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Weakly Prime Modules*

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Abstract. A left R-module M is called weakly prime module if the annihilator of any nonzero submodule of M is a prime ideal and a proper submodule P of M is called weakly prime submodule if the quotient module M/P is a weakly prime module. This notion is introduced and extensively studied. The module in which, weakly prime submodules and the prime submodules coincide, are studied, and it is shown that multiplicative modules have this property called compatibility property. It is also shown that each R-module is compatible if and only if each prime ideal is maximal or if and only if the R-module $R \oplus R$ is compatible. Over commutative rings the modules in which every proper submodule (proper nonzero submodules) is weakly prime are characterized. It is proved that if $\dim R < \infty$, then each R-module has a prime submodule if and only if it has weakly prime submodule.

1. Introduction

Let M be a left R-module. Then a proper submodule P of M is called prime if for any $r \in R$ and $m \in M$ such that $rRm \subseteq P$, we have $m \in P$ or $rM \subseteq P$. Equivalently, P is prime if for any ideal A of R and any submodule N of M such that $AN \subseteq P$, either $AM \subseteq P$ or $N \subseteq P$. Recently, this notion of prime submodule has been extensively studied by various authors (cf. [5, 3, 9, 14, 12, 11, 1, 2] and many others).

In this article we introduce a slightly different notion of prime submodule and call it weakly prime submodule. A proper submodule P of M is called a weakly prime submodule if whenever $K \subseteq M$ and $rRsK \subseteq P$, where $r, s \in R$,

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then either $rK \subseteq P$ or $sK \subseteq P$. Clearly, by comparing with the usual definition of prime ideals in rings, our definition of weakly prime submodule seems more natural. It is also clear that each prime submodule is weakly prime but not conversely. Our aim in this article is to investigate weakly prime submodules over a ring which is not necessarily commutative. We call an R-module M compatible if its prime submodules and its weakly prime submodules are the same. A ring R is said to be compatible if every R-module is compatible. We show that multiplicative modules are compatible and a ring R is compatible if and only if every prime ideal is maximal or equivalently, if and only if the R-module $R \oplus R$ is compatible. We also prove that if every torsion R-module is compatible, then dim $R \leq 1$ and in the case of commutativity the converse also holds. We show that if R has finite dimension (classical Krull-dimension), then each R-module has a weakly prime submodule if and only if every R-module has a prime submodule.

All rings in this paper are associative with identity and modules are unitary left modules. Let us also call a left ideal I of R a prime left ideal, if for every two left ideals A, B of R with $AB \subseteq I$, either $A \subseteq I$ or $B \subseteq I$. An outline of this paper is as follows.

This article consists of six sections. In Sec. 2 give some preliminary facts about weakly prime modules. Sec. 3 is devoted to compatible rings and modules. In Sec. 4 we give several equivalent conditions for a module to be fully (almost fully) weakly prime module. In Sec. 5 a brief study of rings over which each module has a weakly prime submodule is made. Finally, in Sec. 6, we give some useful miscellaneous facts about weakly prime modules and some open problems.

2. Weakly Prime Modules

Definition. *A left R-module M is called weakly prime if the annihilator of any nonzero submodule of M is a prime ideal and a proper submodule $P \subset M$ is called weakly prime submodule if the quotient module M/P is a weakly prime module (i.e, if $rRsK \subseteq P$, where $r, s \in R$, $K \leq M$, implies either $rK \subseteq P$ or $sK \subseteq P$).*

Before we continue, let us show that a weakly prime submodule need not be a prime submodule.

Example 1. Assume that R is a commutative domain and P is a non-zero prime ideal in R. It is trivial to see that $P \oplus 0$, $0 \oplus P$ and $P(1,1)$ are weakly prime submodules in the free module $M = R \oplus R$, but they are not prime submodules.

Example 2. Assume that R is a domain that is not a simple ring and $M = N \oplus D$, where N is a simple module and D is a torsion-free R-module, then M is weakly prime module which is not prime.

Clearly any ideal P in a ring R is a prime ideal if and only if it is a prime (weakly prime) submodule of the R-module R, but we may have a left ideal P in a ring R such that it is a weakly prime submodule of R and is not a prime submodule (i.e., it is not a prime left ideal). To see this, consider the next example.

Example 3. Take V to be an infinite dimensional vector space over a division ring D. It is well-known that each non-zero ideal of $R = \mathrm{End}_D(V)$ is of the form

$$I_c = \{f \in R \,:\, \dim(f(V)) < c\}$$

where c is a cardinal number with $c \leq \dim V$, i.e., each ideal of R is prime. Clearly, R is not a simple ring, i.e., not every left ideal of R is a prime left ideal; see [4] and [13] and note that in any ring, prime left ideads and prime submodules coincide. But by our definition of weakly prime submodules, each left ideal of R is a weakly prime submodule of R. Therefore R has a left ideal which is a weakly prime submodule of R and not a prime submodule.

It is trivial to see that M is a faithful prime module if and only if each non-zero submodule of M is faithful (i.e., M is fully faithful) and N is a prime submodule of M if and only if $P = Ann_l(M/N)$ is a prime ideal and M/N is fully faithful as an R/P-module, this means that an R-module M is prime if and only if $P = Ann_l(M) = Ann_l(Rm)$ is a prime ideal for all $0 \neq m \in M$. But for the weakly prime modules we have the following evident result.

Proposition 2.1. *Let M be an R-module. Then the following are equivalent.*
1. *M is a weakly prime module.*
2. *Each direct summand of M including the zero summand is a weakly prime submodule.*
3. *For each $0 \neq m \in M$, $Ann_l(Rm)$ is a prime ideal.*
4. *$P = Ann_l(M)$ is a prime ideal and the set $\{Ann_l(Rm) : 0 \neq m \in M\}$ is a chain of prime ideals.* ■

If M is a homogeneous semisimple module, then every submodule of M is a weakly prime submodule, in fact a prime submodule.

Next, we determine weakly prime modules over zero dimensional rings.

Proposition 2.2. *Let every nonzero prime ideal of a ring R be maximal. Then M is a weakly prime R-module, if and only if one of the following statements holds.*
1. *M is a fully faithful module.*
2. *$Ann_l(M)$ is a maximal ideal.*
3. *$M \cong N + K$, where $Ann_l(K)$ is a maximal ideal and N is a fully faithful R-module (in this case if R is not a simple ring then M is not a prime module).*

Proof. Evident. ■

Corollary 2.3. *Let every nonzero prime ideal in a commutative ring R be maximal. Then M is a weakly prime R-module if and only if one of the following statements holds.*
1. *M is a torsion free R-module.*
2. *M is a homogeneous semisimple R-module.*

3. *M is a sum of a homogeneous semisimple module and a torsion free module.* ■

Clearly, each summand of a weakly prime R-module is a weakly prime R-module. Therefore it is natural to consider modules which are not indecomposable and not weakly prime but their nonzero summands are weakly prime modules. The following characterizes these modules.

Proposition 2.4. *Let M be an R-module, which is not weakly prime and not indecomposable. Then every non-zero summand of M is weakly prime module if and only if every decomposition of M is of the form $M = N \oplus K$, where N, K are indecomposable weakly prime modules.*

Proof. Assume that $M = N \oplus K$, where $N \neq (0) \neq K$ are submodules of M. By our hypothesis N and K are weakly prime modules, i.e., by Proposition 2.1, $\{\mathrm{Ann}_l(Rx) : 0 \neq x \in N\}$ and $\{\mathrm{Ann}_l(Ry) : 0 \neq y \in K\}$ are chains of prime ideals of R. Now we claim that N, K are indecomposable. To see this, let us take, for example, N to be decomposable and seek a contradiction. Now put $N = N_1 \oplus N_2$, $N_1 \neq (0) \neq N_2$, i.e., $M = N_1 \oplus N_2 \oplus K$ and we show that the annihilator of each nonzero cyclic submodule of M is prime. Let $m = x_1 + x_2 + k \in M$, where $0 \neq x_1 \in N_1$, $0 \neq x_2 \in N_2$, $0 \neq k \in K$ and $aRb(Rm) = (0)$, $a, b \in R$. Thus for each $r, s \in R$, we have $arbsx_1 = arbsx_2 = arbsk = 0$, i.e., $aRb(Rx_1) = aRb(Rx_2) = aRb(Rk) = (0)$. But N_1, N_2, K, $N_1 \oplus N_2$, $N_2 \oplus K$ and $N_1 \oplus K$ are all weakly prime modules by our hypothesis, i.e., $aRb(Rx_1 + Rx_2) = aRb(Rx_1 + Rk) = aRb(Rx_2 + Rk) = (0)$, which implies that either $aRx_1 = aRx_2 = aRk = (0)$ or $bRx_1 = bRx_2 = bRk = (0)$. This shows that $a(Rm) = (0)$ or $b(Rm) = (0)$, i.e., $\mathrm{Ann}(Rm)$ is prime and by Proposition 2.1, M becomes weakly prime module, which is a contradiction. The converse is evident. ■

Remark 1. In [2], the previous result is proved for prime modules over commutative rings, and in particular it is shown that the above decomposition is unique. But we admit that we were not able to show the validity of this uniqueness for weakly prime modules over commutative rings.

3. Compatible Modules

We recall that an R-module M is called compatible if its weakly prime submodules and prime submodules coincide. The field of fractions of a domain R is a compatible R-module. All commutative rings, semisimple modules and multiplicative modules are examples of compatible modules. We also recall that a ring R is compatible if any R-module is compatible. Clearly, each factor ring of a compatible ring is a compatible ring. Finally, an R-module M is called multiplicative if each submodule of M is of the form IM, where I is a left ideal, i.e., without losing generality we may assume that I is a two-sided ideal.

The following lemma which is well-known when R is commutative, is needed.

Lemma 3.1. *Let M be a multiplicative R-module. Then the following statements are equivalent.*
1. *$N \subset M$ is a prime submodule.*
2. *$Ann_l(M/N)$ is a prime ideal.*
3. *$N = PM$, where P is a prime ideal which is maximal with respect to this property (i.e., $IM \subseteq N$ implies that $I \subseteq P$).*

Proof. Evident. ∎

Proposition 3.2. *Every multiplicative R-module is a compatible module.*

Proof. Evident. ∎

The following result shows compatible rings are abundant.

Theorem 3.3. *Let R be a ring. Then the following are equivalent.*
1. *R is a compatible ring.*
2. *The R-module $R \oplus R$ is compatible.*
3. *Every prime ideal is maximal.*

Proof. (1) $\Rightarrow$ (2) is evident. ∎

(2) $\Rightarrow$ (3). Let P_1 be a prime ideal in R and let P_2 be a maximal ideal with $P_1 \subseteq P_2$. Now we claim that $N = P_1 \oplus P_2$ is a weakly prime submodule of $M = R \oplus R$. To see this, let $rRs(R(x,y)) \subseteq P_1 \oplus P_2$, where $(x,y) \in M \setminus P_1 \oplus P_2$. Thus $rRs(Rx) \subseteq P_1$ and $rRs(Ry) \subseteq P_2$. But either we have $x \notin P_1$ or $y \notin P_2$. This implies as in the former case that we have either $r \in P_1$ or $s \in P_1$, i.e., $rR(x,y) \subseteq P_1 \oplus P_2$ or $sR(x,y) \subseteq P_1 \oplus P_2$, which means that $P_1 \oplus P_2$ is a weakly prime submodule. Similarly, if $y \notin P_2$, then $N = P_1 \oplus P_2$ becomes a weakly prime submodule. Now by our hypothesis N is a prime submodule of M. Finally, we note that $P_2R(0,1) \subseteq P_1 \oplus P_2$, and since $(0,1) \notin P_1 \oplus P_2$, we must have $P_2M = P_2(R \oplus R) \subseteq P_1 \oplus P_2$, i.e., $P_1 = P_2$ and we are through.
(3) $\Rightarrow$ (1) is evident by Proposition 2.1. ∎

In what follows by $\dim R$ we mean the classical Krull-dimension of R.

Proposition 3.4. *Let each R-module whose cyclic submodules have nonzero annihilator be compatible. Then* $\dim R \leq 1$.

Proof. Let $P_1 \subseteq P_2$ be nonzero prime ideals in R. Now put $M = R/P_1 \oplus R/P_2$ and note that P_1 annihilates M. Clearly, for each $0 \neq m \in M$, we have either $\mathrm{Ann}_l(Rm) = P_1$ or $\mathrm{Ann}_l(Rm) = P_2$, i.e., $\{\mathrm{Ann}_l(Rm) : 0 \neq m \in M\} = \{P_1, P_2\}$, thus M is a weakly prime R-module by Proposition 2.1 and therefore it is a prime R-module. Thus we must have $P_1 = P_2$ and the proof is complete. ∎

The next two corollaries are immediate.

Corollary 3.5. *Let R be a commutative ring. Then each R-module M with $Ann(x) \neq (0)$ for all $x \in M$ is compatible if and only if* $\dim R \leq 1$.

Corollary 3.6. *Let R be a commutative domain. Then each torsion R-module is compatible if and only if R is a field.*

4. Fully Weakly Prime Modules and Almost Fully Weakly Prime Modules

A ring R is called a fully prime ring if each ideal of R is prime and it is called an almost fully prime ring if each nonzero ideal of R is prime. These rings are fully investigated in [4] and [13]. For example, in [4] it is shown that R is a fully prime ring if and only if each ideal of R is idempotent and the set of ideals of R is a chain. It is also shown in [13] that R is a simple ring if and only if each left ideal of R is a prime left ideal. Similarly, we call an R-module M a fully weakly prime module if every submodule of M is a weakly prime submodule and M is called almost fully weakly prime if each nonzero submodule of M is a weakly prime submodule. In this section we try to characterize fully weakly prime and almost fully weakly prime R-modules in the same vein.

The following shows that fully prime rings give us a big set of fully weakly prime modules. We should note that our third example $R = \mathrm{End}_D(V)$ shows that, although R is a fully prime ring, it is not a fully prime module.

Lemma 4.1. *Let R be a fully prime ring. Then each R-module is fully weakly prime R-module.* ■

The comparison of the following two immediate corollaries is of interest.

Corollary 4.2. *Let R be a ring. Then the following statements are equivalent.*
1. *Each R-module is fully weakly prime.*
2. *The R-module R is fully weakly prime.*
3. *R is a fully prime ring.* ■

Corollary 4.3. *Let R be a ring. Then the following statements are equivalent.*
1. *Each R-module is prime.*
2. *The R-module R is fully prime.*
3. *Each left ideal of R is a prime left ideal.*
4. *R is a simple ring.* ■

The next result extends a fact of fully prime rings to fully weakly prime modules; see [4], [13].

Proposition 4.4. *Let M be an R-module. Then M is a fully weakly prime R-module if and only if for each submodule $K \subseteq M$ and each ideal I of R, $IK = I^2K$ and also for any two ideals A, B, AK and BK are comparable.*

Proof. Let M be a fully weakly prime R-module, $K \subseteq M$ a submodule, and I an ideal of R. If $I^2K = M$, then clearly $I^2K = IK = M$. Thus we may assume that $I^2K \neq M$, i.e., I^2K is a weakly prime submodule, i.e., $I \cdot IK \subseteq I^2K$

implies that $IK = I^2K$. Now if A, B are two ideals of R, then we may assume that $AK \neq M \neq BK$. But $AK \cap BK$ is a weakly prime submodule of M, i.e., $ABK \subseteq AK \cap BK$ implies that $AK \subseteq BK$ or $BK \subseteq AK$. Conversely, we have to show that each submodule N of M is a weakly prime submodule. To see this, let $ABK \subseteq N$, where A, B are ideals of R. By our hypothesis we may assume that $AK \subseteq BK$, i.e., $AK = A^2K \subseteq ABK \subseteq N$, which means that N is a weakly prime submodule.

In [2], we have shown that over commutative rings, fully prime modules coincide with homogeneous semisimple modules. But Corollary 4.3 shows that, generally fully prime modules need not be semisimple.

The following lemma is needed.

Lemma 4.5. *Let M be a faithful R-module which is also a fully prime module. Then for each $0 \neq m \in M$, $Ann_l(m)$ is either zero or does not contain a proper nonzero ideal and is contained in at most a unique proper ideal.*

Proof. We have $Rm \cong R/I$, where $I = \mathrm{Ann}_l(m)$. Let us assume that $I \neq (0)$. Now we have $\mathrm{Ann}_l(Rm) = \mathrm{Ann}_l(M) = (0)$, i.e., I does not contain a nonzero ideal. Now let $I \subsetneq A$, where A is an ideal of R, then $(R/I)/(A/I)$ is fully prime R-module, i.e., R/A is fully prime R-module. Clearly, R/A is also fully prime as an R/A-module. Thus by Corollary 4.3, R/A is a simple ring, i.e., A is maximal. The uniqueness of A is now evident by the first part of the proof.

The next result, whose proof is similar to the proof of Proposition 1.10 in [2] gives us some information about fully weakly prime modules over commutative rings.

Proposition 4.6. *Let M be a module over a commutative ring R. Then the following are equivalent.*

1. *M is a fully weakly prime module.*
2. *Each cyclic submodule of M is a weakly prime submodule.*
3. *Each cyclic submodule of M is semiprime and M is a weakly prime module.*
4. *M is a homogeneous semisimple module.*
5. *M is both a semisimple and a weakly prime module.*
6. *M is a weakly prime module and the set of cyclic submodules satisfies the minimum condition.*
7. *M is a weakly prime module and is co-semisimple (each submodule is an intersection of maximal submodules).*
8. *M is a weakly prime module and Von-Neumann regular (i.e., each cyclic submodule of M is a summand).*

Remark 2. Clearly, each submodule of a homogeneous semisimple R-module, where R is commutative, is a prime submodule, i.e., in view of the previous result we infer that M is a fully weakly prime module if and only if it is a fully prime module; see also Proposition 1.10 in [2]. Finally, it is observed in [2], that if M is a prime R-module and $\mathrm{Socle}(M) \neq (0)$, then M is a homogeneous semisimple module. We observe that this is not true for weakly prime modules, for let $R = \mathbb{Z}$ and $M = \mathbb{Z} \oplus \mathbb{Z}_p$,where p is a prime number. Then clearly M is a weakly prime $\mathbb{Z}$-module and $\mathrm{Socle}(M) = \mathbb{Z}_p$, but M is not very semisimple.

Next, we aim to provide some information on almost fully weakly prime modules. Clearly, each nonzero submodule of an almost fully weakly prime module is an almost fully weakly prime module. The following lemma is also trivial in view of the previous proposition.

Lemma 4.7. *Let R be a commutative ring. Then the following statements are equivalent.*
1. *M is an almost fully weakly prime R-module.*
2. *For each proper nonzero submodule N of M, M/N is a homogeneous semisimple module.*
3. *For each nonzero cyclic submodule Rm of M, $M/(Rm)$ is a homogeneous semisimple module.* ■

The following shows that over commutative rings, non-prime almost fully prime modules and non-weakly prime almost fully weakly prime modules are the same.

Proposition 4.8. *Let M be a non-prime (non-weakly prime) R-module, where R is a commutative ring. Then the following are equivalent,*
1. *M is an almost fully weakly prime R-module.*
2. *M is an almost fully prime R-module.*
3. *M is either a direct sum of two simple modules or the Jacobson radical of M is nonzero and the smallest submodule of M. Moreover each nonzero cyclic submodule of M contains at most one nonzero proper submodule.*

Proof. (1) $\Rightarrow$ (2). Let $N \neq (0)$ be a proper submodule of M, i.e., by Lemma 4.7, M/N is homogeneous semisimple, i.e., M/N is a fully prime module and we are through.
(2) $\Leftrightarrow$ (3) is Theorem 1.13, in [2].
(2) $\Rightarrow$ (1) is trivial. ■

5. Rings Over Which Every Module Contains a Weakly Prime Submodule

Let us call a ring R a P-ring (weakly P-ring) if every R-module has a prime (weakly prime) submodule. In [2] it is shown that commutative P-rings coincide with Max-rings (modules have maximal submodules). Clearly, simple rings are P-ring, i.e., they are also weakly P-ring. It is well-known that $\mathbb{Z}_\infty$ does not contain a prime submodule; see [2, 3, 11]. The next result shows in fact $\mathbb{Z}_\infty$ has also no weakly prime submodules.

Proposition 5.1. *Let R be a ring with $\dim R < \infty$, and M be an R-module. Then M has a prime R-submodule if and only if it has a weakly prime R-module.*

Proof. Let M have a weakly prime submodule. We have to show that M has a prime submodule. If K is a weakly prime submodule of M, then M/K is a weakly prime R-module. It suffices to show that M/K has a prime submodule. Since the annihilator of M/K is a prime ideal, without losing any generality, we may assume that M is a faithful weakly prime R-module and R is a prime ring. Therefore the set $\{\mathrm{Ann}(Rm) : 0 \neq m \in M\} = T$ is a chain of prime ideals. If T is a singleton, then $T = (0)$, i.e., M becomes a prime R-module and we are through. Thus the chain T contains a nonzero element and clearly $N = \{x \in M : (0) \neq \mathrm{Ann}\,(Rx) \in T\}$ is a submodule of M. Since $\dim R < \infty$ we infer that T has both *acc* and *dcc* condition, i.e., if P is the minimal element among the nonzero elements of T, then we have $P = \mathrm{Ann}\,(N)$. Now we claim that N is a prime submodule of M. First, we note that $P = \mathrm{Ann}\,(N) \neq (0)$, which implies that N is a proper submodule of M. Then, let $IRm \subseteq N$ and we show that either $m \in N$ or $IM \subseteq N$. Thus we may assume that $I \neq (0)$, i.e., $PI(Rm) = (0)$. Since R is a prime ring we infer that $PI \neq (0)$, i.e., $\mathrm{Ann}(Rm) \neq (0)$ which means that $m \in N$ and the proof is complete. ∎

The following is now immediate.

Theorem 5.2. *Let R be a ring with $\dim R < \infty$. Then R is a P-ring if and only if it is a weakly P-ring.* ∎

6. Some Odds and Ends about Weakly Prime Submodules

In this section we try to extend some useful and well-known facts on prime ideals to weakly prime submodule. Minimal weakly prime submodules are defined in a natural way. It is clear that whenever $\{P_i\}_{i\in I}$ is a chain of weakly prime submodules of an R-module M, then $\bigcap_{i\in I} P_i$ is always a weakly prime submodule, i.e., by Zorn's lemma each weakly prime submodule of M contains a minimal one. It is also evident that if $\bigcup_{i\in I} P_i \neq M$, then $\bigcup_{i\in I} P_i$ is a weakly prime submodule.

The following is of interest. For its proof we just mimic the usual proof in the case of prime ideals.

Proposition 6.1. *Let M be an R-module, where R is a commutative ring, and let $P \subset Q$ be two distinct weakly prime submodules of M. Then there are two distinct weakly prime submodules P_1, Q_1 with $P \subseteq P_1 \subset Q_1 \subseteq Q$, where there is no proper weakly prime submodule between P_1 and Q_1.*

It is well-known that Cohen's theorem is extended to prime submodules, see [11, 1], i.e., for a commutative ring, it is trivially valid for weakly prime submodules. In [11] it is shown that Noetherian modules contain only finitely many minimal prime submodules. From this fact one cannot infer that in these modules the set of minimal weakly prime submodules are also finite. But using the same method of [11] we have the following.

Proposition 6.2. *Every Noetherian R-module M, where R is a commutative* *ring, contains only finitely many weakly prime submodules.*

Proof. Let us put $T = \{N \subseteq M : M/N$ has an infinite number of minimal weakly prime submodules$\}$. If $(0) \notin T$, then we are through, i.e., we may assume that $(0) \in T \neq \emptyset$ and seek a contradiction. Now let K be a maximal element of T. Clearly, K cannot be weakly prime. Thus there exist $m \in M \setminus K$ and ideals I, J in R with $IJm \subseteq K$ such that $Im \nsubseteq K$, $Jm \nsubseteq K$. The maximality of K implies that both $M/(Jm + K)$ and $M/(Im + K)$ have only finitely many minimal weakly prime submodules. Now let P/K be a minimal weakly prime submodule of M/K, i.e., $IJm \subseteq K \subseteq P$, which implies that either $Im \subseteq P$ or $Jm \subseteq P$. Thus either $P/(Im + K)$ is a minimal weakly prime submodule of $M/(Im + K)$ or $P/(Jm + K)$ is a minimal weakly prime submodule of $M/(Jm + K)$. This means that the number of choices for P is finite, which is a contradiction. ∎

Next, we observe that weakly prime modules behave naturally under localization. In the next result which is given without proof, R is a commutative ring, S is a multiplicatively closed set in R with $0 \notin S$, $1 \in S$ and M_S denotes the module of fractions with respect to S and $f : M \to M_S$ denotes the natural homomorphism.

Proposition 6.3. *If P is a weakly prime submodule of an R-module M with $P_S \neq M_S$, then P_S is also a weakly prime submodule of M_S. Moreover if Q is a prime R_S-submodule of M_S, then $Q^c = \{m \in M : f(m) \in Q\}$ denotes a weakly prime submodule of M.*

Remark 3. In [8] the prime avoidance lemma is extended to prime submodules. The same proof can be applied to show the validity of this fundamental result for the weakly prime submodules.

Finally, we conclude this article with a remark which gives good reason for the study of weakly prime modules.

Remark 4. Clearly, over a commutative ring R each weakly prime module is a semiprime module, i.e., the class of weakly prime modules lies properly between the class of semiprime modules and the class of prime modules. It is well-known that not every semiprime submodule of an R-module M is an intersection of prime submodules, see [6]. We will observe shortly that the example given in [6] for that purpose is in fact a weakly prime submodule. Thus, it is interesting to characterize commutative rings over which each weakly prime submodule is an intersection of prime submodules (see [10], where commutative rings over which semiprime submodules are such an intersection, are characterized). It is also of interest to see whether each semiprime submodule of an R-module is an intersection of weakly prime submodules. Finally, let us present the proof which we mentioned. Let $R = \mathbb{Z}[x]$ and $F = R \oplus R$. If $f = (2, x) \in F$, and $P = 2R + Rx$ which is a maximal ideal of R. In [6] it is shown that $N = Pf$ is a semiprime submodule of F which is not an intersection of prime submodules. We claim that N is in fact a weakly prime submodule of F. For, let $rs(g, h) \in N = (2R+Rx)(2, x)$, where $r, s \in R$, $(g, h) \in F$, i.e., there exists $p \in P$

with $rsg = 2p$, $rsh = px$. Thus, we have $2rsh = rsgx$, i.e., $rs(2h - gx) = 0$. Since R is a domain, we must have $rs = 0$ or $2h = gx$. But if $rs = 0$, then clearly N becomes weakly prime. Thus we may assume that $2h = gx$, i.e., $x|h$. This means that $h = h'x$ and $g = 2h'$, i.e., $(g,h) = (2h', h'x) = h'(2,x)$. But $rs(g,h) = rsh'(2,x) \in N = P(2,x)$ implies that $rsh' \in P$, i.e., $rh' \in P$ or $sh' \in P$. Thus, either $r(g,h) = rh'(2,x) \in N$ or $s(g,h) = sh'(2,x) \in N$, and the proof is complete.

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