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Borel Exceptional Values of Meromorphic Functions

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Abstract. In the paper we discuss the Borel exceptional value of a transcendental meromorphic function and its relation with Picard and Nevanlinna exceptional values.

1. Introduction and Definitions

Let f be a nonconstant meromorphic function defined in the open complex plane $\mathbb{C}\cup\{\infty\}$. We use the standard notations and definitions of the value distribution theory (cf. [4]). Let k be a positive integer or infinity. We denote by $\overline{N}(r,a;f\mid\leq k)$ the counting function of distinct a-points of f whose multiplicities do not exceed k. Clearly $\overline{N}(r,a;f\mid\leq\infty)\equiv\overline{N}(r,a;f)$.

We put

$$\overline{\rho}_k(a;f)=\limsup_{r\to\infty}\frac{\log^+\overline{N}(r,a;f\mid\leq k)}{\log r},\ \overline{\rho}(a;f)=\limsup_{r\to\infty}\frac{\log^+\overline{N}(r,a;f)}{\log r},$$

and

$$\rho(a;f)=\limsup_{r\to\infty}\frac{\log^+ N(r,a;f)}{\log r}.$$

If f is a meromorophic function of order $\rho, 0\leq\rho\leq\infty$, $a\in\mathbb{C}\cup\{\infty\}$ and k is a positive integer then we say that a is

(i) an exceptional value in the sense of Borel (evB for short) for f for distinct zeros of multiplicities not exceeding k if $\overline{\rho}_k(a;f)<\rho$;

(ii) an evB for f for distinct zeros if $\overline{\rho}(a;f)<\rho$;

(iii) an evB for f for the whole aggregate of zeros if $\rho(a; f) < \rho$.

We call a an evB for f for simple zeros if $\overline{\rho}_1(a; f) < \rho$ and an evB for f for simple and double zeros if $\overline{\rho}_2(a; f) < \rho$.

The quantity

$$\delta(a; f) = 1 - \limsup_{r\to\infty} \frac{N(r, a; f)}{T(r, f)}$$

is called Nevanlinna deficiency of the value $a \in \mathbb{C} \cup \{\infty\}$. If $\delta(a; f) > 0$ then a is called an exceptional value in the sense of Nevanlinna, in short evN. If $\delta(a; f) = 0$ then a is called a normal value in the sense of Nevanlinna, in short nvN.

A value $a \in \mathbb{C} \cup \{\infty\}$ is called an exceptional value for a transcendental meromorphic function in the sense of Picard, in short evP, if f has at most a finite number of a-points.

Let k be a nonnegative integer or infinity. We denote by $N_k(r, a; f)$ the counting function of a-points of f where an a-point of multiplicity ν is counted ν times if $\nu \le k$ and $1 + k$ times if $\nu > k$.

We put

$$\delta_k(a; f) = 1 - \limsup_{r\to\infty} \frac{N_k(r, a; f)}{T(r, f)}.$$

Then clearly $\delta(a; f) \le \delta_k(a; f) \le \delta_{k-1}(a; f) \le \cdots \le \delta_1(a; f) \le \delta_0(a; f) = \Theta(a; f) \le 1$, where

$$\Theta(a; f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, a; f)}{T(r, f)}$$

is called the ramification index.

Valiron (cf. [9, pp. 72-78]) proved the following generalization of the classical theorem of Borel for entire functions of finite order.

Theorem A. *Let f be an entire function of finite order ρ. Then*

(i) *there exist at most two distinct elements of $\mathbb{C}$ which are evB for f for simple zeros,*

(ii) *if there exists $a \in \mathbb{C}$ such that a is an evB for f for the joint sequence of simple and double zeros (double zeros being counted twice) then $\overline{\rho}_1(b; f) = \rho$ for all $b \in \mathbb{C}\backslash\{a\}$,*

(iii) *there exists at most one element of $\mathbb{C}$ which is an evB for f for the joint sequence of simple and double zeros.*

In [7] Singh and Gopalakrishna obtained stronger results than above for meromorphic functions of finite order. Gopalakrishna and Bhoosnurmath [2] improved the results of Valiron and Singh-Gopalakrishna to meromorphic functions of unrestricted order. The result of Gopalakrishna and Bhoosnurmath [2] can be stated as follows:

Theorem B. *Let f be a meromorphic function of order ρ, $0 \le \rho \le \infty$. If there exist distinct elements $a_1, a_2, \dots, a_p, b_1, b_2, \dots, b_q, c_1, c_2, \dots, c_s$ in $\mathbb{C}\cup\{\infty\}$ such that $a_1, a_2, \dots, a_p$ are evB for f for distinct zeros of multiplicity $\le k$, $b_1, b_2, \dots, b_q$ are evB for f for distinct zeros of multiplicity $\le l$ and $c_1, c_2, \dots, c_s$*

are evB for f for distinct zeros of multiplicity $\le m$, where k, l, m are positive integers, then

$$\frac{pk}{1+k} + \frac{ql}{1+l} + \frac{sm}{1+m} \le 2.$$

In the paper we improve Theorem B and discuss the relation between the evB for a meromorphic function and the Nevanlinna deficiency of those values. We also improve some other results on the exceptional values in the sense of Borel.

Definition 1. [1, 8] *We put for $a \in \mathbb{C} \cup \{\infty\}$*

$$T_o(r, f) = \int_1^r \frac{T(t, f)}{t} dt,$$

$$N_o(r, a; f) = \int_1^r \frac{N(t, a; f)}{t} dt, \quad m_o(r, a; f) = \int_1^r \frac{m(t, a; f)}{t} dt,$$

$$\overline{N}_o(r, a; f) = \int_1^r \frac{\overline{N}(t, a; f)}{t} dt, \quad \overline{N}_o(r, a; f \mid\le k) = \int_1^r \frac{\overline{N}(t, a; f \mid\le k)}{t} dt,$$

$$N_k^o(r, a; f) = \int_1^r \frac{N_k(t, a; f)}{t} dt, \quad S_o(r, f) = \int_1^r \frac{S(t, f)}{t} dt \quad etc.$$

Further we put for $a \in \mathbb{C} \cup \{\infty\}$

$$\delta_o(a; f) = 1 - \limsup_{r\to\infty} \frac{N_o(r, a; f)}{T_o(r, f)},$$
$$\Theta_o(a; f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}_o(r, a; f)}{T_o(r, f)},$$
$$\delta_o^k(a; f) = 1 - \limsup_{r\to\infty} \frac{N_k^o(r, a; f)}{T_o(r, f)}.$$

Throughout the paper we assume that f is a transcendental meromorphic function of finite or infinite order ρ defined in the open complex plane $\mathbb{C}$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1.

(i) $$\limsup_{r\to\infty} \frac{\log T_o(r, f)}{\log r} = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}$$

and

$$\liminf_{r\to\infty}\frac{\log T_o(r,f)}{\log r}=\liminf_{r\to\infty}\frac{\log T(r,f)}{\log r}.$$

(ii) *For any $a\in\mathbb{C}\cup\{\infty\}$ and for any k, a positive integer or infinity,*

$$\limsup_{r\to\infty}\frac{\log^+\overline{N}_o(r,a;f\mid\le k)}{\log r}=\limsup_{r\to\infty}\frac{\log^+\overline{N}(r,a;f\mid\le k)}{\log r}$$

and

$$\liminf_{r\to\infty}\frac{\log^+\overline{N}_o(r,a;f\mid\le k)}{\log r}=\liminf_{r\to\infty}\frac{\log^+\overline{N}(r,a;f\mid\le k)}{\log r}.$$

Proof. Since for all large values of r

$$T_o(r,f)\le T(r,f)\log r\quad\text{and}\quad T_o(2r,f)\ge T(r,f)\log 2,$$

(i) follows easily.

Again since for all large values of r

$$\overline{N}_o(r,a;f\mid\le k)\le\overline{N}(r,a;f\mid\le k)\log r$$

and

$$\overline{N}_o(2r,a;f\mid\le k)\ge\overline{N}(r,a;f\mid\le k)\log 2,$$

(ii) follows easily. This proves the lemma. ∎

Lemma 2. *Let k be a positive integer or infinity. Then for $a\in\mathbb{C}\cup\{\infty\}$*

$$\overline{N}_o(r,a;f)\le\frac{k}{1+k}\overline{N}_o(r,a;f\mid\le k)+\frac{1}{1+k}N_k^o(r,a;f),$$

where we assume that $\frac{k}{1+k}=1$ and $\frac{1}{k+1}=0$ if $k=\infty$.

Proof. Since we know that (cf. [5])

$$N(r,a;f)\le\frac{k}{1+k}\overline{N}(r,a;f\mid\le k)+\frac{1}{1+k}N_k(r,a;f),$$

the lemma follows on integration. This proves the lemma. ∎

Lemma 3. [1] *Let $a_1,a_2,\dots,a_q$ be $q(\ge 2)$ distinct elements of $\mathbb{C}\cup\{\infty\}$. Then for all $r>1$*

$$(q-2)T_o(r,f)\le\sum_{i=1}^{q}\overline{N}_o(r,a_i;f)+S_o(r,f),$$

where

$$\lim_{r\to\infty}\frac{S_o(r,f)}{T_o(r,f)}=0\tag{1}$$

through all values of r.

Lemma 4. [8] *If f is transcendental meromorphic then*

$$\lim_{r\to\infty}\frac{T_o(r,f)}{(\log r)^2}=\infty.$$

Lemma 5. (cf. [8]) *For* $a \in \mathbb{C}\cup\{\infty\}$ *we get* $\delta(a; f) \le \delta_o(a; f)$, $\delta_k(a; f) \le \delta_k^o(a; f)$ *and* $\Theta(a; f) \le \Theta_o(a; f)$.

Lemma 6. *If f is a transcendental meromorphic function then*

$$\limsup_{r\to\infty} \frac{N_o(r, 0; f')}{T_o(r, f)} \le 2 - \Theta(\infty; f) - \sum_{b\in\mathbb{C}} \delta(b; f).$$

Proof. Let $b_1, b_2, \dots, b_p$ be distinct finite complex numbers. Then on integration we get from Littlewood's inequality and the first fundamental theorem

$$\begin{aligned}\sum_{\nu=1}^{p} m_o(r, b_\nu; f) &\le m_o(r, 0; f') + S_o(r, f)\\ &= T_o(r, f') - N_o(r, 0; f') + O(\log r) + S_o(r, f)\end{aligned}$$

and so by Lemma 4 we obtain

$$\sum_{\nu=1}^{p} m_o(r, b_\nu; f) \le T_o(r, f') - N_o(r, 0; f') + S_o(r, f).$$

Since $T_o(r, f') \le T_o(r, f) + \overline{N}_o(r, f) + S_o(r, f)$, it follows from above that

$$\sum_{\nu=1}^{p} m_o(r, b_\nu; f) + N_o(r, 0; f) \le T_o(r, f) + \overline{N}_o(r, f) + S_o(r, f)$$

and so by (1) we get

$$\sum_{\nu=1}^{p} \delta_o(b_\nu; f) + \limsup_{r\to\infty} \frac{N_o(r, 0; f')}{T_o(r, f)} \le 2 - \Theta_o(\infty; f).$$

Since p is arbitrary, it follows in view of Lemma 5 that

$$\limsup_{r\to\infty} \frac{N_o(r, 0; f')}{T_o(r, f)} \le 2 - \Theta(\infty; f) - \sum_{b\in\mathbb{C}} \delta(b; f).$$

This proves the lemma. ∎

3. Main Results

In this section we discuss the main results of the paper.

Theorem 1. *Let f be a meromorphic function of order* ρ, $0 \le \rho \le \infty$. *If there exist distinct elements* $a_1, a_2, \dots, a_p, b_1, b_2, \dots, b_q, c_1, c_2, \dots, c_s$ *in* $\mathbb{C}\cup\{\infty\}$ *such that* $a_1, a_2, \dots, a_p$ *are evB for f for distinct zeros of multiplicity* $\le k$, $b_1, b_2, \dots, b_q$ *are evB for f for distinct zeros of multiplicity* $\le l$ *and* $c_1, c_2, \dots, c_s$ *are evB for f for distinct zeros of multiplicity* $\le m$, *where* k, l, m *are positive integers or infinity, then*

$$\frac{pk}{1+k}+\frac{ql}{1+l}+\frac{sm}{1+m}+\frac{1}{1+k}\sum_{i=1}^{p}\delta_k(a_i;f)$$
$$+\frac{1}{1+l}\sum_{j=1}^{q}\delta_l(b_j;f)+\frac{1}{1+m}\sum_{t=1}^{s}\delta_m(c_t;f)\leq 2.$$

Proof. From Lemmas 2 and 3 we get

$$\begin{aligned}(p+q+s-2)T_o(r,f)\leq{}&\frac{k}{1+k}\sum_{i=1}^{p}\overline{N}_o(r,a_i;f\mid\leq k)+\frac{1}{1+k}\sum_{i=1}^{p}N_k^o(r,a_i;f)\\&+\frac{l}{1+l}\sum_{j=1}^{q}\overline{N}_o(r,b_j;f\mid\leq l)+\frac{1}{1+l}\sum_{j=1}^{q}N_l^o(r,b_j;f)\\&+\frac{m}{1+m}\sum_{t=1}^{s}\overline{N}_o(r,c_t;f\mid\leq m)+\frac{1}{1+m}\sum_{t=1}^{s}N_m^o(r,c_t;f)\\&+S_o(r,f).\end{aligned}\tag{2}$$

Since a_i, b_j, c_t are evB for f for distinct zeros of multiplicities not exceeding k, l and m respectively, by Lemma 1 there exists a number $\alpha(0<\alpha<\rho)$ such that

$$\overline{N}_o(r,a_i;f\mid\leq k)<r^{\alpha},\ \overline{N}_o(r,b_j;f\mid\leq l)<r^{\alpha}\quad\text{and}\quad\overline{N}_o(r,c_t;f\mid\leq m)<r^{\alpha}$$

for all large values of r.

So from (2) we obtain for all large values of r

$$\begin{aligned}(p+q+s-2)T_o(r,f)\leq{}&O(r^{\alpha})+\frac{1}{1+k}\sum_{i=1}^{p}N_k^o(r,a_i;f)+\frac{1}{1+l}\sum_{j=1}^{q}N_l^o(r,b_j;f)\\&+\frac{1}{1+m}\sum_{t=1}^{s}N_m^o(r,c_t;f)+S_o(r,f).\end{aligned}\tag{3}$$

Now for $\varepsilon(>0)$ arbitrary we get from (3) for all large values of r

$$\begin{aligned}(p+q+s-2)T_o(r,f)\leq{}&O(r^{\alpha})+\frac{1}{1+k}\{p-\sum_{i=1}^{p}\delta_k^o(a_i;f)+\varepsilon\}T_o(r,f)\\&+\frac{1}{1+l}\{q-\sum_{j=1}^{q}\delta_l^o(b_j;f)+\varepsilon\}T_o(r,f)\\&+\frac{1}{1+m}\{s-\sum_{t=1}^{s}\delta_m^o(c_t;f)+\varepsilon\}T_o(r,f)+S_o(r,f).\end{aligned}\tag{4}$$

Now we choose a number β such that $\alpha<\beta<\rho$. Then in view of Lemma 1 there exists a sequence of values of r tending to infinity such that

$$T_o(r,f) > r^{\beta}. \tag{5}$$

Since $\varepsilon(>0)$ is arbitrary, it follows from (4) in view of (1) and (5) that

$$\frac{pk}{1+k} + \frac{ql}{1+l} + \frac{sm}{1+m} + \frac{1}{1+k}\sum_{i=1}^{p} \delta_k^o(a_i;f)$$
$$+ \frac{1}{1+l}\sum_{j=1}^{q} \delta_l^o(b_j;f) + \frac{1}{1+m}\sum_{t=1}^{s} \delta_m^o(c_t;f) \le 2$$

from which the theorem follows by Lemma 5. This proves the theorem. ∎

We now discuss some consequences of Theorem 1.

Consequence 1.1. *For $k=1$ we get*

$$p + \sum_{i=1}^{p} \delta_1(a_i;f) \le 4.$$

This shows that there exist at most four elements of $\mathbb{C}\cup\{\infty\}$ which are evB for f simple zeros. If there exist four evB for f for simple zeros then all these values are nvN for f.

If f has an evB for simple zeros which is also an evN for f then f has at most three evB for simple zeros.

If a_1, a_2 are two evP for f then no element of $\mathbb{C}\cup\{\infty\}\backslash\{a_1,a_2\}$ is an evB for f for simple zeros.

Consequence 1.2. *For $k=1$, $p=1$, $l=3$ we get*

$$\frac{3q}{4} + \frac{1}{2}\delta_1(a_1;f) + \frac{1}{4}\sum_{j=1}^{q} \delta_3(b_j;f) \le \frac{3}{2}\,. \tag{6}$$

This shows that if f has one evB for simple zeros, say a_1, then there exist at most two elements in $\mathbb{C}\cup\{\infty\}\backslash\{a_1\}$ which are evB for f for distinct zeros of multiplicity ≤ 3.

Further it follows from (6) that if f has an evB for simple zeros and two other evB for distinct zeros of multiplicity ≤ 3 then all these values are nvN for f.

Consequence 1.3. For $k=1$, $p=1$, $l=2$ we get

$$\frac{2q}{3} + \frac{1}{2}\delta_1(a_1;f) + \frac{1}{3}\sum_{j=1}^{q} \delta_2(b_j;f) \le \frac{3}{2}\,. \tag{7}$$

This shows that if f has one evB for simple zeros, say a_1, then there exist at most two elements in $\mathbb{C}\cup\{\infty\}\backslash\{a_1\}$ which are evB for f for distinct simple and double zeros.

If a_1 is an evB for f for simple zeros and b_1, b_2 are evB for f for distinct simple and double zeros, it follows from (7) that

$$3\delta_1(a_1;f)+2\delta_2(b_1;f)+2\delta_2(b_2;f)\le 1.$$

Hence $\delta_1(a_1;f)\le 1/3$, $\delta_2(b_1;f)\le 1/2$, $\delta_2(b_2;f)\le 1/2$ and if the equality holds for any one then the other two are nvN for f. In particular none of these exceptional values is an evP for f.

Consequence 1.4. *For $k=2$, $p=1$, $l=3$, $q=1$, $m=1$ we get*

$$s+\frac{2}{3}\delta_2(a_1;f)+\frac{1}{2}\delta_3(b_1;f)+\sum_{t=1}^{s}\delta_1(c_t;f)\le\frac{7}{6}\,. \tag{8}$$

This shows that if f has one evB, say a_1, for distinct simple and double zeros and has one evB, say b_1, for distinct zeros of multiplicity ≤ 3 then f has at most one evB for simple zeros.

If f has one evB, say c_1, for simple zeros then we see from (8) that

$$4\delta_2(a_1;f)+3\delta_3(b_1;f)+6\delta_1(c_1;f)\le 1.$$

Hence $\delta_2(a_1;f)\le 1/4$, $\delta_3(b_1;f)\le 1/3$, $\delta_1(c_1;f)\le 1/6$ and if the equality holds for any one then the other two are nvN for f. In particular none of these exceptional values is an evP for f.

Consequence 1.5. *For $k=2$ we get*

$$\frac{2p}{3}+\frac{ql}{1+l}+\frac{1}{3}\sum_{i=1}^{p}\delta_2(a_i;f)+\frac{1}{1+l}\sum_{j=1}^{q}\delta_l(b_j;f)\le 2. \tag{9}$$

This shows that

$$p+\frac{1}{2}\sum_{i=1}^{p}\delta_2(a_i;f)\le 3.$$

So f has at most three evB for distinct simple and double zeros. Also if there exist three evB for f for distinct simple and double zeros then all the three exceptional values are nvN for f. Further in this case (9) shows that there exists no other element of $\mathbb{C}\cup\{\infty\}$ which is an evB for f for simple zeros.

If there exists an evB for f for distinct simple and double zeros which is also an evN then f has at most two evB for distinct simple and double zeros.

Consequence 1.6. *For $k=2$, $p=1$, $l=1$ we get*

$$q+\frac{2}{3}\delta_2(a_1;f)+\sum_{j=1}^{q}\delta_1(b_j;f)\le\frac{8}{3}\,. \tag{10}$$

From (10) it follows that if there exists an element of $\mathbb{C}\cup\{\infty\}$ which is an evB for f for distinct simple and double zeros then there exist at most two other elements of $\mathbb{C}\cup\{\infty\}$ which are evB for f for simple zeros.

If a_1 is an evB for f for distinct simple and double zeros, b_1, b_2 are evB for f for simple zeros then we get from (10)

$$2\delta_2(a_1;f)+3\delta_1(b_1;f)+3\delta_1(b_2;f)\le 2.$$

This shows that no one of b_1, b_2 can be an evP for f. Further it follows that if a_1 is an evP for f then b_1, b_2 are nvN for f.

Consequence 1.7. *For $k = 5$, $p = 1$, $l = 2$, $q = 1$, $m = 1$ we get*

$$s + \frac{1}{3}\delta_5(a_1; f) + \frac{2}{3}\delta_2(b_1; f) + \sum_{t=1}^{s} \delta_1(c_t; f) \leq 1.$$

This shows that if a_1 is an evB for f for distinct zeros of multiplicity ≤ 5 and $b_1(\neq a_1)$ is an evB for f for distinct simple and double zeros then there exists at most one element c_1, say, of $\mathbb{C} \cup \{\infty\} \backslash \{a_1, b_1\}$ which is an evB for f for simple zeros. If f actually has an evB simple zeros then all the exceptional values a_1, b_1, c_1 are nvN for f. Consequently if any one of a_1 and b_1 is an evN then there exist no element of $\mathbb{C} \cup \{\infty\} \backslash \{a_1, b_1\}$ which is an evB for f for simple zeros.

Theorem 2. *If there exist $a \in \mathbb{C} \cup \{\infty\}$ and positive integers k and q such that*

$$(1 + k)\Theta(a; f) + \sum_{b \neq a} \delta(b; f) > 2 - k(q - 1)$$

then there exist at most q elements of $\mathbb{C} \cup \{\infty\} \backslash \{a\}$ which are evB for f for distinct zeros of multiplicity not exceeding k.

Proof. We assume, without loss of generality, that $a = \infty$. If possible suppose that there exist $q + 1$ elements $a_1, a_2, \ldots, a_{q+1}$ in $\mathbb{C}$ which are evB for f for distinct zeros of multiplicity $\leq k$. Then by Lemma 1 there exists $\alpha(\alpha < \rho)$ such that for all large values of r

$$\overline{N}_o(r, a_i; f \mid \leq k) < r^{\alpha}$$

for $i = 1, 2, \ldots, q + 1$.

Since for $b \in \mathbb{C}$, a zero of $f - b$ of multiplicity $m(> 1)$ is a zero of f' of multiplicity $m - 1$, we clearly have

$$\sum_{i=1}^{q+1} \overline{N}(r, a_i; f) \leq \sum_{i=1}^{q+1} \overline{N}(r, a_i; f \mid \leq k) + \frac{1}{k} N(r, 0; f')$$

which on integration gives

$$\sum_{i=1}^{q+1} \overline{N}_o(r, a_i; f) \leq \sum_{i=1}^{q+1} \overline{N}_o(r, a_i; f \mid \leq k) + \frac{1}{k} N_o(r, 0; f').$$

Hence by Lemma 3 we get for all large values of r

$$\begin{aligned} qT_o(r, f) &\leq \sum_{i=1}^{q+1} \overline{N}_o(r, a_i; f) + \overline{N}_o(r, \infty; f) + S_o(r, f) \\ &\leq \sum_{i=1}^{q+1} \overline{N}_o(r, a_i; f \mid \leq k) + \frac{1}{k} N_o(r, 0; f') + \overline{N}_o(r, \infty; f) + S_o(r, f) \\ &\leq (q + 1)r^{\alpha} + \frac{1}{k} N_o(r, 0; f') + \overline{N}_o(r, \infty; f) + S_o(r, f) \end{aligned}$$

and so by (1)

$$q \leq (q+1)\liminf_{r\to\infty}\frac{r^\alpha}{T_o(r,f)} + \frac{1}{k}\limsup_{r\to\infty}\frac{N_o(r,0;f')}{T_o(r,f)} + \limsup_{r\to\infty}\frac{\overline{N}_o(r,\infty;f)}{T_o(r,f)}. \quad (11)$$

By Lemma 1 there exists $\beta, \alpha < \beta < \rho$, such that for a sequence of values of r tending to infinity we get $T_o(r,f) > r^\beta$. So it follows that $\liminf_{r\to\infty}\frac{r^\alpha}{T_o(r,f)} = 0$. Hence we get from (11) by Lemma 6 and Lemma 5

$$\begin{aligned} q &\leq \frac{1}{k}\{2-\Theta(\infty;f) - \sum_{b\neq\infty}\delta(b;f)\} + 1 - \Theta_o(\infty;f) \\ &\leq \frac{1}{k}\{2-\Theta(\infty;f) - \sum_{b\neq\infty}\delta(b;f)\} + 1 - \Theta(\infty;f), \end{aligned}$$

i.e.,

$$(1+k)\Theta(\infty;f) + \sum_{b\neq\infty}\delta(b;f) \leq 2 - k(q-1),$$

which is a contradiction. This proves the theorem. ∎

Remark 1. Gopalakrishna and Bhoosnurmath [2] proved Theorem 2 for functions of finite order by a different method using the notion of proximate order.

Now we discuss some consequences of Theorem 2.

Consequence 2.1. *For $k = 1$, $q = 3$ we see that if $\Theta(a;f) > 0$ for some $a \in \mathbb{C}\cup\{\infty\}$ then there exist at most three elements of $\mathbb{C}\cup\{\infty\}\backslash\{a\}$ which are evB for f for simple zeros.*

Hence it follows that if there exist distinct elements a_1, a_2, a_3, a_4 in $\mathbb{C}\cup\{\infty\}$ which are evB for f for simple zeros then $\Theta(a;f) = 0$ for all $a \in \mathbb{C}\cup\{\infty\}\backslash\{a_1,a_2,a_3,a_4\}$.

So in view of the consequence 1.1 it follows that if f has four distinct evB for simple zeros then f has no evN.

Consequence 2.2. *For $k = 1$, $q = 2$ we see that if*

$$2\Theta(a;f) + \sum_{b\neq a}\delta(b;f) > 1$$

for some $a \in \mathbb{C}\cup\{\infty\}$ then there exist at most two elements of $\mathbb{C}\cup\{\infty\}\backslash\{a\}$ which are evB for f for simple zeros. In particular, this holds if there exists an $a \in \mathbb{C}\cup\{\infty\}$ such that $\Theta(a;f) > 1/2$.

It, therefore, follows that if there exist distinct elements a_1, a_2, a_3, a_4 in $\mathbb{C}\cup\{\infty\}$ which are evB for f for simple zeros then $\Theta(a_i;f) \leq 1/2$ for $i = 1,2,3,4$.

Hence in view of Consequence 2.1 we see that if a_1, a_2, a_3, a_4 are evB for f for simple zeros then $\Theta(a;f) = 0$ if $a \neq a_i (i = 1,2,3,4)$ and $\Theta(a_i;f) \leq 1/2$ for $i = 1,2,3,4$.

We know that ∞, a_1, a_2, a_3 are four distinct evB for the Weierstrass elliptic function $\mathcal{P}(z)$ for simple zeros where a_1, a_2, a_3 are given by

$$\{\mathcal{P}'(z)\}^2 = (\mathcal{P}(z) - a_1)(\mathcal{P}(z) - a_2)(\mathcal{P}(z) - a_3).$$

In view of Lemma 5 [6] we see that $\Theta(\infty; \mathcal{P}) = \Theta(a_1; \mathcal{P}) = \Theta(a_2; \mathcal{P}) = \Theta(a_3; \mathcal{P}) = 1/2$. So the above estimation is sharp.

Consequence 2.3. *For $k = 2$, $q = 2$ we see that if $\Theta(a; f) > 0$ for some $a \in \mathbb{C} \cup \{\infty\}$ then f has at most two evB for distinct simple and double zeros. So if a_1, a_2, a_3 are three evB for f for distinct simple and double zeros then $\Theta(a; f) = 0$ for all $a \in \mathbb{C} \cup \{\infty\} \backslash \{a_1, a_2, a_3\}$.*

Hence in view of Consequence 1.5 it follows that if f has three evB for distinct simple and double zeros then f has no evN.

Consequence 2.4. *For $k = 2$, $q = 1$ we see that if*

$$3\Theta(a; f) + \sum_{b \neq a} \delta(b; f) > 2$$

for some $a \in \mathbb{C} \cup \{\infty\}$ then there exists at most one element of $a \in \mathbb{C} \cup \{\infty\} \backslash \{a\}$ which is an evB for f for distinct simple and double zeros. In particular, this holds if there exists an $a \in \mathbb{C} \cup \{\infty\}$ such that $\Theta(a; f) > 2/3$.

It, therefore , follows that if there exist distinct elements a_1, a_2, a_3 in $\mathbb{C} \cup \{\infty\}$ which are evB for f for distinct simple and double zeros then $\Theta(a_i; f) \leq 2/3$ for $i = 1, 2, 3$.

Hence in view of Consequence 2.3 we see that if a_1, a_2, a_3 are evB for f for distinct simple and double zeros then $\Theta(a; f) = 0$ if $a \neq a_i (i = 1, 2, 3)$ and $\Theta(a_i; f) \leq 2/3$ for $i = 1, 2, 3$.

The following example shows that the above estimation is sharp.

Example 1. (cf. [4, p. 45]) We set

$$z = \phi(z) = \int_0^w (t - a_1)^{-2/3} (t - a_2)^{-2/3} (t - a_3)^{-2/3} dt$$

where a_1, a_2, a_3 are distinct finite complex numbers. The inverse function can be analytically continued over the whole plane as a single valued meromorphic function by Schwarz's reflection principle and the resulting function $w = f(z)$ is doubly periodic. Also we can verify that a_1, a_2, a_3 are evB for f for distinct simple and double zeros and $\Theta(a_i; f) = 2/3$ for $i = 1, 2, 3$. Further we note that f has no evN.

Singh and Gopalakrishna [7] proved the following theorem.

Theorem B. *Let f be a meromorphic function of finite order. If a and ∞ are evB for f for distinct zeros for some $a \in \mathbb{C}$ then for each integer $k (\geq 1)$ $f^{(k)}$ has no evB for simple zeros in $\mathbb{C} \backslash \{a\}$.*

Improving Theorem B Gopalakrishna and Bhoosnurmath [3] proved the following result.

Theorem C. *Let f be a meromorphic function of finite hyperorder, i.e.,*

$$\limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r} < \infty.$$

If 0 and ∞ are evB for f for distinct zeros then any homogeneous differential polynomial P generated by f has no evB for simple zeros in $\mathbb{C} \cup \{\infty\} \backslash \{0, \infty\}$.

If P is generated by f' then the result holds if a and ∞ are evB for f for distinct zeros for some $a \in \mathbb{C}$.

We improve Theorem C by completely withdrawing the order restriction on f.

Theorem 3. *If 0 and ∞ are evB for f for distinct zeros then any homogeneous differential polynomial P generated by f has no evB for simple zeros in $\mathbb{C}\backslash\{0\}$.*

If P is generated by f' then the result holds if a and ∞ are evB for f for distinct zeros for some $a \in \mathbb{C}$.

Proof. First we note that under the hypothesis of the theorem the order of P is equal to the order of f (cf. [3 Theorem 1]). Let n be the degree of P.

Since 0 and ∞ are evB for distinct zeros for f, by Lemma 1 we get for all large values of r

$$\overline{N}_o(r, 0; f) < r^{\alpha} \quad \text{and} \quad \overline{N}_o(r, \infty; f) < r^{\alpha},$$

where $\alpha < \rho$.

Also for all large values of r we get

$$\begin{aligned} nT_o(r, f) &\le T_o(r, P) + N_o(r, \frac{P}{f^n}) + S_o(r, f) \\ &= T_o(r, P) + O(\overline{N}_o(r, 0; f) + \overline{N}_o(r, \infty; f)) + S_o(r, f) \\ &= O(r^{\alpha}) + T_o(r, P) + S_o(r, f), \end{aligned}$$

i.e.,

$$\{n + o(1)\}T_o(r, f) \le T_o(r, P) + O(r^{\alpha}). \tag{12}$$

We see that

$$\begin{aligned} \overline{N}(r, 0; P) &\le \overline{N}(r, \frac{f^n}{P}) + \overline{N}(r, 0; f) \\ &\le T(r, \frac{P}{f^n}) + \overline{N}(r, 0; f) + S(r, f) \\ &= N(r, \frac{P}{f^n}) + \overline{N}(r, 0; f) + S(r, f) \end{aligned}$$

and so

$$\begin{aligned}\overline{N}_o(r,0;P) &\le N_o(r,\frac{P}{f^n}) + \overline{N}_o(r,0;f) + S_o(r,f)\\ &= O(\overline{N}_o(r,0;f) + \overline{N}_o(r,\infty;f)) + S_o(r,f)\\ &= O(r^\alpha) + S_o(r,f).\end{aligned}$$

Now for $b \in \mathbb{C}\backslash\{0\}$ we get from Lemmas 2 and 3

$$\begin{aligned}T_o(r,P) &\le \overline{N}_o(r,0;P) + \overline{N}_o(r,\infty;P) + \overline{N}_o(r,b;P) + S_o(r,P)\\ &\le \overline{N}_o(r,0;P) + \overline{N}_o(r,\infty;f) + \frac{1}{2}\overline{N}_o(r,b;P\mid\le 1)\\ &+ \frac{1}{2}N_1^o(r,b;P) + S_o(r,P) + S_o(r,f)\\ &\le O(r^\alpha) + \frac{1}{2}\overline{N}_o(r,b;P\mid\le 1) + \frac{1}{2}T_o(r,P) + S_o(r,P) + S_o(r,f),\end{aligned}$$

i.e.,

$$\frac{1}{2}T_o(r,P) \le O(r^\alpha) + \frac{1}{2}\overline{N}_o(r,b;P\mid\le 1) + S_o(r,P) + S_o(r,f). \tag{13}$$

If possible, suppose that b is an evB for P for simple zeros. Then from (13) we get for all large values of r

$$\frac{1}{2}T_o(r,P) \le O(r^\alpha) + S_o(r,P) + S_o(r,f). \tag{14}$$

Since ρ is the order of P, by Lemma 1 it follows that for $\alpha < \beta < \rho$ there exists a sequence $\{r_n\}$ of values of r tending to infinity such that

$$T_o(r_n,P) > r_n^\beta \quad \text{for} \quad n = 1,2,3,\dots .$$

Hence for $r = r_n$ we get from (12) and (14)

$$\frac{1}{2}T_o(r_n;P) \le o\{T_o(r_n;P)\},$$

which is a contradiction. So no element of $\mathbb{C}\backslash\{0\}$ is an evB for P for simple zeros.

The second part of the theorem is obvious because if P is generated by f' then P is also generated by g' where $g = f - a$ for any $a \in \mathbb{C}$. This proves the theorem. ■

Remark 2. Theorem 3 does not hold for a non-homogeneous differential polynomial. For, let $f = \exp(z)$ and $P = f' + 1$. Then 0 and ∞ are evB for f for distinct zeros but 1 is an evB for P simple zeros.

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