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On Stability in Periodic Multi-Wavelet Decompositions

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1. A non-periodic multi-wavelet decomposition of functions on $\mathbb{R}^d$ is L_p-stable if its scaling functions have the l_p-stable integer translates. The problem of l_p-stability of integer translates of scaling functions has been studied in [3]. As well known, in contrast to non-periodic wavelet decompositions there is no single scaling function for a periodic wavelet decomposition and the scaling functions are different at different dyadic levels. They form a sequence of scaling functions. A periodic multi-wavelet decomposition are based in such several sequences of scaling functions. These make the problem of L_p-stability of periodic multi-wavelet decompositions more complicate and interesting. The last problem plays an important role in using multi-wavelet decompositions for nonlinear approximations of functions (see [1, 2]). A necessary and sufficient condition for L_2-stability of periodic wavelet decompositions has been given in [4].

2. Let multivariate periodic functions be represented as functions defined on the d-torus $\mathbb{T}^d := [-\pi, \pi]^d$. Let $\{\varphi_k^j\}_{k\in\mathbb{Z}_+^d}$, $j \in J$, be a finite family of sequences of functions defined on $\mathbb{T}^d$ and $\gamma \in \mathbb{N}^d$, where J is a finite set of indices. For $k \in \mathbb{Z}_+^d$ and $s \in Q(\gamma 2^k)$, we define the h^k-step translates $\varphi_{k,s}^j$ by

$$\varphi_{k,s}^j(x) := \varphi_k^j(x - sh^k),\ x \in \mathbb{T}^d,$$

where $h^k := 2\pi/\gamma 2^k$ and

$$Q(m) := \{\, k \in \mathbb{Z}^d : 0 \le k < m\},\ m \in \mathbb{N}^d.$$

Here and later, for $x, y \in \mathbb{R}^d$, we use the following notations: x_j is the j-th coordinate of $x \in \mathbb{R}^d$, i.e., $x = (x_1, ..., x_d)$; $2^x := (2^{x_1}, \dots, 2^{x_d})$; $xy := (x_1y_1, ..., x_dy_d)$; $1/x := (1/x_1, ..., 1/x_d)$; $\mathbb{Z}^d_+ = \{\, k \in \mathbb{Z}^d : k \geq 0, \}$; the inequality $x \leq y$ $(x < y)$ means $x_j \leq y_j$ $(x_j < y_j)$, $j = 1, ..., d$.

Suppose that every function $f \in L_p := L_p(\mathbb{T}^d)$, $1 \leq p \leq \infty$, can be decomposed into a series:

$$f = \sum_{k \in \mathbb{Z}^d_+} \sum_{j \in J} \sum_{s \in Q(\gamma 2^k)} f^j_{k,s} \varphi^j_{k,s}, \tag{1}$$

converging in the norm of L_p, where $f^j_{k,s} = f^j_{k,s}(f)$ are certain coefficient fucntionals of f. This decomposition is called *a periodic multi-wavelet decomposition of* f. The functions $\varphi^j_k, j \in J$, are called *k-th scaling functions* and $\varphi^j_{k,s}$ *multi-wavelets.*

We say that the periodic multi-wavelet decomposition (1) is *L_p-stable* if there exist positive constants C, C' depending only on p, γ, d and J such that for each linear combination of the multi-wavelets at any dyadic level

$$f = \sum_{j \in J} \sum_{s \in Q(\gamma 2^k)} a^j_s \varphi^j_{k,s},$$

there hold the inequalities

$$C\|f\|_p \leq (\sum_{j \in J} \sum_{s \in Q(\gamma 2^k)} 2^{-|k|} |a^j_s|^p)^{1/p} \leq C'\|f\|_p \tag{2}$$

(the sum is changed to max when $p = \infty$), where $\|\cdot\|_p$ is the usual p-integral norm in L_p, and $|k| := k_1 + \cdots + k_d$. In the present paper we give sufficient conditions of L_p-stability for periodic multi-wavelet decompositions in the case when the sequences of scaling functions of multi-wavelet decompositions are periodized from non-periodic functions by

$$\varphi^j_k(x) := \pi_{2^k}(\psi^j, x) = \sum_{s \in \mathbb{Z}^d} \psi^j(2^k(x + 2\pi s)),\ j \in J, \tag{3}$$

where

$$\psi^j(x) \ = \ \prod_{i=1}^d \psi^j_i(x_i) \tag{4}$$

are tensor products of univariate functions ψ^j_i.

3. Let us preliminarly recall some results on l_p-stability of integer translates of non-periodic functions which are directly related to our problem. Let ψ be a function on $\mathbb{R}^d$ and $u \in \mathbb{R}^d$ with $u > 0$. We set

$$\psi_u := \sum_{s \in \mathbb{Z}^d} |\psi(\cdot - us)|.$$

Let $\mathcal{L}_{p,u}(\mathbb{R}^d)$ be the normed space of all functions ψ for which the norm

$$|\psi|_{p,u} := \|\psi_u\|_{L_p(\Pi(u))}$$

is finite, where $\Pi(u) := \{ x \in \mathbb{R}^d : 0 \le x < u\}$.

Given a function $\psi \in \mathcal{L}_{p,u}(\mathbb{R}^d)$ and a sequence $a \in l_\infty(\mathbb{Z}^d)$, the semi-discrete convolution $\psi * a$ is defined by

$$\psi * a := \sum_{s\in\mathbb{Z}^d} a(s)\psi(\cdot - us).$$

For a family $\psi^1, ..., \psi^n$ of functions on $\mathbb{R}^d$, denote by $S_{p,u}(\psi^1, ..., \psi^n)$ the space of all linear combinations

$$f = \sum_{i=1}^n \psi^i * a^i = \sum_{i=1}^n \sum_{s\in\mathbb{Z}^d} a^i(s)\psi^i(\cdot - us) \tag{5}$$

with $a^1, ..., a^n \in l_p(\mathbb{Z}^d)$.

There holds the following inequality

$$\|f\|_{L_p(\mathbb{R}^d)} \ \le \ \max_{1\le i\le n} |\psi^i|_{p,u} \sum_{i=1}^n \|a^i\|_{l_p(\mathbb{Z}^d)}. \tag{6}$$

We say that the *u-step integer translates* of $\psi^1, ..., \psi^n$ $\psi^i(\cdot - us), s \in \mathbb{Z}^d$, are l_p*-stable* if there exists a positive constant $C = C(p, d, u, \psi^1, ..., \psi^n)$ such that

$$\sum_{i=1}^n \|a^i\|_{l_p(\mathbb{Z}^d)} \ \le \ C\|f\|_{L_p(\mathbb{R}^d)},$$

for all linear combinations (5). For $1 \le p \le \infty$, we denote $p' := (p-1)/p$.

Theorem 1. *Let $1 \le p \le \infty$ and $\psi^1, ..., \psi^n \in \mathcal{L}_{\infty,u}(\mathbb{R}^d)$. Then the following conditions are equivalent:*

(i) *The u-step integer translates of $\psi^1, ..., \psi^n$, are l_p-stable,*

(ii) *For any $y \in \Pi(2\pi/u)$ the sequences $\{\hat{\psi}^i(y + 2\pi s/u)\}_{s\in\mathbb{Z}^d}$, $i = 1, 2, ...n$, are linearly independent, and*

(iii) *There exist $g^1, ..., g^n \in S_{1,u}(\psi^1, ..., \psi^n)$ such that*

$$\langle g^i, \psi^j(\cdot - us)\rangle = \delta_{ij}\delta_{0s} \ \text{ for } i, j = 1, ..., n \text{ and } s \in \mathbb{Z}^d.$$

Moreover, if the u-step integer translates of $\psi^1, ..., \psi^n$, are l_p-stable, then

$$\sum_{i=1}^n \|a^i\|_{l_p(\mathbb{Z}^d)} \ \le \ \|f\|_{L_p(\mathbb{R}^d)} \sum_{i=1}^n |g^i|_{p',u},$$

for all linear combinations (5).

Theorem 1 as well as the inequality (6) were proved in [3] for $u = (1, 1, ..., 1)$. The general case can be proved similarly.

4. We now give sufficient conditions of L_p-stability for periodic multi-wavelet decompositions when the sequences of scaling functions of multi-wavelet decompositions are periodized from non-periodic functions.

Theorem 2. *Let $1 \le p \le \infty$, $\gamma \in \mathbb{N}^d$. Let $\psi^j \in \mathcal{L}_{\infty,2\pi/\gamma}(\mathbb{R}^d), j \in J$, where J is a finite set of indices. Assume that every function $f \in L_p$ has a periodic multiwavelet decomposition (1) and the sequences of scaling functions $\{\varphi_k^j\}_{k \in \mathbb{Z}_+^d}$, $j \in J$, are periodized from non-periodic functions $\psi^j, j \in J$, as in (3). If $\psi^j, j \in J$, have l_p-stable $2\pi/\gamma$-step integer translates, then the decomposition (1) is L_p-stable. Moreover, contants C, C' in (2) can be chosen as follows*

$$C = \left(\max_{j \in J} |\psi^i|_{p,2\pi/\gamma} \right)^{-1}, \quad C' = \#J \max_{j \in J} |g^j|_{p',2\pi/\gamma},$$

where $g^j \in S_{1,2\pi/\gamma}(\psi^1, ..., \psi^n), j \in J$, are functions such that

$$\langle g^i, \psi^j(\cdot - 2\pi s/\gamma) \rangle = \delta_{ij}\delta_{0s} \ \textit{for } i,j \in J \ \textit{and } s \in \mathbb{Z}^d.$$

Theorem 3. *Under the assumptions of Theorem 2, if ψ^j, $j \in J$, are tensor products of univariate functions ψ_l^j, $l = 1, 2, ..., d$, as in (4), which have l_p-stable $2\pi/\gamma_i$-step integer translates, then the decomposition (1) is L_p-stable. Moreover, contants C, C' in (2) can be chosen as follows*

$$C = \prod_{l=1}^d \left(\max_{j \in J} |\psi_l^j|_{p,2\pi/\gamma} \right)^{-1}, \quad C' = \#J \prod_{l=1}^d \max_{j \in J} |g_l^j|_{p',2\pi/\gamma},$$

where $g_l^j \in S_{1,2\pi/\gamma_l}(\psi_l^1, ..., \psi_l^n), j \in J$, are functions such that

$$\langle g_l^i, \psi_l^j(\cdot - 2\pi s/\gamma) \rangle = \delta_{ij}\delta_{0s_l}, \ i,j \in J, \ s_l \in \mathbb{Z}.$$

5. Let us construct a L_p-stable periodic wavelet decomposition with the scaling function periodized from a non-periodic function. Given natural numbers β, η, γ with $\beta > \eta$, we let

$$\psi(x) := \beta\gamma^{-1}\mathrm{sinc}(\beta x/2)\mathrm{sinc}(\eta x/2).$$

The functions

$$\varphi_k, \ k \in \mathbb{Z}_+,$$

are periodized from non-periodic functions by:

$$\varphi_k(x) := \pi_{2^k}(\psi, x), \ k = 1, 2, ...$$

If $\beta + \eta$ is even, then a straightforward calculation shows that

$$\varphi_k(x) = \frac{\sin(\beta 2^k x/2)}{\gamma 2^k \sin(x/2)} \frac{\sin(\eta 2^k x/2)}{\eta 2^k \sin(x/2)}.$$

In particular,

$$v_k(x) := \pi_{2^k}(v; x) = \frac{\sin(3 \times 2^k x/2)}{3 \times 2^k \sin(x/2)} \frac{\sin(2^k x/2)}{2^k \sin(x/2)}$$

is the classical de la Vallée Pussin kernel where

$$v(x) := \mathrm{sinc}(3x/2)\mathrm{sinc}(x/2)$$

is the non-periodic de la Vallée Pussin kernel.

The sequences of scaling functions for a wavelet decomposition

$$\{\varphi_k\}_{k\in\mathbb{Z}^d_+}$$

are defined by:

$$\varphi_k(x) = \prod_{j=1}^{d} \varphi_{k_j}(x_j),$$

as the tensor product of φ_{k_j}. Further, we define the wavelets as follows:

$$\varphi_{k,s} := \varphi_k(\cdot - sh^k),\ s \in Q(\gamma 2^k),\ k \in \mathbb{Z}^d_+.$$

If $\beta \geq 3\eta$ and $\gamma > (\beta+\eta)/2$, then a continuous function f on $\mathbb{T}^d$ has a wavelet decomposition into the series

$$f(x) = \sum_{k\in\mathbb{Z}^d_+} \sum_{s\in Q(\gamma 2^k)} f_{k,s}\varphi_{k,s}(x), \tag{7}$$

converging uniformly on $\mathbb{T}^d$, and consequently in the norm of L_p, where $f_{k,s} = f_{k,s}(f)$ are certain coefficient functionals of f. For $u = 2\pi/\gamma$, by Theorem 1 it is easy to check that the u-step integer translates of ψ are l_p-stable if and only if $\gamma < \beta + \eta$. Hence and from Theorems 2 and 3 we obtain

Corollary. *Let $1 \leq p \leq \infty$, and β, η, γ satisfy the conditions $\beta \geq 3\eta$ and $(\beta+\eta)/2 < \gamma < \beta + \eta$. Then the decomposition (7) is L_p-stable, that is, there exist positive constants C, C' depending only on β, η, γ and p such that for each linear combination of the wavelets at any dyadic level*

$$f = \sum_{s\in Q(\gamma 2^k)} a_s\varphi_{k,s},$$

there hold the inequalities

$$C\|f\|_p \leq \Big(\sum_{s\in Q(\gamma 2^k)} 2^{-|k|}|a_s|^p\Big)^{1/p} \leq C'\|f\|_p$$

(*the sum is changed to* max *when* $p = \infty$).

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