Borel Exceptional Values of Meromorphic Functions

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Abstract. In the paper we discuss the Borel exceptional value of a transcendental meromorphic function and its relation with Picard and Nevanlinna exceptional values.

1. Introduction and Definitions

Let \( f \) be a nonconstant meromorphic function defined in the open complex plane \( \mathbb{C} \cup \{ \infty \} \). We use the standard notations and definitions of the value distribution theory (cf. \([4]\)). Let \( k \) be a positive integer or infinity. We denote by \( N(r, a; f \mid \leq k) \) the counting function of distinct \( a \)-points of \( f \) whose multiplicities do not exceed \( k \). Clearly \( N(r, a; f \mid \leq \infty) \equiv N(r, a; f) \).

We put

\[
\mu_k(a; f) = \limsup_{r \to \infty} \frac{\log^+ N(r, a; f \mid \leq k)}{\log r}, \quad \mu(a; f) = \limsup_{r \to \infty} \frac{\log^+ N(r, a; f)}{\log r},
\]

and

\[
\rho(a; f) = \limsup_{r \to \infty} \frac{\log^+ N(r, a; f)}{\log r}.
\]

If \( f \) is a meromorphic function of order \( \rho \), \( 0 \leq \rho \leq \infty \), \( a \in \mathbb{C} \cup \{ \infty \} \) and \( k \) is a positive integer then we say that \( a \) is

(i) an exceptional value in the sense of Borel (evB for short) for \( f \) for distinct zeros of multiplicities not exceeding \( k \) if \( \mu_k(a; f) < \rho \);
(ii) an evB for \( f \) for distinct zeros if \( \mu(a; f) < \rho \);
(iii) an evB for $f$ for the whole aggregate of zeros if $\rho(a; f) < \rho$.

We call $a$ an evB for $f$ for simple zeros if $\rho_1(a; f) < \rho$ and an evB for $f$ for simple and double zeros if $\rho_2(a; f) < \rho$.

The quantity

$$\delta(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)}$$

is called Nevanlinna deficiency of the value $a \in \mathbb{C} \cup \{\infty\}$. If $\delta(a; f) > 0$ then $a$ is called an exceptional value in the sense of Nevanlinna, in short evN. If $\delta(a; f) = 0$ then $a$ is called a normal value in the sense of Nevanlinna, in short nvN.

A value $a \in \mathbb{C} \cup \{\infty\}$ is called an exceptional value for a transcendental meromorphic function in the sense of Picard, in short evP, if $f$ has at most a finite number of $a$-points.

Let $k$ be a nonnegative integer or infinity. We denote by $N_k(r, a; f)$ the counting function of $a$-points of $f$ where an $a$-point of multiplicity $\nu$ is counted $\nu$ times if $\nu \leq k$ and $1 + k$ times if $\nu > k$.

We put

$$\delta_k(a; f) = 1 - \limsup_{r \to \infty} \frac{N_k(r, a; f)}{T(r, f)}.$$

Then clearly $\delta(a; f) \leq \delta_k(a; f) \leq \delta_{k-1}(a; f) \leq \cdots \leq \delta_1(a; f) \leq \delta_0(a; f) = \Theta(a; f) \leq 1$, where

$$\Theta(a; f) = 1 - \limsup_{r \to \infty} \frac{N(r, a; f)}{T(r, f)},$$

is called the ramification index.

Valiron (cf. [9, pp. 72-78]) proved the following generalization of the classical theorem of Borel for entire functions of finite order.

**Theorem A.** Let $f$ be an entire function of finite order $\rho$. Then

(i) there exist at most two distinct elements of $\mathbb{C}$ which are evB for $f$ for simple zeros,

(ii) if there exists $a \in \mathbb{C}$ such that $a$ is an evB for $f$ for the joint sequence of simple and double zeros (double zeros being counted twice) then $\rho_1(b; f) = \rho$ for all $b \in \mathbb{C} \setminus \{a\}$,

(iii) there exists at most one element of $\mathbb{C}$ which is an evB for $f$ for the joint sequence of simple and double zeros.

In [7] Singh and Gopalakrishna obtained stronger results than above for meromorphic functions of finite order. Gopalakrishna and Bhosmurth [2] improved the results of Valiron and Singh-Gopalakrishna to meromorphic functions of unrestricted order. The result of Gopalakrishna and Bhosmurth [2] can be stated as follows:

**Theorem B.** Let $f$ be a meromorphic function of order $\rho$, $0 \leq \rho \leq \infty$. If there exist distinct elements $a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_q, c_1, c_2, \ldots, c_s$ in $\mathbb{C} \cup \{\infty\}$ such that $a_1, a_2, \ldots, a_p$ are evB for $f$ for distinct zeros of multiplicity $\leq k$, $b_1, b_2, \ldots, b_q$ are evB for $f$ for distinct zeros of multiplicity $\leq l$ and $c_1, c_2, \ldots, c_s$
are evB for f for distinct zeros of multiplicity \(\leq m\), where \(k, l, m\) are positive integers, then
\[
\frac{pk}{1+k} + \frac{ql}{1+l} + \frac{sm}{1+m} \leq 2.
\]

In the paper we improve Theorem B and discuss the relation between the evB for a meromorphic function and the Nevanlinna deficiency of those values. We also improve some other results on the exceptional values in the sense of Borel.

**Definition 1.** [1, 8] We put for \(a \in \mathbb{C} \cup \{\infty\}\)
\[
T_o(r, f) = \int_1^r \frac{T(t, f)}{t} dt,
\]
\[
N_o(r, a; f) = \int_1^r \frac{N(t, a; f)}{t} dt, \quad m_o(r, a; f) = \int_1^r \frac{m(t, a; f)}{t} dt,
\]
\[
N_o(r, a; f) = \int_1^r \frac{N(t, a; f)}{t} dt, \quad N_o(r, a; f) = \int_1^r \frac{N(t, a; f \leq k)}{t} dt,
\]
\[
N_o^k(r, a; f) = \int_1^r \frac{N_k(t, a; f)}{t} dt, \quad S_o(r, f) = \int_1^r \frac{S(t, f)}{t} dt \quad \text{etc.}
\]

Further we put for \(a \in \mathbb{C} \cup \{\infty\}\)
\[
\delta_o(a; f) = 1 - \limsup_{r \to \infty} \frac{N_o(r, a; f)}{T_o(r, f)},
\]
\[
\Theta_o(a; f) = 1 - \limsup_{r \to \infty} \frac{N_o(r, a; f)}{T_o(r, f)},
\]
\[
\delta_k^*(a; f) = 1 - \limsup_{r \to \infty} \frac{N_o^k(r, a; f)}{T_o(r, f)}.
\]

Throughout the paper we assume that \(f\) is a transcendental meromorphic function of finite or infinite order \(\rho\) defined in the open complex plane \(\mathbb{C}\).

## 2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

**Lemma 1.**

(i) \[
\limsup_{r \to \infty} \frac{\log T_o(r, f)}{\log r} = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r}
\]
and
\[
\liminf_{r \to \infty} \frac{\log T_o(r, f)}{\log r} = \liminf_{r \to \infty} \frac{\log T(r, f)}{\log r}.
\]

(ii) For any \(a \in \mathbb{C} \cup \{\infty\}\) and for any \(k, a\) positive integer or infinity,
\[
\limsup_{r \to \infty} \frac{\log^+ N_o(r, a; f| \leq k)}{\log r} = \limsup_{r \to \infty} \frac{\log^+ N(r, a; f| \leq k)}{\log r}
\]
and
\[
\liminf_{r \to \infty} \frac{\log^+ N_o(r, a; f| \leq k)}{\log r} = \liminf_{r \to \infty} \frac{\log^+ N(r, a; f| \leq k)}{\log r}.
\]

Proof. Since for all large values of \(r\)
\[
T_o(r, f) \leq T(r, f) \log r \quad \text{and} \quad T_o(2r, f) \geq T(r, f) \log 2,
\]
(i) follows easily.

Again since for all large values of \(r\)
\[
N_o(r, a; f| \leq k) \leq N(r, a; f| \leq k) \log r
\]
and
\[
N_o(2r, a; f| \leq k) \geq N(r, a; f| \leq k) \log 2,
\]
(ii) follows easily. This proves the lemma. \(\square\)

Lemma 2. Let \(k\) be a positive integer or infinity. Then for \(a \in \mathbb{C} \cup \{\infty\}\)
\[
\frac{1}{1+k} N_o(r, a; f| \leq k) \leq N(r, a; f| \leq k) \log r
\]
where we assume that \(\frac{1}{1+k} = 1\) and \(\frac{1}{k+1} = 0\) if \(k = \infty\).

Proof. Since we know that (cf. [5])
\[
N(r, a; f) \leq \frac{k}{1+k} N_o(r, a; f| \leq k) + \frac{1}{1+k} N_o(r, a; f),
\]
the lemma follows on integration. This proves the lemma. \(\square\)

Lemma 3. [1] Let \(a_1, a_2, \ldots, a_q\) be \(q(\geq 2)\) distinct elements of \(\mathbb{C} \cup \{\infty\}\). Then for all \(r > 1\)
\[
(q-2)T_o(r, f) \leq \sum_{i=1}^{q} N_o(r, a_i; f) + S_o(r, f),
\]
where
\[
\lim_{r \to \infty} \frac{S_o(r, f)}{T_o(r, f)} = 0 \quad \text{(1)}
\]
through all values of \(r\).

Lemma 4. [8] If \(f\) is transcendental meromorphic then
\[
\lim_{r \to \infty} \frac{T_o(r, f)}{(\log r)^2} = \infty.
\]
Lemma 5. (cf. [8]) For \( a \in \mathbb{C} \cup \{ \infty \} \) we get \( \delta(a; f) \leq \delta_0(a; f), \delta_k(a; f) \leq \delta_k^0(a; f) \) and \( \Theta(a; f) \leq \Theta_0(a; f) \).

Lemma 6. If \( f \) is a transcendental meromorphic function then
\[
\limsup_{r \to \infty} \frac{N_o(r, 0; f')}{T_o(r, f)} \leq 2 - \Theta(\infty; f) - \sum_{b \in \mathbb{C}} \delta(b; f).
\]

Proof. Let \( b_1, b_2, \ldots, b_p \) be distinct finite complex numbers. Then on integration we get from Littlewood’s inequality and the first fundamental theorem
\[
\sum_{\nu=1}^p m_o(r, b_\nu; f) \leq m_o(r, 0; f') + S_o(r, f)
\]

and so by Lemma 4 we obtain
\[
\sum_{\nu=1}^p m_o(r, b_\nu; f) \leq T_o(r, f') - N_o(r, 0; f') + S_o(r, f).
\]
Since \( T_o(r, f') \leq T_o(r, f) + \overline{N}_o(r, f) + S_o(r, f) \), it follows from above that
\[
\sum_{\nu=1}^p m_o(r, b_\nu; f) + N_o(r, 0; f) \leq T_o(r, f) + \overline{N}_o(r, f) + S_o(r, f)
\]
and so by (1) we get
\[
\sum_{\nu=1}^p \delta_\nu(b_\nu; f) + \limsup_{r \to \infty} \frac{N_o(r, 0; f')}{T_o(r, f)} \leq 2 - \Theta_0(\infty; f).
\]
Since \( p \) is arbitrary, it follows in view of Lemma 5 that
\[
\limsup_{r \to \infty} \frac{N_o(r, 0; f')}{T_o(r, f)} \leq 2 - \Theta(\infty; f) - \sum_{b \in \mathbb{C}} \delta(b; f).
\]
This proves the lemma.

3. Main Results

In this section we discuss the main results of the paper.

Theorem 1. Let \( f \) be a meromorphic function of order \( \rho, 0 \leq \rho \leq \infty \). If there exist distinct elements \( a_1, a_2, \ldots, a_p, b_1, b_2, \ldots, b_q, c_1, c_2, \ldots, c_s \) in \( \mathbb{C} \cup \{ \infty \} \) such that \( a_1, a_2, \ldots, a_p \) are evB for \( f \) for distinct zeros of multiplicity \( \leq k \), \( b_1, b_2, \ldots, b_q \) are evB for \( f \) for distinct zeros of multiplicity \( \leq l \) and \( c_1, c_2, \ldots, c_s \) are evB for \( f \) for distinct zeros of multiplicity \( \leq m \), where \( k, l, m \) are positive integers or infinity, then
\[
\frac{pk}{1+k} + \frac{ql}{1+l} + \frac{sm}{1+m} + \frac{1}{1+k} \sum_{i=1}^{p} \delta_k(a_i; f) \\
+ \frac{1}{1+l} \sum_{j=1}^{q} \delta_l(b_j; f) + \frac{1}{1+m} \sum_{t=1}^{s} \delta_m(c_t; f) \leq 2.
\]

Proof. From Lemmas 2 and 3 we get

\[
(p + q + s - 2)T_o(r, f) \leq \frac{k}{1+k} \sum_{i=1}^{p} N_o(r, a_i; f | \leq k) + \frac{1}{1+k} \sum_{i=1}^{p} N_o^0(r, a_i; f) \\
+ \frac{l}{1+l} \sum_{j=1}^{q} N_o(r, b_j; f | \leq l) + \frac{1}{1+l} \sum_{j=1}^{q} N_o^0(r, b_j; f) \\
+ \frac{m}{1+m} \sum_{t=1}^{s} N_o(r, c_t; f | \leq m) + \frac{1}{1+m} \sum_{t=1}^{s} N_m^0(r, c_t; f) \\
+ S_o(r, f).
\]

Since \(a_i, b_j, c_t\) are evB for \(f\) for distinct zeros of multiplicities not exceeding \(k, l\) and \(m\) respectively, by Lemma 1 there exists a number \(\alpha(0 < \alpha < \rho)\) such that

\[\overline{N}_o(r, a_i; f | \leq k) < r^\alpha, \overline{N}_o(r, b_j; f | \leq l) < r^\alpha\text{ and } \overline{N}_o(r, c_t; f | \leq m) < r^\alpha\]

for all large values of \(r\).

So from (2) we obtain for all large values of \(r\)

\[
(p + q + s - 2)T_o(r, f) \leq O(r^\alpha) + \frac{1}{1+k} \sum_{i=1}^{p} N_o^0(r, a_i; f) + \frac{1}{1+l} \sum_{j=1}^{q} N_o^0(r, b_j; f) \\
+ \frac{1}{1+m} \sum_{t=1}^{s} N_m^0(r, c_t; f) + S_o(r, f).
\]

(3)

Now for \(\varepsilon(>0)\) arbitrary we get from (3) for all large values of \(r\)

\[
(p + q + s - 2)T_o(r, f) \leq O(r^\alpha) + \frac{1}{1+k} \{p - \sum_{i=1}^{p} \delta_k^0(a_i; f) + \varepsilon\}T_o(r, f) \\
+ \frac{1}{1+l} \{q - \sum_{j=1}^{q} \delta_l^0(b_j; f) + \varepsilon\}T_o(r, f) \\
+ \frac{1}{1+m} \{s - \sum_{t=1}^{s} \delta_m^0(c_t; f) + \varepsilon\}T_o(r, f) + S_o(r, f).
\]

(4)

Now we choose a number \(\beta\) such that \(\alpha < \beta < \rho\). Then in view of Lemma 1 there exists a sequence of values of \(r\) tending to infinity such that
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\[ T_o(r, f) > r^\alpha. \]  

(5)

Since \( \varepsilon(>0) \) is arbitrary, it follows from (4) in view of (1) and (5) that

\[
\frac{pk}{1+k} + \frac{ql}{1+l} + \frac{sm}{1+m} + \frac{1}{1+k} \sum_{i=1}^{p} \delta^i_k(a_i; f) \\
+ \frac{1}{1+l} \sum_{j=1}^{q} \delta^j_l(b_j; f) + \frac{1}{1+m} \sum_{t=1}^{s} \delta^t_m(c_t; f) \leq 2
\]

from which the theorem follows by Lemma 5. This proves the theorem. \( \blacksquare \)

We now discuss some consequences of Theorem 1.

**Consequence 1.1.** For \( k = 1 \) we get

\[
p + \sum_{i=1}^{p} \delta_1(a_i; f) \leq 4.
\]

This shows that there exist at most four elements of \( \mathbb{C} \cup \{ \infty \} \) which are evB for \( f \) simple zeros. If there exist four evB for \( f \) for simple zeros then all these values are nvN for \( f \).

If \( f \) has an evB for simple zeros which is also an evN for \( f \) then \( f \) has at most three evB for simple zeros.

If \( a_1, a_2 \) are two evP for \( f \) then no element of \( \mathbb{C} \cup \{ \infty \} \{a_1, a_2\} \) is an evB for \( f \) for simple zeros.

**Consequence 1.2.** For \( k = 1, p = 1, l = 3 \) we get

\[
\frac{3q}{4} + \frac{1}{2} \delta_1(a_1; f) + \frac{1}{2} \sum_{j=1}^{q} \delta_3(b_j; f) \leq \frac{3}{2}.
\]  

(6)

This shows that if \( f \) has one evB for simple zeros, say \( a_1 \), then there exist at most two elements in \( \mathbb{C} \cup \{ \infty \} \{a_1\} \) which are evB for \( f \) for distinct zeros of multiplicity \( \leq 3 \).

Further it follows from (6) that if \( f \) has an evB for simple zeros and two other evB for distinct zeros of multiplicity \( \leq 3 \) then all these values are nvN for \( f \).

**Consequence 1.3.** For \( k = 1, p = 1, l = 2 \) we get

\[
\frac{2q}{3} + \frac{1}{2} \delta_1(a_1; f) + \frac{1}{3} \sum_{j=1}^{q} \delta_2(b_j; f) \leq \frac{3}{2}.
\]  

(7)

This shows that if \( f \) has one evB for simple zeros, say \( a_1 \), then there exist at most two elements in \( \mathbb{C} \cup \{ \infty \} \{a_1\} \) which are evB for \( f \) for distinct simple and double zeros.

If \( a_1 \) is an evB for \( f \) for simple zeros and \( b_1, b_2 \) are evB for \( f \) for distinct simple and double zeros, it follows from (7) that
\[3\delta_1(a_1; f) + 2\delta_2(b_1; f) + 2\delta_2(b_2; f) \leq 1.\]

Hence \(\delta_1(a_1; f) \leq 1/3, \ \delta_2(b_1; f) \leq 1/2, \ \delta_2(b_2; f) \leq 1/2\) and if the equality holds for any one then the other two are \(\text{nvN}\) for \(f\). In particular none of these exceptional values is an \(\text{evP}\) for \(f\).

**Consequence 1.4.** For \(k = 2, \ p = 1, \ l = 3, \ q = 1, \ m = 1\) we get

\[
s + \frac{2}{3}\delta_2(a_1; f) + \frac{1}{2}\delta_3(b_1; f) + \sum_{i=1}^{s} \delta_1(c_i; f) \leq \frac{7}{6}. \tag{8}\]

This shows that if \(f\) has one \(\text{evB}\), say \(a_1\), for distinct simple and double zeros and has one \(\text{evB}\), say \(b_1\), for distinct zeros of multiplicity \(\leq 3\) then \(f\) has at most one \(\text{evB}\) for simple zeros.

If \(f\) has one \(\text{evB}\), say \(c_1\), for simple zeros then we see from (8) that

\[4\delta_2(a_1; f) + 3\delta_3(b_1; f) + 6\delta_1(c_1; f) \leq 1.\]

Hence \(\delta_2(a_1; f) \leq 1/4, \ \delta_3(b_1; f) \leq 1/3, \ \delta_1(c_1; f) \leq 1/6\) and if the equality holds for any one then the other two are \(\text{nvN}\) for \(f\). In particular none of these exceptional values is an \(\text{evP}\) for \(f\).

**Consequence 1.5.** For \(k = 2\) we get

\[
\frac{2p}{3} + \frac{ql}{1 + l} + \frac{1}{3} \sum_{i=1}^{p} \delta_2(a_i; f) + \frac{1}{1 + l} \sum_{j=1}^{q} \delta_1(b_j; f) \leq 2. \tag{9}\]

This shows that

\[p + \frac{1}{2} \sum_{i=1}^{p} \delta_2(a_i; f) \leq 3.\]

So \(f\) has at most three \(\text{evB}\) for distinct simple and double zeros. Also if there exist three \(\text{evB}\) for \(f\) for distinct simple and double zeros then all the three exceptional values are \(\text{nvN}\) for \(f\). Further in this case (9) shows that there exists no other element of \(\mathbb{C} \cup \{\infty\}\) which is an \(\text{evB}\) for \(f\) for simple zeros.

If there exists an \(\text{evB}\) for \(f\) for distinct simple and double zeros which is also an \(\text{evN}\) then \(f\) has at most two \(\text{evB}\) for distinct simple and double zeros.

**Consequence 1.6.** For \(k = 2, \ p = 1, \ l = 1\) we get

\[
q + \frac{2}{3}\delta_2(a_1; f) + \sum_{j=1}^{q} \delta_1(b_j; f) \leq \frac{8}{3}. \tag{10}\]

From (10) it follows that if there exists an element of \(\mathbb{C} \cup \{\infty\}\) which is an \(\text{evB}\) for \(f\) for distinct simple and double zeros then there exist at most two other elements of \(\mathbb{C} \cup \{\infty\}\) which are \(\text{evB}\) for \(f\) for simple zeros.

If \(a_1\) is an \(\text{evB}\) for \(f\) for distinct simple and double zeros, \(b_1, b_2\) are \(\text{evB}\) for \(f\) for simple zeros then we get from (10)

\[2\delta_2(a_1; f) + 3\delta_1(b_1; f) + 3\delta_1(b_2; f) \leq 2.\]
This shows that no one of $b_1, b_2$ can be an evP for $f$. Further it follows that if $a_1$ is an evP for $f$ then $b_1, b_2$ are nvN for $f$.

**Consequence 1.7.** For $k = 5$, $p = 1$, $l = 2$, $q = 1$, $m = 1$ we get

$$s + \frac{1}{3} \delta_5(a_1; f) + \frac{2}{3} \delta_2(b_1; f) + \sum_{i=1}^{q} \delta_1(c_i; f) \leq 1.$$  

This shows that if $a_1$ is an evB for $f$ for distinct zeros of multiplicity $\leq 5$ and $b_1(\neq a_1)$ is an evB for $f$ for distinct simple and double zeros then there exists at most one element $c_1$, say, of $\mathbb{C} \cup \{\infty\} \setminus \{a_1, b_1\}$ which is an evB for $f$ for simple zeros. If $f$ actually has an evB simple zeros then all the exceptional values $a_1, b_1, c_1$ are nvN for $f$. Consequently if any one of $a_1$ and $b_1$ is an evN then there exist no element of $\mathbb{C} \cup \{\infty\} \setminus \{a_1, b_1\}$ which is an evB for $f$ for simple zeros.

**Theorem 2.** If there exist $a \in \mathbb{C} \cup \{\infty\}$ and positive integers $k$ and $q$ such that

$$(1 + k) \Theta(a; f) + \sum_{b \neq a} \delta(b; f) > 2 - k(q - 1)$$

then there exist at most $q$ elements of $\mathbb{C} \cup \{\infty\} \setminus \{a\}$ which are evB for $f$ for distinct zeros of multiplicity not exceeding $k$.

**Proof.** We assume, without loss of generality, that $a = \infty$. If possible suppose that there exist $q + 1$ elements $a_1, a_2, \ldots, a_{q+1}$ in $\mathbb{C}$ which are evB for $f$ for distinct zeros of multiplicity $\leq k$. Then by Lemma 1 there exists $\alpha(\alpha < \rho)$ such that for all large values of $r$

$$\overline{N}_o(r, a_i; f | \leq k) < r^\alpha$$

for $i = 1, 2, \ldots, q + 1$.

Since for $b \in \mathbb{C}$, a zero of $f - b$ of multiplicity $m(> 1)$ is a zero of $f'$ of multiplicity $m - 1$, we clearly have

$$\sum_{i=1}^{q+1} \overline{N}(r, a_i; f) \leq \sum_{i=1}^{q+1} \overline{N}(r, a_i; f | \leq k) + \frac{1}{k} N(r, 0; f')$$

which on integration gives

$$\sum_{i=1}^{q+1} \overline{N}_o(r, a_i; f) \leq \sum_{i=1}^{q+1} \overline{N}_o(r, a_i; f | \leq k) + \frac{1}{k} N_o(r, 0; f').$$

Hence by Lemma 3 we get for all large values of $r$

$$qT_o(r, f) \leq \sum_{i=1}^{q+1} \overline{N}_o(r, a_i; f) + \overline{N}_o(r, \infty; f) + S_o(r, f)$$

$$\leq \sum_{i=1}^{q+1} \overline{N}_o(r, a_i; f | \leq k) + \frac{1}{k} N_o(r, 0; f') + \overline{N}_o(r, \infty; f) + S_o(r, f)$$

$$\leq (q + 1)r^\alpha + \frac{1}{k} N_o(r, 0; f') + \overline{N}_o(r, \infty; f) + S_o(r, f)$$
and so by (1)

\[ q \leq (q + 1) \liminf_{r \to \infty} \frac{r^\alpha}{T_\alpha(r, f)} + \frac{1}{k} \limsup_{r \to \infty} \frac{N_\alpha(r, 0; f')}{T_\alpha(r, f)} + \limsup_{r \to \infty} \frac{N_\alpha(r, \infty; f)}{T_\alpha(r, f)}. \quad (11) \]

By Lemma 1 there exists \( \beta, \alpha < \beta < \rho \), such that for a sequence of values of \( r \) tending to infinity we get \( T_\alpha(r, f) > r^\beta \). So it follows that \( \liminf_{r \to \infty} \frac{r^\alpha}{T_\alpha(r, f)} = 0 \).

Hence we get from (11) by Lemma 6 and Lemma 5

\[ q \leq \frac{1}{k} \left( 2 - \Theta(\infty; f) - \sum_{b \neq \infty} \delta(b; f) \right) + 1 - \Theta(\infty; f) \]

\[ \leq \frac{1}{k} \left( 2 - \Theta(\infty; f) - \sum_{b \neq \infty} \delta(b; f) \right) + 1 - \Theta(\infty; f), \]

i.e.,

\[ (1 + k)\Theta(\infty; f) + \sum_{b \neq \infty} \delta(b; f) \leq 2 - k(q - 1), \]

which is a contradiction. This proves the theorem.

\[ \square \]

**Remark 1.** Gopalakrishna and Bhosmurth [2] proved Theorem 2 for functions of finite order by a different method using the notion of proximate order.

Now we discuss some consequences of Theorem 2.

**Consequence 2.1.** For \( k = 1, q = 3 \) we see that if \( \Theta(a; f) > 0 \) for some \( a \in \mathbb{C} \cup \{ \infty \} \) then there exist at most three elements of \( \mathbb{C} \cup \{ \infty \} \setminus \{ a \} \) which are evB for \( f \) for simple zeros.

Hence it follows that if there exist distinct elements \( a_1, a_2, a_3, a_4 \) in \( \mathbb{C} \cup \{ \infty \} \) which are evB for \( f \) for simple zeros then \( \Theta(a; f) = 0 \) for all \( a \in \mathbb{C} \cup \{ \infty \} \setminus \{ a_1, a_2, a_3, a_4 \} \).

So in view of the consequence 1.1 it follows that if \( f \) has four distinct evB for simple zeros then \( f \) has no evN.

**Consequence 2.2.** For \( k = 1, q = 2 \) we see that if

\[ 2\Theta(a; f) + \sum_{b \neq a} \delta(b; f) > 1 \]

for some \( a \in \mathbb{C} \cup \{ \infty \} \) then there exist at most two elements of \( \mathbb{C} \cup \{ \infty \} \setminus \{ a \} \) which are evB for \( f \) for simple zeros. In particular, this holds if there exists an \( a \in \mathbb{C} \cup \{ \infty \} \) such that \( \Theta(a; f) > 1/2 \).

It, therefore, follows that if there exist distinct elements \( a_1, a_2, a_3, a_4 \) in \( \mathbb{C} \cup \{ \infty \} \) which are evB for \( f \) for simple zeros then \( \Theta(a_i; f) \leq 1/2 \) for \( i = 1, 2, 3, 4 \).

Hence in view of Consequence 2.1 we see that if \( a_1, a_2, a_3, a_4 \) are evB for \( f \) for simple zeros then \( \Theta(a; f) = 0 \) if \( a \neq a_i (i = 1, 2, 3, 4) \) and \( \Theta(a_i; f) \leq 1/2 \) for \( i = 1, 2, 3, 4 \).
We know that $\infty, a_1, a_2, a_3$ are four distinct evB for the Weierstrass elliptic function $P(z)$ for simple zeros where $a_1, a_2, a_3$ are given by

$$\{P'(z)\}^2 = (P(z) - a_1)(P(z) - a_2)(P(z) - a_3).$$

In view of Lemma 5 [6] we see that $\Theta(\infty; P) = \Theta(a_1; P) = \Theta(a_2; P) = \Theta(a_3; P) = 1/2$. So the above estimation is sharp.

**Consequence 2.3.** For $k = 2, q = 2$ we see that if $\Theta(a; f) > 0$ for some $a \in \mathbb{C} \cup \{\infty\}$ then $f$ has at most two evB for distinct simple and double zeros. So if $a_1, a_2, a_3$ are three evB for $f$ for distinct simple and double zeros then $\Theta(a; f) = 0$ for all $a \in \mathbb{C} \cup \{\infty\} \setminus \{a_1, a_2, a_3\}$.

Hence in view of Consequence 1.5 it follows that if $f$ has three evB for distinct simple and double zeros then $f$ has no evN.

**Consequence 2.4.** For $k = 2, q = 1$ we see that if

$$3\Theta(a; f) + \sum_{b \neq a} \delta(b; f) > 2$$

for some $a \in \mathbb{C} \cup \{\infty\}$ then there exists at most one element of $a \in \mathbb{C} \cup \{\infty\} \setminus \{a\}$ which is an evB for $f$ for distinct simple and double zeros. In particular, this holds if there exists an $a \in \mathbb{C} \cup \{\infty\}$ such that $\Theta(a; f) > 2/3$.

It, therefore, follows that if there exist distinct elements $a_1, a_2, a_3$ in $\mathbb{C} \cup \{\infty\}$ which are evB for $f$ for distinct simple and double zeros then $\Theta(a_i; f) \leq 2/3$ for $i = 1, 2, 3$.

Hence in view of Consequence 2.3 we see that if $a_1, a_2, a_3$ are evB for $f$ for distinct simple and double zeros then $\Theta(a_i; f) = 0$ if $a \neq a_i (i = 1, 2, 3)$ and $\Theta(a_i; f) \leq 2/3$ for $i = 1, 2, 3$.

The following example shows that the above estimation is sharp.

**Example 1.** (cf. [4, p. 45]) We set

$$z = \phi(z) = \int_0^w (t - a_1)^{-2/3}(t - a_2)^{-2/3}(t - a_3)^{-2/3}dt$$

where $a_1, a_2, a_3$ are distinct finite complex numbers. The inverse function can be analytically continued over the whole plane as a single valued meromorphic function by Schwarz’s reflection principle and the resulting function $w = f(z)$ is doubly periodic. Also we can verify that $a_1, a_2, a_3$ are evB for $f$ for distinct simple and double zeros and $\Theta(a_i; f) = 2/3$ for $i = 1, 2, 3$. Further we note that $f$ has no evN.

Singh and Gopalakrishna [7] proved the following theorem.

**Theorem B.** Let $f$ be a meromorphic function of finite order. If $a$ and $\infty$ are evB for $f$ for distinct zeros for some $a \in \mathbb{C}$ then for each integer $k(\geq 1)$ $f^{(k)}$ has no evB for simple zeros in $\mathbb{C} \setminus \{a\}$.
Improving Theorem B Gopalakrishna and Bhoosnurmath [3] proved the following result.

**Theorem C.** Let \( f \) be a meromorphic function of finite hyperorder, i.e.,
\[
\limsup_{r \to \infty} \frac{\log \log T(r, f)}{\log r} < \infty.
\]
If 0 and \( \infty \) are evB for \( f \) for distinct zeros then any homogeneous differential polynomial \( P \) generated by \( f \) has no evB for simple zeros in \( \mathbb{C} \cup \{ \infty \} \setminus \{0, \infty\} \).

If \( P \) is generated by \( f' \) then the result holds if \( a \) and \( \infty \) are evB for \( f \) for distinct zeros for some \( a \in \mathbb{C} \).

We improve Theorem C by completely withdrawing the order restriction on \( f \).

**Theorem 3.** If 0 and \( \infty \) are evB for \( f \) for distinct zeros then any homogeneous differential polynomial \( P \) generated by \( f \) has no evB for simple zeros in \( \mathbb{C} \setminus \{0\} \).

If \( P \) is generated by \( f' \) then the result holds if \( a \) and \( \infty \) are evB for \( f \) for distinct zeros for some \( a \in \mathbb{C} \).

**Proof.** First we note that under the hypothesis of the theorem the order of \( P \) is equal to the order of \( f \) (cf. [3 Theorem 1]). Let \( n \) be the degree of \( P \).

Since 0 and \( \infty \) are evB for distinct zeros for \( f \), by Lemma 1 we get for all large values of \( r \)
\[
N_0(r, 0; f) < r^\alpha \quad \text{and} \quad N_0(r, \infty; f) < r^\alpha,
\]
where \( \alpha < \rho \).

Also for all large values of \( r \) we get
\[
nT_o(r, f) \leq T_o(r, P) + N_o(r, \frac{P}{f^n}) + S_o(r, f)
\]
\[
= T_o(r, P) + O(N_0(r, 0; f) + N_0(r, \infty; f)) + S_o(r, f)
\]
\[
= O(r^\alpha) + T_o(r, P) + S_o(r, f),
\]
i.e.,
\[
\{n + o(1)\}T_o(r, f) \leq T_o(r, P) + O(r^\alpha). \tag{12}
\]
We see that
\[
N(r, 0; P) \leq N(r, \frac{f^n}{P}) + N(r, 0; f)
\]
\[
\leq T(r, \frac{P}{f^n}) + N(r, 0; f) + S(r, f)
\]
\[
= N(r, \frac{P}{f^n}) + N(r, 0; f) + S(r, f)
\]
and so
Now for \( b \in \mathbb{C}\backslash\{0\} \) we get from Lemmas 2 and 3

\[
T_o(r, P) \leq N_o(r, 0; P) + N_o(r, \infty; P) + N_o(r, b; P) + S_o(r, P) \\
\leq N_o(r, 0; P) + N_o(r, \infty; f) + \frac{1}{2} N_o(r, b; P \leq 1) \\
+ \frac{1}{2} N_o^b(r, b; P) + S_o(r, P) + S_o(r, f) \\
\leq O(r^\alpha) + \frac{1}{2} N_o(r, b; P \leq 1) + \frac{1}{2} T_o(r, P) + S_o(r, P) + S_o(r, f),
\]

i.e.,

\[
\frac{1}{2} T_o(r, P) \leq O(r^\alpha) + \frac{1}{2} N_o(r, b; P \leq 1) + S_o(r, P) + S_o(r, f). \quad (13)
\]

If possible, suppose that \( b \) is an evB for \( P \) for simple zeros. Then from (13) we get for all large values of \( r \)

\[
\frac{1}{2} T_o(r, P) \leq O(r^\alpha) + \frac{1}{2} N_o(r, b; P \leq 1) + S_o(r, P) + S_o(r, f). \quad (14)
\]

Since \( \rho \) is the order of \( P \), by Lemma 1 it follows that for \( \alpha < \beta < \rho \) there exists a sequence \( \{r_n\} \) of values of \( r \) tending to infinity such that

\[
T_o(r_n, P) > r_n^\beta \text{ for } n = 1, 2, 3, \ldots.
\]

Hence for \( r = r_n \) we get from (12) and (14)

\[
\frac{1}{2} T_o(r_n; P) \leq o(T_o(r_n; P)),
\]

which is a contradiction. So no element of \( \mathbb{C}\backslash\{0\} \) is an evB for \( P \) for simple zeros.

The second part of the theorem is obvious because if \( P \) is generated by \( f' \) then \( P \) is also generated by \( g' \) where \( g = f - a \) for any \( a \in \mathbb{C} \). This proves the theorem. \( \blacksquare \)

**Remark 2.** Theorem 3 does not hold for a non-homogeneous differential polynomial. For, let \( f = \exp(z) \) and \( P = f' + 1 \). Then \( 0 \) and \( \infty \) are evB for \( f \) for distinct zeros but \( 1 \) is an evB for \( P \) simple zeros.

**References**