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A Property of Entire Functions of Exponential Type for Lorentz Spaces*

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Abstract. In this paper we extend a result on the existence of the limit $d_f = \lim\limits_{m\to\infty} \|D^m f\|_p^{1/m}$ and the equality $d_f = \sup\{|\xi| : \xi \in \text{supp}\hat{f}\}$ to n-dimensional case and for Lorentz spaces.

1. Introduction and Preliminaries

Let $\Phi : [0, \infty) \to [0, \infty)$ be a non-zero concave function, which is non-decreasing and $\Phi(0) = \Phi(0+) = 0$. We put $\Phi(\infty) = \lim\limits_{t\to\infty} \Phi(t)$. For an arbitrary measurable on $\mathbb{R}^n$ function f we define

$$\|f\|_{N_\Phi} = \int\limits_0^\infty \Phi\big(\lambda_f(y)\big)dy,$$

where $\lambda_f(y) = \text{mes}\{x \in \mathbb{R}^n : |f(x)| > y\}$, $(y \geq 0)$. If the space $N_\Phi(\mathbb{R}^n)$ consists of measurable functions f, such that $\|f\|_{N_\Phi} < \infty$ then $N_\Phi(\mathbb{R}^n)$ is a Banach space. Denote by $M_\Phi(\mathbb{R}^n)$ the space of measurable functions g such that

$$\|g\|_{M_\Phi} = \sup\Big\{\frac{1}{\Phi(\text{mes }\Delta)} \int\limits_\Delta |g(x)|dx : \Delta \subset \mathbb{R}^n,\ 0 < \text{mes }\Delta < \infty\Big\} < \infty.$$

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Then $M_\Phi(\mathbb{R}^n)$ is a Banach space, too [7, 6]. The $N_\Phi(\mathbb{R}^n)$ and $M_\Phi(\mathbb{R}^n)$ are called Lorentz spaces.

We have the following results.

Lemma 1. [7] *If $f \in N_\Phi(\mathbb{R}^n)$, $g \in M_\Phi(\mathbb{R}^n)$ then $fg \in L_1(\mathbb{R}^n)$ and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leq \|f\|_{N_\Phi}\|g\|_{M_\Phi}.$$

Lemma 2. [7] *If $f \in M_\Phi(\mathbb{R}^n)$ then*

$$\|f\|_{M_\Phi} = \sup_{\|g\|_{N_\Phi}\leq 1} \Big| \int_{\mathbb{R}^n} f(x)g(x)dx \Big|.$$

Using Lemma 2, it is easy to prove the following:

Lemma 3. *If $f \in M_\Phi(\mathbb{R}^n), g \in L_1(\mathbb{R}^n)$ then $f * g \in M_\Phi(\mathbb{R}^n)$ and*

$$\|f * g\|_{M_\Phi} \leq \|f\|_{M_\Phi}\|g\|_{L_1}.$$

Let $m \in \mathbb{Z}_+$. Denote by $W_{m,2}$ the usual Sobolev space, i.e., the set of all $f \in S'$ such that

$$||f||_{m,2} = \big(\sum_{|\alpha|\leq m} ||D^\alpha f||_2^2 \big)^{1/2} < \infty \ .$$

We have the topological equality $H_{(m)} = W_{m,2}$ (see [5, 7.9]), where

$$H_{(m)} = \Big\{ f \in S' : \ ||f||_{(m)} = \Big(\int_{\mathbb{R}^n} (1 + |\xi|^2)^m |\hat{f}(\xi)|^2 d\xi \Big)^{1/2} < \infty \Big\}$$

and $\hat{f} = Ff$ is the Fourier transform of f.

2. Results

Theorem 1. *Let $P(\xi)$ be a polynomial with constant coefficients, $f \in M_\Phi(\mathbb{R}^n)$ and let* supp$\hat{f}$ *be bounded. Then there always exists the limit*

$$d_f = \lim_{m\to\infty} ||P^m(D)f||_{M_\Phi}^{1/m},$$

and moreover,

$$d_f = \sup_{\xi\in \mathrm{supp}\hat{f}} |P(\xi)| \, .$$

Note that Theorem 1 is an extension of a result obtained in [1], which is very helpful for us to study imbedding theorems for Sobolev spaces of infinite order [2 - 4].

Proof of Theorem 1

We shall begin by showing that

$$\varliminf_{m\to\infty} ||P^m(D)f||_{M_\Phi}^{1/m} \geq \sup_{\mathrm{sp}(f)} |P(\xi)|, \tag{1}$$

where we denote supp$\hat{f}$ by sp(f) for simplicity.

Let $\xi^o \in \mathrm{sp}(f)$ such that $|P(\xi^o)| = \sup\limits_{\mathrm{sp}(f)} |P(\xi)|$. Without loss of generality we may assume that $P(\xi^o) > 0$. Further, we fix a number $0 < \epsilon < P(\xi^o)/4$ and choose a domain G such that $\xi^o \in G$ and

$$|P(\xi)| > P(\xi^o) - \epsilon,\ \xi \in G. \tag{2}$$

Fix $\hat{v}, \hat{w}_0 \in C_0^\infty(G)$ such that $\xi^o \in \mathrm{supp}\hat{v}\hat{f}$ and $< \hat{v}\hat{f}, \hat{w}_0 > \neq 0$. And let $\psi \in C_0^\infty(G)$ and $\psi = 1$ in some neighborhood of supp$\hat{w}_0$. Then for any $m \geq 1$ we get

$$\begin{aligned} |< \hat{v}\hat{f}, \hat{w}_0 >| &= |< \psi(\cdot)P^{-m}(\cdot)P^m(\cdot)\hat{v}(\cdot)\hat{f}(\cdot), \hat{w}_0(\cdot) >| \\ &= |< P^m(\cdot)\hat{v}(\cdot)\hat{f}(\cdot), \psi(\cdot)P^{-m}(\cdot)\hat{w}_0(\cdot) >| \\ &= |< P^m(\cdot)\hat{v}(\cdot)\hat{f}(\cdot), P^{-m}(\cdot)\hat{w}_0(\cdot) >| \\ &= |< F^{-1}(P^m\hat{v}\hat{f}), F(P^{-m}\hat{w}_0) >| \\ &= |< P^m(D)(v * f), F\hat{w}_m >|, \end{aligned}$$

where $\hat{w}_m(\xi) = P^{-m}(\xi)\hat{w}_0(\xi)$. Therefore, by Lemmas 1 and 3 we get for all $m \geq 1$

$$|< \hat{v}\hat{f}, \hat{w}_0 >| \leq ||P^m(D)f||_{M_\Phi} ||v||_{L_1} ||F\hat{w}_m||_{N_\Phi}. \tag{3}$$

Next we prove

$$||F\hat{w}_m||_{N_\Phi} \leq C(P(\xi^o) - 2\epsilon)^{-m},\ m \geq 2n^2. \tag{4}$$

Actually, let $|\alpha| \leq 2n^2$. Since $P(\xi) \neq 0$ in G and the Leibniz formula, we get

$$D^\alpha\big(P^{-m}(\xi)\hat{w}_0(\xi)\big) = \sum_{\beta\leq\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} D^{\alpha-\beta}\hat{w}_0(\xi) D^\beta P^{-m}(\xi), \tag{5}$$

$$D^\beta P^{-m}(\xi) = \sum_{\gamma^1+\cdots+\gamma^m=\beta} \frac{\beta!}{\gamma^1!\ldots\gamma^m!} D^{\gamma^1}P^{-1}(\xi)\ldots D^{\gamma^m}P^{-1}(\xi). \tag{6}$$

Therefore,

$$\begin{aligned} |x^\alpha F\hat{w}_m(x)| &= \Big|\int_G e^{-ix\xi} D^\alpha\big(P^{-m}(\xi)\hat{w}_0(\xi)\big)d\xi\Big| \\ &\leq \sum_{\beta\leq\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} \sum_{\gamma^1+\cdots+\gamma^m=\beta} \frac{\beta!}{\gamma^1!\ldots\gamma^m!} \\ &\times \int_G \big|D^{\alpha-\beta}\hat{w}_0(\xi) D^{\gamma^1}P^{-1}(\xi)\ldots D^{\gamma^m}P^{-1}(\xi)\big| d\xi \end{aligned} \tag{7}$$

for all $x \in \mathbb{R}^n$. On the other hand, we have

$$\sum_{\beta \le \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} = 2^{|\alpha|} \le 2^{2n^2},$$

$$\sum_{\gamma^1+\cdots+\gamma^m=\beta} \frac{\beta!}{\gamma^1!\ldots\gamma^m!} = m^{|\beta|} \le m^{2n^2}.$$

Further, let $m \ge 2n^2$. We remark that for $\gamma^1+\cdots+\gamma^m=\beta$, there are at least $m-|\beta| \ge m-2n^2$ multi-indices among $\gamma^1,\ldots,\gamma^m$ equal zero. Therefore, by (2), (5) - (7) and $\hat{w}_0 \in C_0^\infty(G)$, we obtain a constant $C_1 = C_1(P, \hat{w}_0, 2n^2)$ such that for all $m \ge 2n^2$

$$|x^\alpha F\hat{w}_m(x)| \le (2m)^{2n^2} C_1 (P(\xi^o)-\epsilon)^{-m+2n^2},$$

where

$$C_1 = \max\{\big(P(\xi^o)-\epsilon\big)^{|\beta|-2n^2} \int_G \big|D^{\alpha-\beta}\hat{w}_0(\xi) D^{\gamma^1}P^{-1}(\xi)\ldots D^{\gamma^{|\beta|}}P^{-1}(\xi)\big| d\xi \ : \ \beta \le \alpha,\ |\alpha| \le 2n^2,\ \gamma^1+\cdots+\gamma^{|\beta|}=\beta\} < \infty.$$

Therefore, since

$$\lim_{m\to\infty} (2m)^{2n^2} \big(\frac{P(\xi^o)-2\epsilon}{P(\xi^o)-\epsilon}\big)^m = 0,$$

we obtain a constant $C_2 = C_2(\epsilon)$ such that

$$\sup_{x\in\mathbb{R}^n} |x^\alpha F\hat{w}_m(x)| \le C_2 (P(\xi^o)-2\epsilon)^{-m}$$

for all $|\alpha| \le 2n^2$ and $m \ge 2n^2$.
Therefore

$$\sup_{x\in\mathbb{R}^n} (1+x_1^{2n})\ldots(1+x_n^{2n})|F\hat{w}_m(x)| \le C_3 (P(\xi^o)-2\epsilon)^{-m} \tag{8}$$

for all $m \ge 2n^2$.
Hence

$$|F\hat{w}_m(x)| \le \frac{C_3(P(\xi^o)-2\epsilon)^{-m}}{(1+x_1^{2n})\ldots(1+x_n^{2n})}$$

for all $m \ge 2n^2, x \in \mathbb{R}^n$.
So, putting $g(x) = [(1+x_1^{2n})\ldots(1+x_n^{2n})]^{-1}$ we obtain

$$||F\hat{w}_m||_{N_\Phi} = \int_0^\infty \Phi\big(\lambda_{F\hat{w}_m}(y)\big)dy \le C_3(P(\xi^o)-2\epsilon)^{-m} \int_0^\infty \Phi\big(\lambda_g(y)\big)dy.$$

On the other hand, we have

$$\int_0^\infty \Phi\big(\lambda_g(y)\big)dy = \int_0^\infty \Phi\Big(\text{mes}\{x\in\mathbb{R}^n : \frac{1}{(1+x_1^{2n})...(1+x_n^{2n})} > y\}\Big)dy$$
$$\le \int_0^1 \Phi\Big(\text{mes}\{x\in\mathbb{R}^n : \frac{1}{1+x_j^{2n}} > y, j=1,...,n\}\Big)dy$$
$$= \int_0^1 \Phi\Big(2^n\Big(\frac{1}{y}-1\Big)^{1/2}\Big)dy$$
$$\le \int_0^1 \Phi\big(2^n\frac{1}{y^{1/2}}\big)dy$$
$$\le \int_0^1 2^n\frac{1}{y^{1/2}}\Phi(1)dy < \infty$$

because $\Phi(t)\le t\Phi(1)$ for all $t\ge 1$. Thus we have proved (4)

$$||F\hat{w}_m||_{N_\Phi} \le C(P(\xi^o)-2\epsilon)^{-m},\ m\ge 2n^2,$$

where C depends on ϵ. Combining (3) - (4), we obtain

$$\varliminf_{m\to\infty} ||P^m(D)f||_{M_\Phi}^{1/m} \ge P(\xi^o)-2\epsilon\ .$$

Letting $\epsilon\to 0$, we get (1).

To complete the proof, it remains to show that

$$\varlimsup_{m\to\infty} ||P^m(D)f||_{M_\Phi}^{1/m} \le \sup_{\text{sp}(f)}|P(\xi)|. \tag{9}$$

Given $\epsilon>0$. We choose a domain $G\supset \text{sp}(f)$ and a function $\psi\in C_0^\infty(G)$ such that $\psi=1$ in some neighborhood of sp(f) and

$$\sup_G|P(\xi)| < \sup_{\text{sp}(f)}|P(\xi)|+\epsilon\ . \tag{10}$$

By Lemma 3 we have

$$||P^m(D)f||_{M_\Phi} = ||F^{-1}\big(\psi(\xi)P^m(\xi)\hat{f}(\xi)\big)||_{M_\Phi} \le ||F^{-1}\big(\psi(\xi)P^m(\xi)\big)||_{L_1}||f||_{M_\Phi} \tag{11}$$

for all $m\ge 0$.

Putting $h_m(\xi)=\psi(\xi)P^m(\xi), m\ge 1$ and $\ell=[n/2]+1$, we get from Holder's inequality that

$$||F^{-1}h_m||_{L_1} = ||Fh_m||_{L_1} = \int_{\mathbb{R}^n}(|\hat{h}_m(\xi)|^2)^{1/2}d\xi$$
$$\le \Big(\int_{\mathbb{R}^n}|\hat{h}_m(\xi)|^2(1+|\xi|^2)^\ell d\xi\Big)^{1/2}\Big(\int_{\mathbb{R}^n}(1+|\xi|^2)^{-\ell}d\xi\Big)^{1/2} = C_4\|h_m\|_{(\ell)},$$

where $C_4 < \infty$ is independent of m. Therefore, by (11) and the topological equality $H_{(\ell)} = W_{\ell,2}$, we get

$$||P^m(D)f||_{M_\Phi} \le C_5 ||h_m||_{\ell,2} ||f||_{M_\Phi}, \ m \ge 1 \ . \tag{12}$$

On the other hand, it follows from the Leibniz formula that

$$D^\alpha h_m(\xi) = \sum_{\beta \le \alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!} D^{\alpha-\beta}\psi(\xi) D^\beta P^m(\xi), \tag{13}$$

$$D^\beta P^m(\xi) = \sum_{\gamma^1+\cdots+\gamma^m=\beta} \frac{\beta!}{\gamma^1!\dots\gamma^m!} D^{\gamma^1}P(\xi)\dots D^{\gamma^m}P(\xi). \tag{14}$$

Further, we again notice that for $|\beta| \le \ell \le m$ and $\gamma^1 + \cdots + \gamma^m = \beta$, there are at least $m - |\beta| \ge m - \ell$ multi-indices among $\gamma^1, \dots, \gamma^m$ equal zero. Therefore, combining (10) and (12) - (14), we obtain a constant $C_6 = C_6(\psi, P, \ell)$ such that

$$\begin{aligned} ||P^m(D)f||_{M_\Phi} &\le C_5 C_6 (\sup_G |P(\xi)|)^{m-\ell} ||f||_{M_\Phi} \\ &\le C_5 C_6 (\sup_{\mathrm{sp}(f)} |P(\xi)| + \epsilon)^{m-\ell} ||f||_{M_\Phi} \end{aligned}$$

for all $m \ge \ell$. Hence

$$\overline{\lim_{m\to\infty}} ||P^m(D)f||_{M_\Phi}^{1/m} \le \sup_{\mathrm{sp}(f)} |P(\xi)| + \epsilon.$$

Letting $\epsilon \to 0$, we get (9). The proof of Theorem 1 is complete. ∎

Remark 1. Theorem 1 still holds when we replace $P(D)$ by the following pseudodifferential operator:

$$A(D)f = F^{-1}a(\xi)Ff(\xi),$$

where $a(\xi)$ is an arbitrary function in $C^\infty(\mathbb{R}^n)$.

3. An Application

Let us now apply Theorem 1 to obtain certain Paley-Wiener-Schwartz theorems.

Theorem 2. *Let $f \in M_\Phi(\mathbb{R}^n)$, $b_j \ne 0, j = 1, \cdots, n$ and $r > 0$. Then* $\mathrm{sp}(f)$ *is contained in the ellipsoid* $\{\xi \in \mathbb{R}^n : \frac{\xi_1^2}{b_1^2} + \cdots + \frac{\xi_n^2}{b_n^2} \le r^2\}$ *if and only if*

$$\lim_{m\to\infty} \Big\| \Big(\frac{1}{b_1^2} D_1^2 + \cdots + \frac{1}{b_n^2} D_n^2 \Big)^m f \Big\|_{N_\Phi}^{1/m} \le r^2.$$

Further, let $P(\xi)$ be a polynomial, $K \subset \mathbb{R}^n, r > 0$, $\mathcal{B}(0, r)$ the ball of radius r centered at zero, $\sigma = (\sigma_1, \dots, \sigma_n), \sigma_j > 0$ and $\Delta_\sigma = \{\xi \in \mathbb{R}^n : |\xi_j| \le \sigma_j, j = 1, \dots, n\}$.

We put

$$Q(K,P) = \{\xi \in \mathbb{R}^n : |P(\xi)| \le \sup_K |P(x)|\},$$
$$Q(K,P,\sigma) = Q(K,P) \cap \Delta_\sigma,$$
$$Q(K,P,r) = Q(K,P)\mathcal{B}(0,r).$$

Clearly, $K \subset Q(K,P), Q(K,P)$ can be noncompact although K is compact, and $Q(K,P), Q(K,P,\sigma)$ and $Q(K,P,r)$ can be nonconvex.

Using Theorem 1, we have the following results:

Theorem 3. *Let $f \in M_\Phi(\mathbb{R}^n)$. Then* $\mathrm{sp}(f) \subset Q(K,P,\sigma)$ *if and only if*

1) $\underline{\lim}_{m\to\infty} ||P^m(D)f||_{M_\Phi}^{1/m} \le \sup_K |P(\xi)|$,
2) $\underline{\lim}_{m\to\infty} \Big\|\dfrac{\partial^m}{\partial x_j^m} f\Big\|_{M_\Phi}^{1/m} \le \sigma_j,\ j = 1,\ldots,n.$

Theorem 4. *Let $f \in M_\Phi(\mathbb{R}^n)$. Then* $\mathrm{sp}(f) \subset Q(K,P,r)$ *if and only if*

1) $\underline{\lim}_{m\to\infty} ||P^m(D)f||_{M_\Phi}^{1/m} \le \sup_K |P(\xi)|$,
2) $\underline{\lim}_{m\to\infty} ||\Delta^m f||_{M_\Phi}^{1/m} \le r^2$,
where Δ is the Laplacian.

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