\frac{2G\nu}{1-\nu}\Big)\frac{\partial f_0}{\partial x} \\ &+ \frac{(1-\nu)\nu}{G}\frac{\partial G}{\partial y}\frac{\partial f_0}{\partial x}\frac{(1-\nu)^2}{4\nu^2 G^2}\frac{\partial}{\partial y}\Big(\frac{2G\nu}{1-\nu}\Big)\overline{Y} + \frac{(1-\nu^2)\overline{Y}}{G^2}\frac{\partial G}{\partial y} \\ &- \frac{1-\nu}{2}\frac{\partial}{\partial x}\Big(\frac{\overline{X}}{G}\Big) + \frac{(1-\nu)\overline{X}}{2G^2}\frac{\partial G}{\partial x} - \frac{(1-\nu)Y}{2G}. \end{aligned} \tag{1.20}$$

From (1.9), (1.16), (1.17), it follows that our problem is a Cauchy-type problem and it is ill-posed. In Lattès–Lions' book [5], Chap. 4, the Cauchy problem for an elliptic equation is regularized by the method of quasi-reversibility. However, (1.1), (1.2), (1.6) - (1.8) were not considered in [5]. In practice, measured values $(f, g, \tilde{X}, \tilde{Y})$ of the exact boundary data $(f_0, g_0, \overline{X}, \overline{Y})$ are given only at a finite set of points. It should be noted that exact solutions of (1.3) - (1.6), with $(f_0, g_0, \overline{X}, \overline{Y})$ replaced by $(f, g, \tilde{X}, \tilde{Y})$, usually do not exist. In fact, the set of boundary data $(f, g, \tilde{X}, \tilde{Y})$ for which our system has no solution is dense in $(L^2(\Gamma_0))^4$. If (1.2) - (1.6) have a solution in $(H^2(\Omega))^2$ (which is a natural solution space) then $(u, v)|_{\Gamma_0} \in (H^{3/2}(\Gamma_0))^2$. Thus if f, g are step functions then (1.3) - (1.6) have no solution in $(H^2(\Omega))^2$. In the present paper, we take the given data $(f, g, \tilde{X}, \tilde{Y})$ as L^2-functions and we shall regularize both the boundary data and the solution of our system. Explicit estimates will be derived.

2. Notations and Main Result

Consider Ω satisfying $\Omega \subset Q = [0, \pi] \times [0, T]$, $\Gamma_0 = \{(x, 0) : 0 < \alpha_0 < x < \beta_0 < \pi\}$, $\Gamma_1 = \partial\Omega \setminus \Gamma_0$.

We assume that there exists a simply connected domain Ω^* satisfying

(P1) The boundary $\partial\Omega^*$ is $C^{1+\alpha}$ $(0 < \alpha < 1)$ and

$$\Omega^* \supset \Omega \cup \Gamma_0, \ \Gamma_1 = \partial\Omega \setminus \Gamma_0 \subset \partial\Omega^*.$$

(P2) For each $x \in \partial\Omega^*$ we can find an open ball ω such that $x \in \partial\omega$ and $\omega \subset \Omega^*$.

For each $\delta > 0$, put

$$\Omega_\delta = \{(x, y) \in \Omega : \text{ dist}\,((x, y), \mathbb{R}^2 \setminus \Omega^*) > \delta\},$$

where dist (ω_1, ω_2) $(\omega_1, \omega_2 \subset \mathbb{R}^2)$ is the distance between ω_1 and ω_2.

Let ρ_δ be a nonegative C^2-function satisfying

$$\rho_\delta(x, y) = \begin{cases} 1, & \text{for} \quad (x, y) \in \Omega_\delta, \\ 0, & \text{for} \quad (x, y) \in \Omega \setminus \overline{\Omega}_{\delta/2}. \end{cases}$$

Put

$$V_\delta = \{V : \ V \in (L^2(\Omega))^3, \ \rho_\delta \frac{\partial V}{\partial \xi} \in (L^2(\Omega))^3, \ \xi = x, y,$$
$$\rho_\delta A V \in (L^2(\Omega))^3, \ V|_{\Gamma_0} = \frac{\partial V}{\partial y}|_{\Gamma_0} = 0\}$$

where $AV = \Delta V + R(V)$ and $R(V)$ is defined in (1.9) - (1.12).

Let $U_0 = (u_0, v_0, e_0)$ be a solution of (1.9), (1.16), (1.17) corresponding to the (possibly unknown) data $F_0 = (f_0, g_0, \overline{X}, \overline{Y})$ defined on Γ_0. Let $F = (f, g, \tilde{X}, \tilde{Y})$ be a "measured" data of F_0. Assume that

$$\|f - f_0\|^2_{L^2(\Gamma_0)} + \|g - g_0\|^2_{L^2(\Gamma_0)} + \|\overline{X} - \tilde{X}\|^2_{L^2(\Gamma_0)} + \|\overline{Y} - \tilde{Y}\|^2_{L^2(\Gamma_0)} < \epsilon^2. \quad (2.1)$$

We shall consider a regularized solution U_ϵ satisfying

$$\frac{1}{\epsilon_1^2} A^*(\rho_\delta^2 A U_\epsilon) - div(\rho_\delta \nabla U_\epsilon) + \delta U_\epsilon = \frac{1}{\epsilon_1^2} A^*(\rho_\delta^2 \chi), \quad (2.2)$$
$$U_\epsilon|_{\Gamma_0} = (f_\epsilon, g_\epsilon, B_0(F_\epsilon)), \quad (2.3)$$
$$\frac{\partial U_\epsilon}{\partial y}|_{\Gamma_0} = B(F_\epsilon), \quad (2.4)$$

where χ, B, B_0 are in (1.9), (1.13), (1.14), (1.17) - (1.20). Here $\epsilon_1 > 0$ (to be defined later) is a function of ϵ such that $\epsilon_1 \to 0$ as $\epsilon \to 0$, and $F_\epsilon = (f_\epsilon, g_\epsilon, \overline{X}_\epsilon, \overline{Y}_\epsilon)$ is defined in terms of $(f, g, \tilde{X}, \tilde{Y})$ in Sec. 3.

Following is the main result of this paper

Theorem 1. *Let* ϵ, δ *be in* $(0, 1)$*, let* Ω *satisfy P1), P2). Suppose that*
(a) $X, Y \in H^1(\Omega)$, $G, \nu \in C^2(\overline{\Omega})$, $G(x) > 0$ *for all* $x \in \overline{\Omega}$.
(b) $(f_0, g_0, \overline{X}, \overline{Y}) \in (H^{5/2}(\Gamma_0))^2 \times (H^{3/2}(\Gamma_0))^2$, $(f, g, \tilde{X}, \tilde{Y}) \in (L^2(\Gamma_0))^4$, *and* (2.1) *holds.*
(c) *System* (1.1), (1.2), (1.6) - (1.8) *has a solution* (u_0, v_0) *in* $(H^3(\Omega))^2$.

Then, from $(f, g, \tilde{X}, \tilde{Y})$*, we can construct* $(f_\epsilon, g_\epsilon, \overline{X}_\epsilon, \overline{Y}_\epsilon)$ *in* $(H^{5/2}(\Gamma_0))^2 \times (H^{3/2}(\Gamma_0))^2$ *and two functions* $\epsilon_1(\epsilon), W_\epsilon$ *such that* $\lim_{\epsilon \to 0} \epsilon_1(\epsilon) = 0$ *and that* $W_\epsilon \in (H^2(Q))^3$ *satisfies*

$$U_\epsilon - W_\epsilon \in V_\delta,$$

where U_ϵ *is the unique solution of* (2.2) - (2.4).

Moreover, there exist positive constants δ_0, k, C, θ_0 independent from ϵ, δ and a function $\eta(\epsilon)$ satisfying $\lim_{\epsilon\downarrow 0}\eta(\epsilon) = 0$ such that

$$\|U_\epsilon - U_0\|_{(L^2(\Omega_{k\delta}))^3} \le C\eta(\epsilon) + C\delta^{-1}\left(\ln\frac{1}{\eta(\epsilon)}\right)^{-3/2}(\eta(\epsilon))^{\theta\delta}M_0, \tag{2.5}$$

where $0 < \delta < \delta_0,\ 0 < \theta < \theta_0$ *and*

$$M_0 = 1 + \|(f_0, g_0)\|_{H^{5/2}(\Gamma_0)} + \|(\overline{X}, \overline{Y})\|_{H^{3/2}(\Gamma_0)}.$$

If, in addition,

$$(f_0, g_0, \overline{X}, \overline{Y}) \in (H^{5/2+s}(\Gamma_0))^2 \times (H^{3/2+s}(\Gamma_0))^2$$

for an $s \in (0, 1/2)$*, then*

$$\|U_\epsilon - U_0\|_{(L^2(\Omega_{k\delta}))^3} \le CM_1\left(\epsilon^{s\theta\delta/9}\delta^{-1}\left(\ln\frac{1}{\epsilon}\right)^{-3/2} + \epsilon^{s/9}\right), \tag{2.6}$$

where

$$M_1 = 1 + \|U_0\|_{(H^2(\Omega_{k\delta}))^3} + \|(f_0, g_0)\|_{H^{5/2+s}(\Gamma_0)} + \|(\overline{X}, \overline{Y})\|_{H^{3/2+s}(\Gamma_0)}.$$

Remark. If

$$\left(\ln\frac{1}{\epsilon}\right)^{-1} \le \delta < \min\{\delta_0,\ e^{-k}, \theta_0^{-1}\},$$

then (2.6) gives

$$\|U_\epsilon - U_0\|_{(L^2(\Omega_{k\delta'}))^3} \le C'M_1\left(\ln\frac{1}{\epsilon}\right)^{-1/2},$$

where $\delta' = \delta\ln\frac{1}{\delta}$. Thus, in this case, we get an estimate independent from δ.

The proof of the theorem is divided into four steps. In Step 1 (Sec. 3), we shall construct $(f_\epsilon, g_\epsilon, \overline{X}_\epsilon, \overline{Y}_\epsilon) \in (H^{5/2}(0,\pi))^2 \times (H^{3/2}(0,\pi))^2$ approximating $(f_0, g_0, \overline{X}, \overline{Y})$ in the norm of $(H^{5/2}(\Gamma_0))^2 \times (H^{3/2}(\Gamma_0))^2$. In Step 2 (Sec. 4), we shall construct $W_\epsilon \in (H^2(Q))^3$ from $(f_\epsilon, g_\epsilon, \overline{X}_\epsilon, \overline{Y}_\epsilon)$ such that $(W_\epsilon|_{\Gamma_0}, \partial W_\epsilon/\partial y|_{\Gamma_0})$ approximates $(U_0|_{\Gamma_0}, \partial U_0/\partial y|_{\Gamma_0})$ in a sense to be specified later. In Step 3 (Sec. 5), we shall find a U_ϵ in the form $U_\epsilon = Z_\epsilon + W_\epsilon$, where Z_ϵ satisfies

$$\frac{1}{\epsilon_1^2}A^*(\rho_\delta^2 AZ_\epsilon) - \mathrm{div}(\rho_\delta^2\nabla Z_\epsilon\epsilon) + \delta Z_\epsilon = \frac{1}{\epsilon_1^2}A^*(\rho_\delta^2(\chi - W_\epsilon)) + div(\rho_\delta^2\nabla W_\epsilon\epsilon) - \delta W_\epsilon$$

subject to the homogeneous condition

$$Z_\epsilon|_{\Gamma_0} = \frac{\partial Z_\epsilon}{\partial y}|_{\Gamma_0} = 0.$$

Finally, in Step 4 (Sec. 6), an error estimate will be given. In the remainder of the paper, all of proofs of Lemmas will be omitted.

Before going to Step 1 of the proof we set a notation. Letting H be a Hilbert space and letting $u_1, u_2, ..., u_m$ be in H, we put

$$\|(u_1, ..., u_m)\|_H^2 = \sum_{i=1}^{m} \|u_i\|_H^2.$$

3. Step 1 of the Proof

Let $F = (f, g, \tilde{X}, \tilde{Y}) \in (L^2(\Gamma_0))^4$, $F_0 = (f_0, g_0, \overline{X}, \overline{Y}) \in (H^{5/2+s}(\Gamma_0))^2 \times (H^{3/2+s}(\Gamma_0))^2$ $(0 \le s < 1/2)$ satisfying

$$\|(f - f_0, g - g_0, \tilde{X} - \overline{X}, \tilde{Y} - \overline{Y})\|_{L^2(\Gamma_0)} < \epsilon. \tag{3.1}$$

From $F = (f, g, \tilde{X}, \tilde{Y})$, we construct $(f_\epsilon, g_\epsilon, \overline{X}_\epsilon, \overline{Y}_\epsilon)$ in $(H^3(0, \pi))^4$ approximating $F_0 = (f_0, g_0, \overline{X}, \overline{Y})$.

We divide Step 1 into two parts. In Part i) we construct an operator P which extends a function $\phi \in H^p(\Gamma_0)$, $0 \le p < 3$, to a function $P(\phi)$ in $H^p(0, \pi)$. In part ii) we shall construct functions $f_\epsilon, g_\epsilon, \overline{X}_\epsilon, \overline{Y}_\epsilon$.

(i) *Construction of the operator P.*

Using the reflexive method (see, e.g., [3], page 10) we can construct $P(\phi) \in H^p(0, \pi)$ for every $\phi \in H^p(\Gamma_0)$, such that supp $\phi \subset [\alpha', \beta'] \subset (0, \pi)$, and that there exists a C independent from ϕ and $p \in [0, 3)$ such that

$$\|P(\phi)\|_{H^p(0,\pi)} \le C\|\phi\|_{H^p(\alpha_0,\beta_0)} \quad \text{for all } \phi \in H^p(\alpha_0, \beta_0). \tag{3.2}$$

(ii) *Construction of* $F_\epsilon = (f_\epsilon, g_\epsilon, \overline{X}_\epsilon, \overline{Y}_\epsilon)$

For $\phi \in L^2(0, \pi)$, one has the Fourier expansion

$$\phi = \sum_{n=0}^{\infty} a_n(\phi) \sin nx$$

with

$$a_n(\phi) = \frac{2}{\pi} \int_0^\pi \phi(x) \sin nx dx.$$

For $\delta > 0$ we put

$$T_\delta \phi = \sum_{n=0}^{\infty} \frac{a_n(\phi)}{1 + \delta n^4} \sin nx, \tag{3.3}$$

and

$$f_\epsilon = T_{\sqrt{\epsilon}}(Pf), \;\; g_\epsilon = T_{\sqrt{\epsilon}}(Pg), \;\; \overline{X}{=}T_{\sqrt{\epsilon}}(P\tilde{X}), \;\; Y_\epsilon = T_{\sqrt{\epsilon}}(P\tilde{Y}).$$

Now, we have the following lemma

Lemma 1.

(a) *If* $\phi \in H^{k/2+s}(\Gamma_0)$, $k = 1, 3, 5$, *for some* $0 \le s < 1/2$ *then there are* C_1, C_2 *independent from* ϕ, s *such that*

$$\sum_{n=0}^{\infty} n^{k+2s} |a_n(P\phi)|^2 \le C_1 \|\phi\|_{H^{k/2+s}(\Gamma_0)} \tag{3.4}$$

and for every $0 < \delta < 1,\ \psi \in L^2(\Gamma_0)$,

$$\|T_\delta P\psi - P\phi\|_{H^{k/2+s}(\Gamma_0)} \leq C_2 \sum_{n=0}^{\infty} n^{k+2s} |a_n(T_\delta P\psi - a_n(P\phi)|^2. \tag{3.5}$$

(b) *If* $(f, g, \tilde{X}, \tilde{Y}) \in (L^2(\Gamma_0))^4$ *and* $(f_0, g_0, \overline{X}, \overline{Y}) \in (H^{5/2}(\Gamma_0))^2 \times (H^{3/2}(\Gamma_0))^2$ *satisfy* (3.1) *then there is a constant* $C > 0$ *independent from* $(f, g, \tilde{X}, \tilde{Y})$,$\epsilon$ *such that*

$$\|(f_\epsilon - Pf_0, g_\epsilon - Pg_0)\|^2_{H^{5/2}(\Gamma_0)} + \|(X_\epsilon - P\overline{X}, Y_\epsilon - P\overline{Y})\|^2_{H^{3/2}(\Gamma_0)} \leq C\eta^2(\epsilon), \tag{3.6}$$

where

$$\begin{aligned}\eta(\epsilon) = \epsilon + \epsilon^{1/9} &\left(\|(f_0, g_0)\|^2_{H^{5/2}(\Gamma_0)} + \|(\overline{X}, \overline{Y})\|^2_{H^{3/2}(\Gamma_0)}\right) \\ &+ \sum_{n \geq [\epsilon^{-1/9}]+1} (|a_n(Pf_0)|^2 + |a_n(Pg_0)|^2) \\ &+ \sum_{n \geq [\epsilon^{-1/9}]+1} (|a_n(P\overline{X})|^2 + |a_n(P\overline{Y})|^2).\end{aligned} \tag{3.7}$$

(c) *Let* $(f_0, g_0, \overline{X}, \overline{Y}) \in (H^{5/2+s}(\Gamma_0))^2 \times (H^{3/2+s}(\Gamma_0))^2$, $0 < s < 1/2$. *If* (3.1) *holds then there is a constant* C *independent from* $(f_0, g_0, \overline{X}, \overline{Y})$ *such that*

$$LHS\ of\ (3.5) + LHS\ of\ (3.6) \leq C\eta_1^2(\epsilon),$$

where

$$\eta_1^2(\epsilon) = \epsilon^{2s/9}(1 + \|(f_0, g_0)\|^2_{H^{5/2+s}(\Gamma_0)} + \|(\overline{X}, \overline{Y})\|^2_{H^{3/2+s}(\Gamma_0)})$$

and LHS denotes the left hand side.

4. Step 2 of the Proof

We shall construct a function $W_\epsilon \in (H^2(Q))^3$ such that $(W_\epsilon|_{\Gamma_0}, \partial W_\epsilon/\partial y|_{\Gamma_0})$ approximates $(U_0|_{\Gamma_0}, \partial U_0/\partial y|_{\Gamma_0})$ in $(H^{5/2}(\Gamma_0))^3 \times (H^{5/2}(\Gamma_0))^3$.

Define $\Phi : L^2(0, \pi) \times L^2(0, \pi) \to H^2(Q)$ as follows

$$\Phi(\phi_0, \psi_0) = \sum_{n=0}^{\infty} e^{-ny} \sin x \Big((1 + \sin ny) a_n(\phi_0) + \frac{a_n(\psi_0)}{n} \sin ny\Big), \tag{4.1}$$

where, we recall

$$a_n(\phi) = \frac{2}{\pi} \int_0^\pi \phi(x) \sin nx dx.$$

We have

Lemma 2. *The operator* Φ *has the following properties*
(a) $\Phi(\phi_0, \psi_0) \in C^2(Q)$, *for all* $(\phi_0, \psi_0) \in (L^2(0, \pi))^2$. *Moreover, if*

$$\sum_{n=0}^{\infty}(n^3|a_n(\phi_0)|^2 + n|a_n(\psi_0)|^2) < \infty,$$

then $\Phi(\phi_0,\psi_0) \in H^2(Q)$ *and*

$$\Phi(\phi_0,\psi_0)|_{(0,\pi)\times\{0\}} = \phi_0,\ \frac{\partial\Phi(\phi_0,\psi_0)}{\partial y}|_{(0,\pi)\times\{0\}} = \psi_0$$

and there is a constant C *independent from* ϕ_0, ψ_0 *such that*

$$\|\Phi(\phi_0,\psi_0)\|^2_{H^2(Q)} \le C\sum_{n-0}^{\infty}(n^3|a_n(\phi_0)|^2 + n|a_n(\psi_0)|^2).$$

(b) *If* $F = (f, g, \tilde{X}, \tilde{Y})$, $F_0 = (f_0, g_0, \overline{X}, \overline{Y})$ *are as in Lemma* 1 (b), *then*

$$W_\epsilon = (\Phi(f_\epsilon, PB_1(F_\epsilon)), \Phi(g_\epsilon, PB_2(F_\epsilon)), \Phi(PB_0(F_\epsilon), PB_3(F_\epsilon)))$$

is in $(H^2(Q))^3$. *Moreover, one has*

$$\|W_\epsilon - W_0\|^2_{(H^2(Q))^3} \le C\eta(\epsilon),$$

where $\eta(\epsilon)$ *as in Lemma* 1 (b) *and*

$$W_0 = (\Phi(f_0, PB_1(F_0)), \Phi(g_0, PB_2(F_0)), \Phi(PB_0(F_0), PB_3(F_0))).$$

(c) *Under the assumptions of Lemma* 1(c), *we have*

$$\|W_\epsilon - W_0\|^2_{(H^2(Q))^3} \le C\eta_1(\epsilon)$$

where $\eta_1(\epsilon)$ *is as in Lemma* 1(c).

5. Step 3 of the Proof: Construction of Regularized Solution by QR Method and Preliminary Error Estimates

5.1. Construction of regularized solution

On V_δ, we consider the norm

$$\|V\|_{V_\delta} = \|(V, \rho_\delta AV, \rho_\delta D_1 V, \rho_\delta D_2 V)\|_{(L^2(\Omega))^3}.$$

It can be shown (cf. [5]) that V_δ with this norm is a Hilbert space. Accordingly we have

Lemma 3. *Let* $\delta > 0$, $\epsilon > 0$. *Let* W_ϵ *be as in Lemmas* 1 *and* 2 *and let* $X, Y \in H^1(\Omega)$. *Put* $\epsilon_1 = \eta(\epsilon)$. *Then the system*

$$\frac{1}{\epsilon_1^2}A^*(\rho_\delta^2 AU_\epsilon) - div(\rho_\delta^2\nabla U_\epsilon) + \delta U_\epsilon = \frac{1}{\epsilon_1^2}A^*(\rho_\delta^2 F) \quad (5.1)$$

$$U_\epsilon|_{\Gamma_0} = (f_\epsilon, g_\epsilon, B_0(F_\epsilon)), \quad (5.2)$$

$$\frac{\partial U_\epsilon}{\partial y}|_{\Gamma_0} = B(F_\epsilon) \quad (5.3)$$

has a unique solution U_ϵ satisfying $U_\epsilon - W_\epsilon \in V_\delta$, where W_ϵ is defined in Lemma 2.

5.2. Error estimates: preliminary results.

We claim that

$$\|\rho_\delta A(Z_\epsilon - Z)\|^2_{(L^2(\Omega))^3} \le C\eta(\epsilon)(1 + \|U_0\|_{(H^2(\Omega))^3}), \tag{5.4}$$

$$\|Z_\epsilon - Z\|^2_{(L^2(\Omega))^3} + \|\rho_\delta \nabla(Z_\epsilon - Z)\|^2_{(L^2(\Omega))^3} \le C\delta^{-1}(1 + \|U_0\|_{(H^2(\Omega))^3}), \tag{5.5}$$

where $Z = U_0 - W_0$, with W_0 defined in Lemma 2(b).

In fact, since $Z_\epsilon, Z \in V_\delta$, one has for every $W \in V_\delta$

$$\begin{aligned}\pi_\epsilon(Z_\epsilon, W) &= \frac{1}{\epsilon_1^2}\langle \rho_\delta(F - AW_\epsilon), \rho_\delta AW\rangle \\ &\quad - \langle \rho_\delta \nabla W_\epsilon, \rho_\delta \nabla W\rangle - \delta\langle W_\epsilon, W\rangle \\ \pi_\epsilon(Z, W) &= \frac{1}{\epsilon_1^2}\langle \rho_\delta(F - AW_0), \rho_\delta AW\rangle.\end{aligned}$$

Taking the difference of the foregoing equalities, letting $W = Z_\epsilon - Z$ and estimating, we get

$$\begin{aligned}\|\rho_\delta A(Z_\epsilon - Z)\|^2_{(L^2(\Omega))^3} &+ \epsilon_1^2\|\rho_\delta \nabla(Z_\epsilon - Z)\|^2_{(L^2(\Omega))^3} + \delta\epsilon_1^2\|Z_\epsilon - Z\|^2_{(L^2(\Omega))^3} \\ &\le C\epsilon_1^2\|U_0\|_{(H^2(\Omega))^3} + C\|W_\epsilon - W_0\|^2_{(L^2(\Omega))^3}.\end{aligned}$$

Since $\epsilon_1 = \eta(\epsilon)$, it follows from Lemma 2(b) that

$$\begin{aligned}\|\rho_\delta A(Z_\epsilon - Z)\|^2_{(L^2(\Omega))^3} &+ \eta^2(\epsilon)\|\rho_\delta \nabla(Z_\epsilon - Z)\|^2_{(L^2(\Omega))^3} + \delta\eta^2(\epsilon)\|Z_\epsilon - Z\|^2_{(L^2(\Omega))^3} \\ &\le C\eta^2(\epsilon)(1 + \|U_0\|_{(H^2(\Omega))^3}).\end{aligned}$$

Hence (5.4), (5.5) hold.

6. Step 4 of the Proof: a Carleman Type Estimate

6.1. We first consider a simple case. One has

Lemma 4. *Let Ω be a simply connected domain satisfying (P1), (P2) as discussed. Then there exist a conformal mapping $\Lambda : \overline{\Omega} \to \Lambda(\overline{\Omega})$ and constants $\delta_0, C_1, C_2 > 0$ such that*

$$\begin{aligned}&\Lambda(\overline{\Omega}) \subset \{(z,t) :\ 1/2 \le t \le 1\}, \\ &\Lambda(\Gamma_1) \subset \{(z,1) :\ z \in \mathbb{R}\}, \\ &\Lambda(\overline{\Omega}_\delta) \subset \{(z,t) :\ t \le 1 - C_2\delta\} \qquad (6.1) \\ &\Lambda(\overline{\Omega}_\delta) \supset \{(z,t) \in \Lambda(\overline{\Omega}) :\ t \le 1 - C_1\delta\}, \qquad (6.2)\end{aligned}$$

for all $0 < \delta < \delta_0$.

We now turn to the derivation of an inequality of the Carleman type. Consider an elliptic operator

$$LV = \mu \Delta V + H(V, \nabla V),$$

where $V = V(z,t) \in (H^2(\Lambda(\Omega)))^3$, $\mu \in C(\overline{\Lambda(\Omega)})$ and H depends linearly on $(V, \nabla V)$. From Lemma 4, one has

$$D \equiv \Lambda(\Omega) \subset (a,b) \times (1/2, 1), \quad a < b.$$

We have

Lemma 5. *Let* $V \in (H^2(D))^3$ *and let*

$$V|_{\partial D} = 0, \ \nabla V|_{\partial D} = 0.$$

Then there exist C, λ_0 *independent from* V *such that*

$$\lambda^3 \int_D |V|^2 e^{2\lambda t^{-m}} dzdt + \lambda \int_D |\nabla V|^2 e^{2\lambda t^{-m}} dzdt$$
$$\leq C \int_D |LV|^2 e^{2\lambda t^{-m}} dzdt, \quad \textit{for all } \lambda \geq \lambda_0,$$

where $|W|^2 = w_1^2 + w_2^2 + w_3^2$ *for* $W = (w_1, w_2, w_3)$.

6.2. Error estimates

Put

$$V = (Z_\epsilon - Z) \circ \Lambda^{-1}, \ Z_\epsilon = U_\epsilon - W_\epsilon, \ Z = U_0 - W_0,$$

where Λ is as in Lemma 4. We have

$$V|_{\Lambda(\Gamma_0)} = 0, \ \nabla V|_{\Lambda(\Gamma_0)} = 0. \tag{6.3}$$

Let $\xi \in C_c^\infty(\mathbb{R}^2)$ satisfy

$$\xi(z,t) = \begin{cases} 1, & \text{for all} \quad (z,t) \in D, 0 < t < 1 - 2C_1\delta, \\ 0, & \text{for all} \quad (z,t) \in D, t > 1 - C_1\delta, \end{cases}$$
$$|\nabla \xi(z,t)| \leq C\delta^{-1}, \quad \text{for all} \quad (z,t) \in D, \tag{6.4}$$

where C_1 is as in Lemma 4.

From (6.3), (6.4), it follows that the function ξV satisfies the conditions of Lemma 5. Put

$$LV = A(V \circ \Lambda)$$

Since $AV = \Delta V + R(V)$ and since Λ is a conformal mapping, LV has the form as in Lemma 5. Hence, Lemma 5 gives after some rearrangements

$$\lambda^3 e^{2\lambda(1-2C_1\delta)^{-m}} \|V\|^2_{(L^2(D_{1\delta}))^3} \leq \frac{C}{\delta} \|V\|^2_{(H^1(D_{2\delta}))^3} e^{2\lambda(1-C_1\delta)^{-m}}$$
$$+ Ce^{2\lambda.2^m} \|LV\|^2_{(L^2(D_{3\delta}))^3}, \tag{6.5}$$

where

$$D_{1\delta} = D \cap \{1/2 < t < 1 - 2C_1\delta\},$$
$$D_{2\delta} = D \cap \{1 - 2C_1\delta < t < 1 - C_1\delta\},$$
$$D_{3\delta} = D \cap \{1/2 < t < 1 - C_1\delta\}.$$

From Lemma 4 and from (6.5), we get in view of the fact $V = (Z_\epsilon - Z) \circ \Lambda^{-1}$ that

$$\|Z_\epsilon - Z\|^2_{(L^2(\Omega_{k\delta}))^3} \leq \lambda^{-3}\frac{C}{\delta}e^{2\lambda((1-C_1\delta)^{-m}-(1-2C_1\delta)^{-m})}\|Z_\epsilon - Z\|^2_{(H^1(\Omega_\delta))^3} + C\lambda^{-3}e^{2\lambda(2^m-(1-2C_1\delta)^{-m})}\|A(Z_\epsilon - Z)\|^2_{(L^2(\Omega_\delta))^3}. \tag{6.6}$$

Now, we choose a λ such that

$$e^{2\lambda((1-C_1\delta)^{-m}-2^{m+1})} = \eta^2(\epsilon),$$

or equivalently that

$$\lambda = (2^{m+1} - (1 - C_1\delta)^{-m})\ln\frac{1}{\eta(\epsilon)}.$$

Using the latter equality, we can find a δ_0 such that

$$\|Z_\epsilon - Z\|^2_{(L^2(\Omega_{k\delta}))^3} \leq 2C\lambda^{-3}\delta^{-2}e^{-\lambda mC_1\delta}(1 + \|U_0\|_{(H^2(\Omega))^3}),$$

for $0 < \delta < \delta_0$.

Hence, if we put $\theta = mC_1/(2^{m+1} - 1)$ then we get after some computations that

$$\|Z_\epsilon - Z\|^2_{(L^2(\Omega_{k\delta}))^3} \leq 2C\lambda^{-3}\delta^{-2}(\eta(\epsilon))^{\theta\delta}(1 + \|U_0\|_{(H^2(\Omega))^3}). \tag{6.7}$$

Now, we have

$$U_\epsilon - U_0 = (Z_\epsilon - Z) + (W_\epsilon - W_0).$$

Hence (6.7) and Lemma 2(b) give

$$\|U_\epsilon - U_0\|^2_{(L^2\Omega_{k\delta}))^3} \leq C(\lambda^{-3}\delta^{-2}(\eta(\epsilon))^{\theta\delta}(1 + \|U_0\|_{(H^2(\Omega))^3}) + \eta^2(\epsilon)),$$

i.e., (2.5) holds. From (2.5) and Lemma 2(c) we get (2.6). This completes the proof of our theorem. ■

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