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# An Algebraic Condition of an Irreducible Variety in $\mathbb{C}^n$*

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**Abstract.** The main aim of this paper is to prove that an irreducible variety $V$ in $\mathbb{C}^n$ is an algebraic one if and only if the space $H(V)$ of holomorphic functions on $V$ has property $(\underline{DNDZ})$.

## 1. Introduction

The algebraicity of an irreducible variety $V$ in $\mathbb{C}^n$ was investigated by some authors. The first result in this direction belongs to W.Stoll. In [10] Stoll proved that an irreducible variety $V$ is algebraic if and only if the projective volume of $V$ is finite, i.e $\int_V \Big(dd^c \log(1+|z|^2)\Big)^n < +\infty$.

Next using methods from Padé approximation Sadulaev gave a beautiful criterion on algebraicity of $V$. Namely, in [8] he has shown that $V$ is algebraic if and only if there exists a compact subset $K \subset V$ such that the Siciak extremal function $L(z, K)$ associated to $K$ is locally bounded on $V$. Recently, from some interested results on properties of plurisubharmonic functions of the Lelong class on a complex space. Zeriahi has obtained a generalization of the above result of Sadulaev [15]. At the same time, by relying heavy on the above characterization of Sadulaev, some years ago, Aytuna has proved that $V$ is algebraic if and only if the restriction map $R : H(\mathbb{C}^n) \to H(V)$ has a linearly tame right inverse for the

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system of semi-norms on $H(\mathbb{C}^n)$ defined by the increasing sequence of polydiscs in $\mathbb{C}^n$ [1] of the form $\mathbb{D}_k = \{z \in \mathbb{C}^n : \|z\| \leq e^k\},\ k = 1, 2, \ldots$.

In this paper by employing the modern theory of Fréchet spaces, mainly, by using the linearly topological invariants $(DNDZ)$ and $(\underline{DNDZ})$ on graded Fréchet spaces and linearly tame operators between graded Fréchet spaces we establish the algebraicity of an irreducible variety $V$ in $\mathbb{C}^n$. Namely the main result of the paper is the following.

**The Main Theorem.** *Let $V$ be an irreducible variety in $\mathbb{C}^n$. The following assertions are equivalent:*
(i) *$V$ is algebraic.*
(ii) *$H(V)$ has property $(DNDZ)$.*
(iii) *$H(V)$ has property $(\underline{DNDZ})$.*

Our paper is organized as follows. Beside the introduction the paper contains three sections. In Sec. 2 we recall some definitions and fix some notations. Mainly in this section we introduce the linearly topological invariant $(\underline{DNDZ})$ on graded Fréchet spaces which is a generalization of property $(DNDZ)$ introduced and investigated by Poppenberg (see [4, 5]). In Sec. 3 we give a characterization of property $(\underline{DNDZ})$ which is also of independent interest. The proof of the main theorem is presented in Sec. 4.

## 2. Preliminaries

2.1. For the usual notions on Fréchet spaces we refer to [9, 11] and to [4-5, 7, 12] for grading Fréchet spaces.
In the linearly tame category the objects are the graded Fréchet spaces $E, F, \ldots$, i.e Fréchet spaces equipped with a fixed sequence of semi-norms

$$\|\cdot\|_0 \leq \|\cdot\|_1 \leq \|\cdot\|_2 \leq \ldots$$

defining the topology, or equivalently, a fixed fundamental sequence of balanced convex neighborhoods

$$U_0 \supseteq U_1 \supseteq U_2 \supseteq \ldots \supseteq U_n \supseteq \ldots .$$

The such sequence is called grading. Graded subspaces and graded quotient spaces are equipped with the induced semi-norms.

The morphisms in this category are linearly tame operators between graded Fréchet spaces. A linear operator $L : E \to F$ is called to be linearly tame if there exist $a \geq 1, b \geq 0$ such that

$$\forall n \ \ \exists c_n > 0 \ \ \|Lx\|_n \leq c_n \|x\|_{an+b}, \ \ \forall x \in E.$$

Notice that $L$ is linearly tame if and only if there exist $a \geq 0, b \geq 0$ such that for each $n \geq 1$ $L$ induces continuous linear operators $L_n : E_{an+b} \to F_n$ where $E_{an+b}$ and $F_n$ are Banach spaces associated to the semi-norms $\|\cdot\|_{an+b}$ and $\|\cdot\|_n$ on $E$ and $F$ respectively.

In the case where $a = 1$, $L$ is said to be tame. The category of graded Fréchet spaces with tame morphisms is called the tame category. A short exact sequence of graded Fréchet spaces

$$0 \longrightarrow E \xrightarrow{\mathrm{e}} F \xrightarrow{\mathrm{g}} G \longrightarrow 0$$

is called linearly tame (resp., tame) exact if $e : E \longrightarrow \operatorname{Im} e$ and $\widehat{g} : F/\ker g \to G$ are linearly tame isomorphic, where $\widehat{g} : F/\ker g \to G$ is the map induced by $g$.

2.2. Let $E$ and $F$ be graded Fréchet spaces with the gradings defined by fundamental systems of neighborhoods

$$U_1 \supset U_2 \supset \dots \supset U_n \dots$$
$$V_1 \supset V_2 \supset \dots V_n \supset \dots$$

of the zero elements in $E$ and $F$ respectively. Then $E\widehat{\otimes}_\Pi F$ is graded by

$$W_1 \supset W_2 \supset \dots W_n \supset \dots$$

where $W_n = \Gamma(U_n \otimes V_n)$ is closure of the balanced convex envelope of $U_n \otimes V_n$ in $E\widehat{\otimes}_\Pi F$.

If either $E$ or $F$ is nuclear, we always assume that the canonical maps between Banach spaces associated to $U_n$ and $V_n$ are nuclear. Then $E\widehat{\otimes}_\Pi F$ is tamely isomorphic to $E\widehat{\otimes}_\varepsilon F$ where $E\widehat{\otimes}_\varepsilon F = L(E'_\beta, F)$, the Fréchet space of continuous linear maps from the strongly dual space $E'_\beta$ of $E$ to $F$. This space is graded by

$$\|f\|_k = \sup\{\|f(u)\|_k : u \in U_k^0\}.$$

Now we consider the special case where $E = \Lambda(A)$ is the space of sequences defined by the Köthe matrix $A = (a_{j,k})_{j,k \geq 1}$

$$\Lambda(A) = \Big\{x = (x_j) \subset \mathbb{C}^{\mathbb{N}} : \|x\|_k = \sum_{j \geq 1} |x_j| a_{j,k} < +\infty \quad \forall k\Big\}.$$

In the case where $\Lambda(A)$ is nuclear we always assume that

$$\sum_{j \geq 1} \frac{a_{j,k}}{a_{j,k+1}} < \infty, \quad \forall k \geq 1.$$

Then $\Lambda(A)\widehat{\otimes}_\Pi X$ is tamely isomorphic to $\Lambda(A, X)$ given by

$$\Lambda(A, X) = \Big\{x = (x_j) \subset X : \|x\|_k = \sum_{j \geq 1} \|x_j\| a_{j,k} < +\infty \quad \forall k\Big\}$$

for every Banach space $X$.

Moreover, $\Lambda(A, X)$ can be graded by

$$|\|x\||_k = \sup_{j \geq 1} \|x_j\| a_{j,k}, \quad \forall k \geq 1.$$

If $\{X_j\}_{j \geq 1}$ is a sequence of Banach spaces then $\sqcap_{j \geq 1} X_j$ is graded by

$$\|(x_1, \dots, x_n, \dots)\|_k = \sum_{j=1}^{k} \|x_j\|, \quad k \geq 1.$$

Obviously it is tamely equivalent to the grading defined by

$$|||(x_1, \dots, x_n, \dots)|||_k = \sup_{1 \le j \le k} \|x_j\|, \quad k \ge 1.$$

2.3. Let $E$ be a graded Fréchet space with the topology defined by an increasing sequence of semi-norms

$$\|\cdot\|_1 \le \|\cdot\|_2 \le \dots \le \|\cdot\|_n \le \dots .$$

For each $n \ge 1$, put $U_n = \{x \in E : \|x\|_n \le 1\}$ and

$$\|u\|_n^* = \sup\Big\{|u(x)| : x \in U_n\Big\},\ u \in E'_\beta,$$

where $E'_\beta$ denotes the topological dual space of $E$ equipped with the strong topology $\beta$.

**Definition 2.3.1.** *We say that $E$ has property $(\underline{DNDZ})$ if there exist $a \ge 1$, $b \ge 0$, $p \ge 0$ and constants $c_{n,m} > 0$ such that for all $n \ge b$ and $r > 0$*

$$U_n^0 \subseteq \bigcap_{m=-p}^{a(n-b)} c_{n,m} r^{m+p} U^0_{a^2n-am} + \bigcap_{k=p}^{\infty} \frac{c_{n,k}}{r^{k-p}} U^0_{a^2n+ak}\,. \tag{1}$$

*If $a$ can be chosen equal to 1 then in* [4] *Poppenberg said that $E$ has property $(DNDZ)$.*

*Remark 1.* With $E$ also every graded subspace has property $(\underline{DNDZ})$.

Next we recall property $(\Omega DZ)$ introduced and investigated by Poppenberg in [5].

Let $E$ be a graded Fréchet space.

$E$ is called to have property $(\Omega DZ)$ if there exist $b, p \ge 0$ and constants $c_n, c_{n,k} > 0$ such that for all $n \ge b+p$ and $r > 0$

$$U_n \subseteq c_n\Big(\bigcap_{i=p}^{n-b} r^{i-p} U_{n-i}\Big) + \bigcap_{k=-p}^{+\infty} \frac{c_{n,k}}{r^{k+p}} U_{n+k}\,.$$

As in [5] Poppenberg showed that every power series space of infinite type $\Lambda_\infty(\alpha)$ has property $(\Omega DZ)$ and with $E$ also every graded quotient space of $E$ has property $(\Omega DZ)$.

## 3. A Characterization of Property $(\underline{DNDZ})$

By relying on the tame splitting theorem of Vogt (see 3.2 in [12]) in [4] Poppenberg has given a nice characterization of graded nuclear Fréchet spaces having property $(DNDZ)$. Namely he proves that a graded nuclear Fréchet space $E$ has property $(DNDZ)$ if and only if there exists some $\varepsilon > 0$ such that $E$ is tamely isomorphic to a graded subspace of $s_\varepsilon$ (Theorem 4.3 in [3]). In this section we also establish a characterization of property $(\underline{DNDZ})$ when $E$ is nuclear in the linearly tame category.

The following result is proved here.

**Theorem 3.1.** *Let $E$ be a graded nuclear Fréchet space. Then $E$ has property ($\underline{DNDZ}$) if and only if $E$ is linearly tame isomorphic to a graded subspace of $s$.*

In order to prove Theorem 3.1 we need some following propositions and auxiliary lemmas.

**Proposition 3.2.** *Let $0 \longrightarrow \ell^\infty(I)\widehat{\otimes}_\Pi s \xrightarrow{e} \widetilde{E} \xrightarrow{q} E \longrightarrow 0$ be a linearly tame exact sequence of graded Fréchet spaces. If $E$ has property ($\underline{DNDZ}$) then $q$ has a linearly tame right inverse.*

*Proof.* Without loss of generality we may assume that the gradings of $\ell^\infty(I)\widehat{\otimes}_\Pi s$ and $E$ are induced by the grading of $\widetilde{E}$ and $E$ satisfies property ($\underline{DNDZ}$) for $b = p = 0$. Hence there exist $c_{n,m} > 0$ such that

$$\begin{aligned} U_n^0 &\subseteq \bigcap_{m=0}^{a_n} c_{n,m} r^m U^0_{a^2 n - a_m} + \bigcap_{k=0}^{\infty} \frac{c_{n,k}}{r^k} U^0_{a^2 n^2 + ak} \\ &\subseteq \bigcap_{p \in A_n} \widetilde{c}_{n,p} r^{an - \frac{p}{a}} U_p^0 + \bigcap_{q \in B_n} \widetilde{c}_{n,q} r^{an - \frac{q}{a}} U_q^0, \end{aligned}$$

where

$$\begin{aligned} A_n &= \{a^2 n - ka : 0 \le k \le na\}, \\ B_n &= \{a^2 n + ka : k \ge 0\}. \end{aligned}$$

For each $(i, j) \in I \times \mathbb{N}$ consider the coefficient functional on $\ell^\infty(I)\widehat{\otimes}_\Pi s$ given by

$$f_{ij}\Big([x_{ij} : I \times \mathbb{N}]\Big) = x_{ij}, \quad [x_{ij} : I \times \mathbb{N}] \in \ell^\infty(I)\widehat{\otimes}_\Pi s.$$

Since $\|f_{ij}\|_n^* = j^{-n} = e^{-n\alpha_j}$, $\alpha_j = \log j$, $j \ge 1$ then according to the Hahn-Banach theorem we can extend $f_{ij}$ to $F_{ij} \in \widetilde{E}'$ such that

$$\|F_{ij}^{(n)}\|_n^* = e^{-n\alpha_j}.$$

We notice that

$$\begin{aligned} \|F_{ij}^{(n+1)} - F_{ij}^{(n)}\|_{n+1}^* &\le \|F_{ij}^{(n)}\|_{n+1}^* + \|F_{ij}^{(n)}\|_{n+1}^* \\ &\le \|F_{ij}^{(n+1)}\|_{n+1}^* + \|F_{ij}^{(n)}\|_n^* = e^{-(n+1)\alpha_j} + e^{-n\alpha_j} \\ &\le 2e^{-n\alpha_j} \end{aligned}$$

for all $i \in I$, $j \ge 1$.

On the other hand, we have $F_{ij}^{(n+1)} - F_{ij}^{(n)} = 0$ on $\ell^\infty(I)\widehat{\otimes}_\Pi s$ then we may choose $G_{ij}^{(n)} \in 2e^{-n\alpha_j} U_{n+1}^0 \subset E'$ such that

$$G_{ij}^{(n)} \circ q = F_{ij}^{(n+1)} - F_{ij}^{(n)}.$$

Next we choose an increasing sequence $1 \le c_n \le c_{n+1}$ such that

$$D_p = 2^p c_p^p \sup_n \frac{\widetilde{c}_{n,p}}{c_n} < +\infty$$

for all $p \in A_n \cup B_n$.

Since

$$U^0_{n+1} \subset \Big( \bigcap_{p \in A_{n+1}} \widetilde{c}_{n+1,p} r^{a(n+1)-\frac{p}{a}} U^0_p \Big) + \Big( \bigcap_{q \in B_{n+1}} \widetilde{c}_{n+1,q} r^{a(n+1)-\frac{q}{a}} U^0_q \Big)$$

it follows that

$$2e^{-n\alpha_j} U^0_{n+1} \subset \Big( \bigcap_{p \in A_{n+1}} 2e^{-n\alpha_j} \widetilde{c}_{n+1,p} r^{a(n+1)-\frac{p}{a}} U^0_p \Big) + \Big( \bigcap_{q \in B_{n+1}} 2e^{-n\alpha_j} \widetilde{c}_{n+1,q} r^{a(n+1)-\frac{q}{a}} U^0_q \Big).$$

Take $r = \dfrac{1}{2^a c^a_{n+1}} e^{\frac{\alpha_j}{a}}$ and choose

$$g^{(n)}_{ij} \in 2e^{-n\alpha_j} \widetilde{c}_{n+1,p} . \frac{1}{2^{a^2(n+1)-p}} . \frac{1}{c_{n+1}^{a^2(n+1)-p}} e^{\left((n+1)-\frac{p}{a^2}\right)\alpha_j} U^0_p$$

for $p \in A_{n+1}$ such that $G^{(n)}_{ij} \in 2e^{-n\alpha_j} \bigcap_{p \in A_{n+1}} \widetilde{C}_{n+1,p} r^{a(n+1)-\frac{p}{a}} U^0_p + g^{(n)}_{ij}$.

Hence

$$\begin{aligned}
\|g^{(n)}_{ij}\|^*_p &\le 2\widetilde{c}_{n+1,p} . \frac{2^p}{2^{a^2(n+1)}} . \frac{c^p_{n+1}}{c^{a^2(n+1)}_{n+1}} e^{\left(1-\frac{p}{a^2}\right)\alpha_j} \\
&\le 2\widetilde{c}_{n+1,p} . \frac{2^p}{2^{(n+1)}} . \frac{c^p_{n+1}}{c^{n+1}_{n+1}} e^{\left(1-\frac{p}{a^2}\right)\alpha_j} \\
&\le 2^p c^p_p . \frac{\widetilde{c}_{n+1,p}}{c_{n+1}} . 2^{-n} . \frac{c^p_{n+1}}{c^p_p c^n_{n+1}} e^{\left(1-\frac{p}{a^2}\right)\alpha_j} \\
&\le D_p 2^{-n} e^{(1-\frac{p}{a^2})\alpha_j}
\end{aligned}$$

for $p \in A_{n+1}$.

On the other hand,

$$G^{(n)}_{ij} \in 2e^{-n\alpha_j} U^0_{n+1}$$

and

$$G^{(n)}_{ij} - g^{(n)}_{ij} \in 2e^{-n\alpha_j} \bigcap_{q \in B_{n+1}} \widetilde{c}_{n+1,q} r^{a(n+1)-\frac{q}{a}} U^0_q .$$

Hence, we have

$$\|G^{(n)}_{ij} - g^{(n)}_{ij}\|^*_q \le D_q 2^{-n} e^{\left(1-\frac{q}{a}^2\right)}\alpha_j \quad \text{for} \quad q \in B_{n+1}.$$

Now we notice that the series

$$g_{ij} = \sum_{n=0}^{\infty} g_{ij}^{(n)}$$

converges in

$$E_0' = \Big\{ u \in E' : \|u\|_0^* = \sup\{|u(x)| : \|x\|_0 \leq 1\} < +\infty \Big\}$$

because

$$\sum_{n=0}^{\infty} \|g_{ij}^{(n)}\|_0^* \leq D_0 e^{\alpha_j} \sum_{n=0}^{\infty} 2^{-n} < +\infty.$$

Hence $g_{ij} \in E'$ for all $i \in I$, $j \geq 1$. Put

$$\varphi_{ij} = F_{ij}^{(0)} + g_{ij}.q = F_{ij}^{(k+1)} - \Big\{ \sum_{n=0}^{k} (G_{ij}^{(n)} - g_{ij}^{(n)}) - \sum_{n=k+1}^{\infty} g_{ij}^{(n)} \Big\} \circ q.$$

We have

$$\begin{aligned}
\|\varphi_{ij}\|_{a^2(k+1)}^* &\leq \|F_{ij}^{(k+1)}\|_{a^2(k+1)}^* + \sum_{n=0}^{k} \|G_{ij}^{(n)} - g_{ij}^{(n)}\|_{a^2(k+1)}^* \\
&\quad + \sum_{n=k+1}^{\infty} \|g_{ij}^{(n)}\|_{a^2(k+1)}^* \\
&\leq \|F_{ij}^{(k+1)}\|_{k+1}^* + D_{a^2(k+1)} \sum_{n=0}^{k} 2^{-n} e^{(1-(k+1))\alpha_j} \\
&\quad + \sum_{n=k+1}^{\infty} D_{a^2(k+1)} 2^{-n} e^{(1-(k+1))\alpha_j} \\
&\leq e^{-(k+1)\alpha_j} + D_{a^2(k+1)} e^{-k\alpha_j} \sum_{n=0}^{\infty} 2^{-n} \\
&\leq \Big(1 + 2D_{a^2(k+1)}\Big) e^{-k\alpha_j}.
\end{aligned}$$

Hence

$$|\varphi_{ij}(x)| \leq \big(1 + 2D_{a^2(k+1)}\big) e^{-k\alpha_j} \; \|x\|_{a^2(k+1)}$$

for $x \in \widetilde{E}$.

Define

$$\varphi(x) = \Big[\varphi_{ij}(x) : (i,j) \in I \times \mathbb{N}\Big], \quad x \in \widetilde{E}.$$

Now we show that $\varphi(x) \in \ell^\infty(I)\widehat{\otimes}_\Pi s$ for $x \in \widetilde{E}$ and $\varphi$ is linearly tame left inverse of $e$. Indeed,

$$\begin{aligned}
\|\varphi(x)\|_k &= \sup_i \sum_{j\geq 1} |\varphi_{ij}(x)| e^{k\alpha_j} \\
&\leq \Big(1 + 2D_{a^2(2k+1)}\Big) \|x\|_{a^2(2k+1)} \sum_{j\geq 1} e^{-k\alpha_j} \\
&\leq \Big((1 + 2D_{a^2(2k+1)}) \sum_{j\geq 1} \frac{1}{j^k}\Big) \|x\|_{2ka^2+a^2} \\
&\leq \widetilde{D}_{a^2(2k+1)} \|x\|_{2ka^2+a^2}.
\end{aligned}$$

Hence $\varphi$ is linearly tame. Moreover,

$$\begin{aligned}\varphi e\Big([x_{ij} : I\times\mathbb{N}]\Big) &= \big[\varphi_{ij}(e[x_{ij} : I\times\mathbb{N}])\big]\\ &= \Big[F_{ij}^{(k+1)}\Big(e[x_{ij} : I\times\mathbb{N}] - \Big\{\sum_{n=0}^{k}(G_{ij}^{(n)} - g_{ij}^{(n)}) - \sum_{n=k+1}^{\infty} g_{ij}^{(n)}\Big\}\\ &\qquad \cdot\ q(e[x_{ij} : I\times\mathbb{N}]) : I\times\mathbb{N}\Big)\Big]\\ &= \Big[f_{ij}([x_{ij} : I\times\mathbb{N}]) : I\times\mathbb{N}\Big]\\ &= \big[x_{ij} : I\times\mathbb{N}\big] = id_{\ell^\infty(I)\widehat{\otimes}_\Pi s}.\end{aligned}$$

Proposition 3.2 is proved. ■

**Lemma 3.3.** [11] *There exists a tame exact sequence*

$$0\to s\to s\to w\to 0. \tag{3}$$

Tensoring the above tame exact sequence with an arbitrary Banach space we get the following.

**Lemma 3.4.** *For every Banach space $B$ there exists a tame exact sequence*

$$0\to s\widehat{\otimes}_\Pi B\to s\widehat{\otimes}_\Pi B\to B^{\mathbb{N}}\to 0. \tag{4}$$

**Proposition 3.5.** *Let $E$ be a graded Fréchet space. Then there exists an index set $I$ and a tame embedding $e : E\to[\ell^\infty(I)]^{\mathbb{N}}$, where $[\ell^\infty(I)]^{\mathbb{N}}$ is graded by the system of semi-norms*

$$\|x\|_n = \sup_{1\le k\le n}\|x_k\|, x_k\in\ell^\infty(I), x=(x_k)\in[\ell^\infty(I)]^{\mathbb{N}}.$$

*Proof.* Let $\{\|\cdot\|_k\}_{k\ge1}$ be a system of semi-norms defining the grading on $E$. For each $k\ge1$, put $I_k = U_k^0$ and define $I = \bigsqcup_{k\ge1} I_k$. According to Hahn-Banach theorem, for $x\in E$, we have

$$\|x\|_k = \sup\{|u(x)| : u\in I_k\}.$$

Let $e_k : E\to\ell^\infty(I_k)$ be given by

$$\ell_k(x) = \big[u(x) : u\in I_k\big].$$

Then

$$\|e_k(x)\|_{\ell^\infty(I_k)} = \|x\|_k \ \text{ for all } x\in E.$$

Define the map $e : E\to\sqcap_{k\ge1}\ell^\infty(I_k)$ by setting

$$e(x) = \big[e_k(x) : k\ge1\big],$$

where $\sqcap_{k\geq 1}\ell^\infty(I_k)$ is graded by the system of semi-norms

$$\|x\|_n = \sup_{1\leq k\leq n} \|x_k\|_k$$

for $x=(x_k)\in \sqcap_{k\geq 1}\ell^\infty(I_k), x_k\in\ell^\infty(I_k)$ and $\|x_k\|_k = \|x_k\|_{\ell^\infty(I_k)}$. Then

$$\begin{aligned}\|e(x)\|_n &= \|[e_k(x):k\geq 1]\|_n = \sup_{1\leq k\leq n}\|e_k(x)\|_k\\ &= \sup\{\|x\|_k : 1\leq k\leq n\} = \|x\|_n.\end{aligned}$$

Hence, $e$ is a tame embedding. On the other hand, $I=\bigsqcup_{k\geq 1} I_k$ and from the gradings defined on $\sqcap_{k\geq 1}\ell^\infty(I_k)$ and $[\ell^\infty(I)]^{\mathbb{N}}$, the form

$$\widetilde{e}[f_k:\mathbb{N}] = \big[\widetilde{e}_k(f_k):\mathbb{N}\big],$$

where

$$\widetilde{e}_k(f_k)(i) = \begin{cases} f_k(i) & \text{if } i\in I_k,\\ 0 & \text{if } i\notin I_k.\end{cases}$$

defines a tame embedding from $\sqcap_{k=1}^\infty \ell^\infty(I_k)$ into $[\ell^\infty I_k)]^{\mathbb{N}}$.

For the proof of Theorem 3.1 we need the following

**Definition 3.6.** [12] *The graded Fréchet space $E$ admits a family of smoothing operators if there exist linear operators $T_\theta : E\to E, \theta>0$, and $p\geq 0, c_{m,n}>0$ such that for all $\theta>0$ and $x\in E$*

$$\begin{aligned}\|T_\theta x\|_n &\leq c_{m,n}\theta^{n+p-m}\|x\|_m \quad \text{if } m\leq n+p\\ \|x-T_\theta x\|_n &\leq c_{m,n}\theta^{n+p-m}\|x\|_m \quad \text{if } m> n+p.\end{aligned}$$

**Proposition 3.7.** *$\ell^\infty(I)\widehat{\otimes}_\Pi s$ has property ($\underline{DNDZ}$) for all index set $I$.*

*Proof.* By [12] $s$ admits a family of smoothing operators $\{T_\theta:\theta>0\}$ satisfying conditions in Definition 3.6. Consider the family

$$\begin{aligned}\widehat{T}_\theta : \ell^\infty(I)\widehat{\otimes}_\Pi s &\longrightarrow \ell^\infty(I)\widehat{\otimes}_\Pi s,\\ [x_i, I] &\longmapsto [T_\theta x_j, I],\end{aligned}$$

where $x_i$ and $T_\theta x_i$ belong to $s$ for all $i\in I$.
We have

$$\begin{aligned}\|\widehat{T}_\theta[x_i,I]\|_n &= \|[T_\theta x_i, I]\|_n = \sup_i \|T_\theta x_i\|_n\\ &\leq c_{m,n}\theta^{n+p-m}\sup_i\|x_i\|_m = c_{m,n}\theta^{n+p-m}\|[x_i,I]\|_m\end{aligned}$$

for $m\leq n+p$.
Similarly,

$$\begin{aligned}&\|[x_i,I]-\widehat{T}_\theta[x_i,I]\|_n = \|[x_i - T_\theta x_i, I]\|_n\\ &=\sup_i\|x_i - T_\theta x_i\|_n \leq c_{m,n}\theta^{n+p-m}\sup_i\|x\|_i\\ &\leq c_{m,n}\theta^{n+p-m}\|[x_i,I]\|_m \quad \text{for } m>n+p.\end{aligned}$$

Hence, by [4] $\ell^\infty(I)\widehat{\otimes}_\Pi s$ has property $(DNDZ)$ and, hence, $(\underline{DNDZ})$.
Now we are able to prove Theorem 3.1.

*Proof of Theorem 3.1.*
Sufficiency is clear.

Now we establish the necessity condition.

From Proposition 3.5 there exists an index set $I$ such that $E$ is tamely embedded into $[\ell^\infty(I)]^{\mathbb{N}}$. Now Lemma 3.3 admits a tame exact sequence of the form

$$0 \to \ell^\infty(I)\widehat{\otimes}_\Pi s \xrightarrow{e} \ell^\infty(I)\widehat{\otimes}_\Pi s \xrightarrow{q} [\ell^\infty(I)]^{\mathbb{N}} \to 0.$$

Put $\widetilde{E} = q^{-1}(E)$. Then we have a tame exact sequence

$$0 \to \ell^\infty(I)\widehat{\otimes}_\Pi s \xrightarrow{e} \widetilde{E} \xrightarrow{q} E \to 0$$

with the graded subspace $\widetilde{E}$ of $\ell^\infty(I)\widehat{\otimes}_\Pi s$. Since $E$ has property $(\underline{DNDZ})$ and from Proposition 3.2 we deduce that $q$ has a linearly tame right inverse. Hence $E$ is linearly tame isomorphic to a graded subspace of $\ell^\infty(I)\widehat{\otimes}_\Pi s$.By [4] $E$ has property $(DNDZ)$ and from a result of Poppenberg (see [4]) we deduce that there exists $\varepsilon > 0$ such that $E$ is tamely isomorphic to a graded subspace of $s_\varepsilon$. However, $s_\varepsilon$ is linearly tame isomorphic to $s$ and, consequently, $E$ is linearly tame isomorphic to a graded subspace of $s$. ■

Now from the above characterization of property $(\underline{DNDZ})$ we prove the main result of the paper.

## 4. Proof of the Main Theorem

First we describe gradings of Fréchet spaces $H(\mathbb{C}^n)$ and $H(V)$ where $V$ is an irreducible subvariety in $\mathbb{C}^n$. For each $k \geq 1$, put

$$\overline{B}_k = \{z \in \mathbb{C}^n : \|z\| \leq e^k\}, \quad \overline{D}_k = \overline{B}_k \cap V,$$

where

$$\|z\| = \sup\{|z_j| : 1 \leq j \leq n\}, \ z = (z_1, \dots, z_n) \in \mathbb{C}^n.$$

Assume that $H(\mathbb{C}^n)$ and $H(V)$ are spaces of holomorphic functions on $\mathbb{C}^n$ and $V$ respectively. These spaces are graded by

$$\|f\|_k = \sup\{|f(z)| : z \in \overline{B}_k\}, \ f \in H(\mathbb{C}^n)$$

and

$$\|g\|_k = \sup\{|g(z)| : z \in \overline{D}_k\}, \ g \in H(V),$$

respectively.

Now we need to use the following result of Djakov and Mitiagin (see [2]) on the structure of polynomial ideals in the proof of (i) ⇒ (ii) of the main theorem.

Let $V$ be an algebraic variety in $\mathbb{C}^n$. Then there exist polynomials $Q_1, \dots, Q_p$ that generate the ideal

$$I^*(V) = \{P \in \mathbb{C}[z_1, \dots, z_n] : P|_V \equiv 0\}$$

and vector $B = (b_1, \dots, b_n)$ and continuous linear operators

$$R_i : H(\mathbb{C}^n) \to H(\mathbb{C}^n),\ 0 \leq i \leq p$$

such that
(a) $f = R_0(f) + \sum_{i=1}^{p} R_i(f)Q_i$ for all $f \in H(\mathbb{C}^n)$;
(b) $\ker R_0 = \{f \in H(\mathbb{C}^n) : f|_V = 0\}, R_0^2 = R_0$;
(c) for every $r \geq 1$

$$|R_i f|_{rB} \leq |f|_{rB},\ i = 0, \dots, p,$$

where if

$$f = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha z^\alpha$$

is a holomorphic function on $\mathbb{C}^n$ and $c = (c_1, \dots, c_n)$ then we set

$$|f|_c = \sum_{\alpha \in \mathbb{Z}_+^n} |c_\alpha| c^\alpha.$$

Now we prove the implication (i) to (ii) of the Main Theorem.

Let $V$ be an algebraic variety in $\mathbb{C}^n$. Consider the restriction map $\mathcal{R} : H(\mathbb{C}^n) \to H(V)$. If $f \in H(V)$ , by Cartan Theorem B,we can choose an entire function $G \in H(\mathbb{C}^n)$ such that $\mathcal{R}(G) = f$. Consider the map $\mathcal{E} : H(V) \to H(\mathbb{C}^n)$ given by

$$\mathcal{E}(f) = R_0(G).$$

From (b) $\mathcal{E}$ is a well defined continuous linear extension operator and $\mathcal{R}\mathcal{E}(f) = f$ for all $f \in H(V)$. We show that $\mathcal{E}$ is a tame right inverse of $\mathcal{R}$. An important feature of the operators $R_i, 0 \leq i \leq p$ of the above mentioned result is that for any $r > 1$ they can be considered as continuous linear operators on $H(\triangle_{rB})$ satisfying

(d) $g = R_0(g) + \displaystyle\sum_{i=1}^{p} R_i(g)Q_i$ for all $g \in H(\triangle_{rB})$;
(e) $\ker R_0 = \{g \in H(\triangle_{rB}) : g|_{V \cap \triangle_{rB}} = 0\}$,
where $\triangle_{rB}$ is the polydisc around zero with polyradii $rB$ (see [2, Corollary 3]). Now for a fixed $k > 1$ we consider the restriction operator from $H(\triangle_{kB})$ to $H(\triangle_{kB} \cap V)$. Here $H(\triangle_{kB})$ and $H(\triangle_{kB} \cap V)$ are graded by systems of seminorms

$$\|F\|_{sB} = \sup_{z \in \triangle_{sB}} |F(z)|, s < k, F \in H(\triangle_{kB})$$

and

$$\|f\|_{\triangle_{sB} \cap V} = \sup_{z \in \triangle_{sB} \cap V} |f(z)|, s < k, f \in H(\triangle_{kB} \cap V),$$

respectively.

Since this restriction operator is a surjection, by the open map theorem ,we can find a $c_0 = c_0(k)$ such that for every $f \in H(V)$ with $\|f\|_{\triangle_{kB} \cap V} \leq 1$ there exists an $F_k \in H(\triangle_{kB})$ such that $F_k|_{V\cap\triangle_{kB}} = f|_{V\cap\triangle_{kB}}$ and $\|F_k\|_{\triangle\left(k-\frac{\varepsilon}{2}\right)B} \le c_0$ with some $0 < \varepsilon < k$ independent of $k$. Then $\mathcal{E}(f)|_{\triangle_{kB}} = R_0(F_k)$ on $\triangle_{kB}$. Hence,

$$\begin{aligned}\|\mathcal{E}(f)\|_{k-\varepsilon} &= \|R_0(F_k)\|_{\triangle(k-\varepsilon)B} \le |R_0(F_k)|_{\triangle(k-\varepsilon)\triangle} \\ &\le |F_k|_{\triangle(k-\varepsilon)B} \le C_1\|F_k\|_{\triangle(k-\frac{\varepsilon}{2})B} \le C_0C_1\|f\|_{\triangle_{kB}\cap V},\end{aligned}$$

where $C_1 = \sum_{|\alpha|\ge 0}\left(\frac{k-\varepsilon}{k-\frac{\varepsilon}{2}}\right)^{\alpha} < \infty$.

Hence $\mathcal{E}$ is a tame right inverse of $\mathcal{R}$ and $H(V)$ is tamely embedded into $H(\mathbb{C}^n) = \Lambda_\infty\big((k^{\frac{1}{n}})\big)$. However, $\Lambda_\infty\big((k^{\frac{1}{n}})\big)$ has property $(DNDZ)$ [4] and, therefore $H(V)$ has property $(DNDZ)$.

(ii) $\Longrightarrow$ (iii) is clear.

Now the Main Theorem is proved if we show the implication (iii) to (i).

Let $H(V)$ has property $(\underline{DNDZ})$. Consider the restriction map $R : H(\mathbb{C}^n) \to H(V)$. First we show that the map $\widehat{R} : H(\mathbb{C}^n)\big/\ker R \to H(V)$ which is induced by $R$, is tamely isomorphic. For each $k \ge 1$, consider the restriction $R_k : H(B_k) \to H(D_k)$. By Cartan Theorem B, $R_k$ is surjective and, hence, it is open. This yields that there exist a compact set $K_{m_k}$ and a constant $C_k^{(1)} > 0$ satisfying

$$\overline{D}_{k-1} \subset K_{m_k} \subset D_k$$

and

$$C_k^{(1)}\widetilde{W}(K_{m_k}) \subset R_k(W(\overline{B}_{k-1})).$$

Hence,

$$\widetilde{W}(\overline{D}_k) \subset \widetilde{W}(K_{m_k}) \subset \frac{1}{C_k^{(1)}}R_k\Big(W(\overline{B}_{k-1})\Big) \subset C_k^{(2)}\widetilde{W}(\overline{D}_{k-1}),$$

where $C_k^{(2)} > 0$ and

$$\begin{aligned}\widetilde{W}(\overline{D}_k) &= \{f \in H(V) : \|f\|_{\overline{D}_k} \le 1\},\\ \widetilde{W}(K_{m_k}) &= \{f \in H(V) : \|f\|_{K_{m_k}} \le 1\},\\ \widetilde{W}(\overline{B}_{k-1}) &= \{f \in H(\mathbb{C}^n) : \|f\|_{\overline{B}_{k-1}} \le 1\},\\ \widetilde{W}(\overline{D}_{k-1}) &= \{f \in H(V) : \|f\|_{\overline{D}_{k-1}} \le 1\}.\end{aligned}$$

This yields that the gradings $\Big\{\widetilde{W}(\overline{D}_k)\Big\}_{k\ge1}$ and $\Big\{R\Big(\widetilde{W}(\overline{B}_k)\Big)\Big\}_{k\ge1}$ on $H(V)$ are tamely equivalent. However if $H(V)$ is graded by $\Big\{R\Big(\widetilde{W}(\overline{B}_k)\Big)\Big\}_{k\ge1}$ then $\widehat{R} : H(\mathbb{C}^n)\Big/\ker R \to H(V)$ is tamely isomorphic. Hence the exact sequence

$$0 \to J(V) \xrightarrow{e} H(\mathbb{C}^n) \xrightarrow{R} H(V) \to 0$$

is tame exact, where

$$J(V) = \{f \in H(\mathbb{C}^n) : f|_V = 0\}.$$

Let $J_V$ denote the coherent ideal subsheaf of the sheaf $H_{\mathbb{C}^n}$ of germs of holomorphic functions on $\mathbb{C}^n$. By [3] there exists a surjective morphism $\theta : H^{\ell^1}_{\mathbb{C}^n} \to J_V$, where $H^{\ell^1}_{\mathbb{C}^n}$ is the sheaf of germs of $\ell^1$ -valued holomorphic fucntions on $\mathbb{C}^n$. Theorem Cartan's B implies that $\theta$ induces a continuous linear map $\widehat{\theta}$ from $H(\mathbb{C}^n, \ell^1)$ onto $J(V)$. Moreover for each $n \geq 1$ , $\theta$ induces continuous linear maps $\widehat{\theta}_n$ from $H(B_n, \ell^1)$ onto $J(D_n)$. As in the above argument, $\widehat{\theta}$ induces a tamely isomorphism from $H(\mathbb{C}^n, \ell^1)/\ker\widehat{\theta}$ onto $J(V)$. Because $H(\mathbb{C}^n, \ell^1) \overset{\text{tame}}{\cong} H(\mathbb{C}^n)\widehat{\otimes}_\Pi \ell^1 \overset{\text{tame}}{\cong} \Lambda_\infty(\alpha)\widehat{\otimes}_\Pi \ell^1 = \Lambda_\infty(\alpha, \ell^1)$ where $\alpha = (\alpha_k), \alpha_k = k^{\frac{1}{n}}$ then $H(\mathbb{C}^n, \ell^1)$ and, hence, $J(V)$ has property $(\Omega DZ)$ (see [5]). On the other hand, $H(V)$ has property $(\underline{DNDZ})$ then Theorem 3.1 implies that $H(V)$ is linearly tame isomorphic to a subspace of $s$. Hence, if $H(V)$ is considered as a graded subspace of $s$, then $H(V)$ has property $(DNDZ)$ and we obtain a linearly tame exact sequence

$$0 \to J(V) \to H(\mathbb{C}^n) \overset{\text{R}}{\to} H(V) \to 0.$$

Using an argument as in [6, p. 157] and Proposition 3.4 in [5, p. 130] we claim that $R$ has a linearly tame right inverse. By [1] $V$ is algebraic. ∎

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