*Vietnam Journal of Mathematics* **32**:2 (2004) 155–160

Vietnam Journal
of
MATHEMATICS


# A Note on Independence Systems and Matroids*

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Received December 6, 2002
Revised June 20, 2003

**Abstract.** In this paper, we study a relationship between a forbidden induced subsystems and matroids. Using our results, we give a new proof of an important result of Zverovich. Some further research problems are given in this paper.

## 1. Introduction

The purpose of this paper is to provide a forbidden induced subsystem characterization of matroids via forbidden induced independence subsystems. The family of "FIST" of independence system are exactly the minimal forbidden subsystems for matroids. Using our results, we give a new proof of a matroid axiom, which can be used to deduce a general method for characterizing new classes of $\mathcal{P}$-matrogenic graphs (see [8]).

We shall start by recalling some definitions and facts about independence systems and matroids.

**Definition 1.1.** *Let $S$ be a finite set and let $\Im$ be a family of subsets of $S$. The pair $\Sigma = (S, \Im)$ is called an independence system if*
(I1) $\emptyset \in \Im$*; and*
(I2) *for every $I \in \Im$ and $J \subseteq I$ it follows that $J \in \Im$.*

*This work was supported partially by Education Ministry of China (No: 02139).

The elements of $\mathfrak{I}$ are called *independent sets* of $\Sigma$ and $S$ is the support of $\Sigma$. A set $X \subseteq S$ is *dependent* if $X \notin \mathfrak{I}$. The maximal with respect independent sets of an independence system $\Sigma$ are called bases, while the inclusion minimal dependent sets are called circuits.

It is clear that any independence system $\Sigma = (S, \mathfrak{I})$ can be defined also as a pair $(S, \mathfrak{C})$, where $\mathfrak{C}$ is the set of all circuits of $\Sigma$. Independence systems defined in terms of their circuits (i.e., given in *circuit form*) will be denoted by $\Sigma^c$. It should be noted that the set $\mathfrak{C}$ of all circuits forms a *clutter*, i.e., no circuit of $\mathfrak{C}$ can be included in another circuit of $\mathfrak{C}$.

**Definition 1.2.** *An independence system $\Sigma = (S, \mathfrak{I})$ is called a matroid if the following 'exchange axiom' holds:*
(M) *for every $I, J \in \mathfrak{I}$ with $|I| < |J|$ there exists an element $m \in J \backslash I$ such that $I \cup \{m\} \in \mathfrak{I}$.*

*Remark.* An independence system with at most one circuit is a matroid.

*Proof.* Suppose $\Sigma(X)$ is an independence system with at most one circuit. If $\Sigma(X)$ is not a matroid i.e., there exist bases $I$ and $J$ with $|I| < |J|$, for which (M) does not hold. In particular, $I \not\subset J$ and $|I \backslash J| \geq 1$. Hence $|J \backslash I| \geq 2$. Let us choose two distinct elements $m$ and $n$ in $J \backslash I$. Since (M) does not hold, $I \cup \{m\}$ contains a circuit $C^m$. Clearly, $m \in C^m$, since otherwise $C^m \subseteq I$, which is impossible. Similarly, $I \cup \{n\}$ contains a circuit $C^n$ with $n \in C^n$. In fact $C^m$ and $C^n$ are distinct circuits, since $n \notin C^m$ and $m \notin C^n$. Since we have seen that $\Sigma(X)$ has at most one circuit, we obtain a contradiction. Thus, $\Sigma(X)$ is a matroid. ■

**Definition 1.3.** *Let $\Sigma = (S, \mathfrak{I})$ be an independence system. For $X \subseteq S$, an independence subsystem $\Sigma(X) = (X, \mathfrak{I}')$ is induced by $X$ as follows:*

$$I \in \mathfrak{I}' \text{ if and only if } I \subseteq X \text{ and } I \in \mathfrak{I}.$$

Accordingly, an *induced subsystem* of $\Sigma = (S, \mathfrak{I})$ is any independence system which is isomorphic to $\Sigma(X)$ for some $X \subseteq S$. (The *isomorphism* of independence systems is defined in the usual way.)

Note that the induced subsystem $\Sigma(X)$ can be equivalently defined as the independence system obtained from $\Sigma$ by deleting $S \backslash X$ (from $S$) along with all circuits which are not contained in $X$.

We recall the following definitions and facts.

**Proposition 1.4.** (see [1]) *Any induced subsystem of a matroid is a matroid.*

**Definition 1.5.** *An independence system is called well-covered if all its bases have the same cardinality.*

Condition (M) obviously implies the following known statement.

**Proposition 1.6.** *Any matroid is a well-covered independence system.*

**Definition 1.7.** *An independence system is said to be strongly well-covered if all its induced subsystems are well-covered.*

Propositions 1.4 and 1.6 imply the following result.

**Corollary 1.8.** *Any matroid is a strongly well-covered independence system.*

**Proposition 1.9.** *An independence system is a matroid if and only if it is a strongly well-covered independence system.*

*Proof.* By Corollary 1.8, it is sufficient to show that a strongly well-covered independence system $\Sigma = (S, \mathfrak{I})$ is a matroid. To check condition (M), we shall consider two arbitrary sets $I_1, I_2 \in \mathfrak{I}$ with $|I_1| < |I_2|$. By definition, the independence subsystem induced by $X = I_1 \cup I_2$ is well-covered. It follows from the fact $|I_1| < |I_2|$ that $I_1$ is not a base of $\Sigma(X)$. So there exists an element $i \in X \backslash I_1$ such that $I' = I_1 \cup \{i\}$ is an independent set of $\Sigma(X)$. Clearly, $i \in I_2$ and $I' \in \mathfrak{I}$. ∎

## 2. Using Forbidden Induced Subsystems to Characterize Matroids

Proposition 1.9 implies that it is possible to characterize matroids *in terms of forbidden induced subsystems*, i.e., in terms of a set $\mathcal{W}$ of independence systems, that an independence system $\Sigma$ is a matroid if and only if $\Sigma$ does not contain an induced subsystem isomorphic to a member of $\mathcal{W}$.

The family (denoted by "FIST") of independence systems is to be shown below to consist of all the minimal forbidden induced subsysems for matroids.

Since an independence system containing at most one circuit is a matroid, we shall require that every independence system $\Sigma^c = (S, \mathfrak{C})$ in the FIST family should contain at least two circuits. For every $\Sigma^c = (S, \mathfrak{C}) \in$ FIST, each element $s \in S$ will be required to be in at least $|\mathfrak{C}| - 1$ circuits. Also, $\Sigma^c$ should have at least one element that belongs to all circuits. Now we state all these conditions more precisely.

**Definition 2.1.** *An independence system $\Sigma^c = (S, \mathfrak{C})$ belongs to the FIST family if and only if*
(N1) $|\mathfrak{C}| \geq 2$
*and there exists a partition $U \cup V = S$ such that*
(N2) $|U| \geq 1$,
(N3) *each element of $U$ belongs to all circuits of $\Sigma^c$, and*
(N4) *for each element of $V$ there exists a unique circuit of $\Sigma^c$ which does not contain it.*

**Theorem 2.2.** *None of the independence systems in the FIST family is a matroid.*

*Proof.* Let $\Sigma^c = (S, \mathfrak{C})$ be any independence system in the FIST family. By (N1) of Definition 2.1, $|\mathfrak{C}| \geq 2$, which implies that $\Sigma^c = (S, \mathfrak{C})$ is not a matroid by the remark in Sec. 1. ■

Now we prove that the FIST family consists of *minimal* non-matroidal independence systems only, i.e., each proper induced subsystem of any $\Sigma$ in the FIST family is a matroid.

**Theorem 2.3.** *Let $\Sigma^c = (S, \mathfrak{C})$ be in the FIST family and let $X$ be a proper subset of $S$. Then $\Sigma(X)$ is a matroid.*

*Proof.* Let us choose an arbitrary element $s \in S \backslash X$. The conditions (N3) and (N4) imply that $\Sigma(X)$ has at most one circuit. By the remark in Sec. 1, we conclude that $\Sigma(X)$ is a matroid. ■

We shall use Theorems 2.2 and 2.3 to prove the following result of [8] in a new way, while Zverovich prove it by using the 'weak circuit elimination axiom' for matroids (see [8]). The result is very interesting, since from it one can deduce a general method for characterizing new classes of $\mathcal{P}$-matrogenic graphs [8].

**Theorem 2.4.**

(i) *An independence system $\Sigma = (S, \mathfrak{I})$ is a matroid if and only if it does not contain any induced independence system from the FIST family.*

(ii) *The FIST family consists of all the minimal forbidden induced subsystem for matroids.*

*Proof.*

(i) (Only if). By Proposition 1.4, any induced subsystem of a matroid is also a matroid. But Theorem 2.2 states that the FIST family consists of non-matroidal independence systems.

(If). We need to show that $\Sigma = (S, \mathfrak{I})$ is a matroid provided that $\Sigma$ does not contain any element of the FIST family as an induced subsystem.

Suppose that it is not true and let $\Sigma = (S, \mathfrak{I})$ be a minimal forbidden induced subsystem for matroids (i.e., $\Sigma$ is not a matroid, but all its proper induced subsystems are matroids). We show that it is in the FIST family.

Since $\Sigma$ is not a matroid, there exist independent sets $I, J \in \mathfrak{I}$ such that $|I| < |J|$, and $I \cup \{m\}$ is a dependent set for each $m \in J \backslash I$. Clearly $I \cup J$ induces a non-matroidal independence system. By the minimality of $\Sigma$, $S = I \cup J$.

By the choice of $I$ and $J$, clearly $I \nsubseteq J$. Let us fix $n \in I \backslash J$. By the minimality of $\Sigma$, $\Sigma - \{n\}$ is a matroid ( $\Sigma - \{n\}$ is the subsystem induced by $S \backslash \{n\}$). Property (M) applied to $I' = I \backslash \{n\}$ and $J$ in the matroid $\Sigma - \{n\}$ shows that there exists an element $m \in J \backslash I$ such that $I_1 = I' \cup \{m\}$ is independent in $\Sigma - \{n\}$.

Further, $|I_1| < |J|$, since $|I'| < |I| < |J|$. Therefore there exists an element $k \in J \backslash I_1$ such that $I_2 = I_1 \cup \{k\}$ is independent in $\Sigma - \{n\}$.

Now we consider the set $X = I \cup \{m, k\}$ in $\Sigma$. We shall draw the following two claims.

(C1) $\Sigma(X)$ has at least two circuits.

In fact, both $I \cup \{m\}$ and $I \cup \{k\}$ are dependent in $\Sigma$, but $I$ is independent.

Therefore there is a circuit which contains $m$ and does not contain $k$ and there exists a circuit which contains $k$ and does not contain $m$.
(C2) $n$ belongs to all circuits of $\Sigma(X)$.

This property holds because $I_2 = (I \backslash \{n\}) \cup \{m, k\}$ is independent in $\Sigma - \{n\}$ and hence in $\Sigma$.

Let us define $X'$ as a subset of $X$ such that both (C1) and (C2) hold (with $X'$ instead of $X$), and which is minimal by inclusion with this property. Putting

$$U = \{x \in X' : x \quad \text{belongs to all circuits of} \quad \Sigma(X')\}$$

and $V = X' \backslash U$, we obtain the partition $U \cup V$ of $X'$.

The existence of $X'$ shows that (N1), (N2) and (N3) hold for $\Sigma(X)$.

In order to check the condition (N4), let us suppose that some element $t \in V$ belongs to at most one circuit of $\Sigma(X')$. Then deleting $t$ from $\Sigma(X')$ produces an independence system for which both (C1) and (C2) hold despite the minimality of $X'$. This shows that $\Sigma(X')$ is in the FIST family, hence matroid. Now the minimality of $X'$ and the minimality $\Sigma$ implies that $S = X'$, i.e., $\Sigma = \Sigma(X')$, therefore $\Sigma$ is in the FIST family.
(ii) By (i), the FIST family must contain all the minimal forbidden induced subsystem for matroids. The result follows from Theorem 2.3. ∎

## 3. Further Research Problems

The poset matroid is a natural generalization of the classic matroid, developed by replacing the underlying set of a matroid by a partially ordered set. Consequently, the notion of a subset of the fundamental set is replaced by a filter of the poset. It is systematically studied in [1, 2, 5, 6].

Some features of matroids can be inserted into the notion of poset matroid. However, not all properties of matroids can be translated in a simple way into this new language. Typical example of this fact is the forbidden induced subsystem axioms of traditional matroid. So we give the first research problem as

**Research Problem 3.1.** *How can we generalize Theorem* 2.4 *to poset matroids*?

In [8], Zverovich uses the forbidden induced subsystem to study the classes of $i$-matrogenic graphs, namely $\mathcal{M}_i$, where $i \geq 0$. Especially, he characterize $\mathcal{M}_1$ as follows.

**Proposition 3.2.** (see [8]) *The set of the minimal forbidden induced subgraphs for $\mathcal{M}_1$ consists of $P_4$, the claw, the paw and the diamond.*

It is natural to ask

**Research Problem 3.3.** *How can we give the characterization for* $\mathcal{M}_0$, $\mathcal{M}_2$, $\ldots$ , $\mathcal{M}_\infty$?

*Acknowledgements.* We thank the referee for his/her helpful comments and corrections.

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