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The Ext-Groups of Generalized Macaulay–Northcott Modules*

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Abstract. Let $(S, \leq)$ be a strictly totally ordered monoid which is finitely generated and artinian, R a left noetherian ring and M and N left R-modules. Then there exists an isomorphism of abelian groups:

$$Ext^i_{[[R^{S,\leq}]]}([M^{S,\leq}], [N^{S,\leq}]) \cong \prod_{s\in S} Ext^i_R(M, N).$$

1. Rings of Generalized Power Series

All rings considered here are associative with identity. Any concept and notation not defined here can be found in [1 - 3].

Let $(S, \leq)$ be an ordered set. Recall that $(S, \leq)$ is artinian if every strictly decreasing sequence of elements of S is finite, and that $(S, \leq)$ is narrow if every subset of pairwise order-incomparable elements of S is finite. Let S be a commutative monoid. Unless stated otherwise, the operation of S will be denoted additively, and the neutral element by 0. The following definition is due to [4].

Let $(S, \leq)$ be a strictly ordered monoid (that is, $(S, \leq)$ is an ordered monoid satisfying the condition that, if $s, s', t \in S$ and $s < s'$, then $s + t < s' + t$), and

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R a ring. Let $[[R^{S,\leq}]]$ be the set of all maps $f : S \to R$ such that $\text{supp}\,(f) = \{s \in S \,|\, f(s) \neq 0\}$ is artinian and narrow. With pointwise addition, and the operation of convolution

$$(fg)(s) = \sum_{(u,v)\in X_s(f,g)} f(u)g(v),$$

where $X_s(f,g) = \{(u,v) \in S \times S | s = u+v, f(u) \neq 0, g(v) \neq 0\}$ is a finite set by [4, 4.1] for every $s \in S$ and $f, g \in [[R^{S,\leq}]]$, $[[R^{S,\leq}]]$ becomes a ring, which is called the ring of generalized power series. The elements of $[[R^{S,\leq}]]$ are called generalized power series with coefficients in R and exponents in S. Many examples and results of rings of generalized power series are given in [1-7].

2. Modules of Generalized Power Series

Let M be a left R-module over a ring R and $(S, \leq)$ a strictly ordered monoid. Let $[[M^{S,\leq}]]$ be the set of all maps $\phi : S \to M$ such that $\text{supp}\,(\phi) = \{s \in S \,|\, \phi(s) \neq 0\}$ is artinian and narrow. With pointwise addition, $[[M^{S,\leq}]]$ is an abelian additive group. For each $f \in [[R^{S,\leq}]]$, each $\phi \in [[M^{S,\leq}]]$, and $s \in S$, denote

$$X_s(f,\phi) = \{(u,v) \in S \times S \,|\, s = u+v, f(u) \neq 0, \phi(v) \neq 0\}.$$

Then, by [8, Lemma 1], $X_s(f,\phi)$ is finite. Now $[[M^{S,\leq}]]$ is a left $[[R^{S,\leq}]]$-module with respect to the scalar multiplication defined by

$$(f\phi)(s) = \sum_{(u,v)\in X_s(f,\phi)} f(u)\phi(v)$$

for each $f \in [[R^{S,\leq}]]$ and each $\phi \in [[M^{S,\leq}]]$. $[[M^{S,\leq}]]$ is called the module of generalized power series over a left R-module M. The elements of $[[M^{S,\leq}]]$ are called generalized power series with coefficients in M and exponents in S. Examples and results of modules of generalized power series are given in [8].

Let M, N be left R-modules and $\alpha : M \to N$ be an R-homomorphism. Define a mapping $[[\alpha^{S,\leq}]] : [[M^{S,\leq}]] \longrightarrow [[N^{S,\leq}]]$ via

$$\begin{aligned} [[\alpha^{S,\leq}]](g) : S &\longrightarrow N, \\ s &\longrightarrow \alpha(g(s)), \end{aligned}$$

for any $g \in [[M^{S,\leq}]]$. Clearly $\text{supp}\,([[\alpha^{S,\leq}]](g)) \subseteq \text{supp}\,(g)$. Thus it follows that $\text{supp}\,([[\alpha^{S,\leq}]](g))$ is artinian and narrow. Hence $[[\alpha^{S,\leq}]](g) \in [[N^{S,\leq}]]$. This means that $[[\alpha^{S,\leq}]]$ is well-defined.

Lemma 2.1 *$[[\alpha^{S,\leq}]]$ is an $[[R^{S,\leq}]]$-homomorphism.*

Proof. For any $f \in [[R^{S,\leq}]]$, $g \in [[M^{S,\leq}]]$ and $s \in S$,

$$
\begin{aligned}
([[\alpha^{S,\leq}]](fg))(s) &= \alpha((fg)(s)) \\
&= \alpha\Big(\sum_{(u,v)\in X_s(f,g)} f(u)g(v)\Big) \\
&= \sum_{(u,v)\in X_s(f,g)} f(u)\alpha(g(v)) \\
&= \sum_{(u,v)\in X_s(f,g)} f(u)([[\alpha^{S,\leq}]](g))(v).
\end{aligned}
$$

Clearly

$$
\begin{aligned}
X_s(f,g) = &\big\{(u,v)|u+v=s, f(u)\neq 0, ([[\alpha^{S,\leq}]](g))(v)\neq 0\big\} \\
&\cup \big\{(u,v)|u+v=s, f(u)\neq 0, ([[\alpha^{S,\leq}]](g))(v)=0, g(v)\neq 0\big\}.
\end{aligned}
$$

Denote $X_1 = \{(u,v)|u+v=s, f(u)\neq 0, ([[\alpha^{S,\leq}]](g))(v)\neq 0\}$. Then

$$
([[\alpha^{S,\leq}]](fg))(s) = \sum_{(u,v)\in X_1} f(u)([[\alpha^{S,\leq}]](g))(v) = (f[[\alpha^{S,\leq}]](g))(s).
$$

Thus $[[\alpha^{S,\leq}]](fg) = f[[\alpha^{S,\leq}]](g)$. Now it is easy to see that $[[\alpha^{S,\leq}]]$ is an $[[R^{S,\leq}]]$ -homomorphism. ■

Lemma 2.2. *If* $M \xrightarrow{\alpha} N \xrightarrow{\beta} L$ *is a complex, then so is*

$$
[[M^{S,\leq}]] \xrightarrow{[[\alpha^{S,\leq}]]} [[N^{S,\leq}]] \xrightarrow{[[\beta^{S,\leq}]]} [[L^{S,\leq}]].
$$

Proof. For any $g \in [[M^{S,\leq}]]$ and any $s \in S$,

$$
\begin{aligned}
(([[\beta^{S,\leq}]][[\alpha^{S,\leq}]])(g))(s) &= \beta(([[\alpha^{S,\leq}]](g))(s)) \\
&= \beta(\alpha(g(s))) = (\beta\alpha)(g(s)) = 0.
\end{aligned}
$$

Lemma 2.3. *The functor* $[[(-)^{S,\leq}]] : R\text{-Mod} \longrightarrow [[R^{S,\leq}]]\text{-Mod}$ *is exact.*

Proof. Let $M \xrightarrow{\alpha} N \xrightarrow{\beta} L$ be an exact sequence of left R-modules. Suppose that $g \in [[N^{S,\leq}]]$ is such that $[[\beta^{S,\leq}]](g) = 0$. Then for any $s \in S$, $\beta(g(s)) = [[\beta^{S,\leq}]](g)(s) = 0$. Thus $g(s) \in \operatorname{Ker}(\beta)$ and, so there exists $m_s \in M$ such that $g(s) = \alpha(m_s)$. Define a mapping $h : S \to N$ via

$$
h(s) = \begin{cases} m_s, & s \in \operatorname{supp}(g), \\ 0, & s \notin \operatorname{supp}(g). \end{cases}
$$

Clearly $h \in [[M^{S,\leq}]]$. Now

$$
([[\alpha^{S,\leq}]](h))(s) = \alpha(h(s)) = \begin{cases} \alpha(m_s) = g(s), & s \in \operatorname{supp}(g), \\ 0 = g(s), & s \notin \operatorname{supp}(g).0 \end{cases}
$$

Thus $g \in \operatorname{Im}([[\alpha^{S,\leq}]])$. Now the result follows from Lemma 2.2. ■

Lemma 2.4. *Let $N \leq M$ be left R-modules. Then $[[M^{S,\leq}]]/[[N^{S,\leq}]] \cong [[M/N^{S,\leq}]]$ as left $[[R^{S,\leq}]]$-modules.*

Proof. It follows from Lemma 2.3. ∎

3. Generalized Macaulay–Northcott Modules

If M is a left R-module, we let $[M^{S,\leq}]$ be the set of all maps $\phi : S \to M$ such that the set $\mathrm{supp}\,(\phi) = \{s \in S \mid \phi(s) \neq 0\}$ is finite. Now $[M^{S,\leq}]$ can be turned into a left $[[R^{S,\leq}]]$-module under some additional conditions. The addition in $[M^{S,\leq}]$ is componentwise and the scalar multiplication is defined as follows

$$(f\phi)(s) = \sum_{t \in S} f(t)\phi(s+t), \quad \text{for every } s \in S,$$

where $f \in [[R^{S,\leq}]]$, and $\phi \in [M^{S,\leq}]$. Since the set $\mathrm{supp}\,(\phi)$ is finite, this multiplication is well-defined. If $(S, \leq)$ is a strictly totally ordered monoid which is also artinian, then, by [6], $[M^{S,\leq}]$ becomes a left $[[R^{S,\leq}]]$-module, which we called the generalized Macaulay–Northcott module.

For example, if $S = \mathbb{N}$ and $\leq$ is the usual order, then $[M^{\mathbb{N},\leq}] \cong M[x^{-1}]$, the usual left $R[[x]]$-module introduced in [10, 11], which is called the Macaulay–Northcott module in [12, 13].

We shall henceforth assume that $(S, \leq)$ is a strictly totally ordered monoid which is also artinian. Then it is easy to see that $(S, \leq)$ satisfies the condition that $0 \leq s$ for every $s \in S$ [9].

For any abelian additive group G, we denote by $[[G^{S,\leq}]]$ the set of all maps $h : S \longrightarrow G$. With pointwise addition, $[[G^{S,\leq}]]$ is an abelian additive group.

Let M be a left R-module. For any $t \in S$ and $m \in M$, define $\phi_{tm} \in [M^{S,\leq}]$ as follows

$$\phi_{tm}(x) = \begin{cases} m, & x = t \\ 0, & x \neq t. \end{cases}$$

Denote $G_t = \{\phi_{tm} \mid m \in M\}$. Clearly there exists an isomorphism of left R-modules $\lambda_t : M \to G_t$ via $\lambda_t(m) = \phi_{tm}$.

Let E be a left R-module. Define a mapping $\delta : [E^{S,\leq}] \to E$ by $\delta(\phi) = \phi(0)$. It is easy to see that δ is an R-homomorphism. Now we define

$$F_M : \mathrm{Hom}_{[[R^{S,\leq}]]}([M^{S,\leq}], [E^{S,\leq}]) \longrightarrow [[\mathrm{Hom}_R(M, E)^{S,\leq}]]$$

via $F_M(h) : S \to \mathrm{Hom}_R(M, E)$ as

$$F_M(h)(s) = \delta h \lambda_s,$$

for any $h \in \mathrm{Hom}_{[[R^{S,\leq}]]}([M^{S,\leq}], [E^{S,\leq}])$. The following result appeared in [6, Lemma 2.3].

Lemma 3.1. *Let M, E be left R-modules. Then*

$$F_M : \mathrm{Hom}_{[[R^{S,\leq}]]}([M^{S,\leq}], [E^{S,\leq}]) \longrightarrow [[\mathrm{Hom}_R(M, E)^{S,\leq}]]$$

defined as above is an isomorphism of abelian groups.

For any R-homomorphism $\alpha : M \to N$, define $f \in [[\mathrm{Hom}_R(M,N)^{S,\leq}]]$ via $f(0) = \alpha$ and $f(x) = 0$ for all $0 \neq x \in S$. By Lemma 3.1 and its proof, there exists $[\alpha^{S,\leq}] \in \mathrm{Hom}_{[[R^{S,\leq}]]}([M^{S,\leq}],[N^{S,\leq}])$ such that for any $\phi \in [M^{S,\leq}]$ and any $s \in S$,

$$[\alpha^{S,\leq}](\phi)(s) = \sum_{u\in S} f(u)(\phi(s+u)) = \alpha(\phi(s)).$$

Lemma 3.2. *The functor* $[(-)^{S,\leq}] : R\text{-Mod} \longrightarrow [[R^{S,\leq}]]\text{-Mod}$ *defined as* $[(-)^{S,\leq}](M) = [M^{S,\leq}]$, $[(-)^{S,\leq}](\alpha) = [\alpha^{S,\leq}]$*, is exact.*

Proof. It follows from [13, Lemma 5]. ■

Let $\eta : N \to M$ be an R-homomorphism. Denote

$$[\eta^{S,\leq}](*) = \mathrm{Hom}_{[[R^{S,\leq}]]}([\eta^{S,\leq}],[E^{S,\leq}]).$$

For any $h \in \mathrm{Hom}_{[[R^{S,\leq}]]}([M^{S,\leq}],[E^{S,\leq}])$, and any $s \in S$, we have

$$\begin{aligned}(F_N[\eta^{S,\leq}](*))(h)(s) &= (F_N[\eta^{S,\leq}](*)(h))(s)\\ &= (F_N(h[\eta^{S,\leq}]))(s) = \delta(h[\eta^{S,\leq}])\lambda_s\\ &= (\delta h)[\eta^{S,\leq}]\lambda_s\end{aligned}$$

and

$$\begin{aligned}([[\eta(*)^{S,\leq}]]F_M)(h)(s) &= ([[\eta(*)^{S,\leq}]](F_M(h)))(s)\\ &= \eta(*)(F_M(h)(s))\\ &= \eta(*)(\delta h\lambda_s) = \delta h\lambda_s\eta,\end{aligned}$$

where $\eta(*) = \mathrm{Hom}_R(\eta, E)$. Thus for any $n \in N$, we have

$$(F_N[\eta^{S,\leq}](*))(h)(s)(n) = ((\delta h)[\eta^{S,\leq}]\lambda_s)(n) = ((\delta h)[\eta^{S,\leq}])(\phi_{sn}),$$

and

$$([[\eta(*)^{S,\leq}]]F_M)(h)(s)(n) = \delta h\lambda_s(\eta(n)) = \delta h(\phi_{s,\eta(n)}).$$

Since

$$\begin{aligned}[\eta^{S,\leq}](\phi_{sn})(t) &= \eta(\phi_{sn}(t))\\ &= \begin{cases}\eta(n), & t = s,\\ \eta(0) = 0, & t \neq s\end{cases}\\ &= \phi_{s,\eta(n)}(t),\end{aligned}$$

thus $[\eta^{S,\leq}](\phi_{sn}) = \phi_{s,\eta(n)}$. Hence $F_N[\eta^{S,\leq}](*) = [[\eta(*)^{S,\leq}]]F_M$. This means that the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{[[R^{S,\leq}]]}([M^{S,\leq}],[E^{S,\leq}]) & \xrightarrow{F_M} & [[\mathrm{Hom}_R(M,E)^{S,\leq}]] \\ {\scriptstyle [\eta^{S,\leq}](*)}\downarrow & & \downarrow{\scriptstyle [[\eta(*)^{S,\leq}]]} \\ \mathrm{Hom}_{[[R^{S,\leq}]]}([N^{S,\leq}],[E^{S,\leq}]) & \xrightarrow{F_N} & [[\mathrm{Hom}_R(N,E)^{S,\leq}]] \end{array}$$

commutes.

If $E \to E'$ is an R-homomorphism, then by analogy with above proof, we can get a corresponding commutative diagram. Thus we have showed the following result:

Lemma 3.3. *Let M, E be left R-modules. Then there exists a natural isomorphism of abelian groups:*

$$\mathrm{Hom}_{[[R^{S,\leq}]]}([M^{S,\leq}],[E^{S,\leq}]) \cong [[\mathrm{Hom}_R(M,E)^{S,\leq}]].$$

The following result appeared in [9, Theorem 6].

Lemma 3.4. *Let S be a finitely generated monoid, R a left noetherian ring and M a left R-module. Then $[M^{S,\leq}]$ is an injective left $[[R^{S,\leq}]]$-module if and only if M is an injective left R-module.*

4. Ext-Groups

Theorem 4.1. *Let S be a finitely generated monoid, R a left noetherian ring and M and N be left R-modules. Then there exists an isomorphism of abelian groups:*

$$\mathrm{Ext}^i_{[[R^{S,\leq}]]}([M^{S,\leq}],[N^{S,\leq}]) \cong \prod_S \mathrm{Ext}^i_R(M,N).$$

Proof. Let $0 \longrightarrow N \longrightarrow E_0 \xrightarrow{\delta_0} E_1 \xrightarrow{\delta_1} E_2 \xrightarrow{\delta_2} \cdots$ be an injective resolution of left R-module N. Then

$$0 \longrightarrow [N^{S,\leq}] \longrightarrow [E_0^{S,\leq}] \longrightarrow [E_1^{S,\leq}] \longrightarrow [E_2^{S,\leq}] \longrightarrow \cdots$$

is an injective resolution of left $[[R^{S,\leq}]]$-module $[N^{S,\leq}]$ by Lemmas 3.4 and 3.2. Consider the deleted injective resolution

$$0 \longrightarrow [E_0^{S,\leq}] \xrightarrow{[\delta_0^{S,\leq}]} [E_1^{S,\leq}] \xrightarrow{[\delta_1^{S,\leq}]} [E_2^{S,\leq}] \xrightarrow{[\delta_2^{S,\leq}]} \cdots,$$

We have the complex

$$0 \longrightarrow \mathrm{Hom}_{[[R^{S,\leq}]]}([M^{S,\leq}],[E_0^{S,\leq}]) \xrightarrow{[\delta_0^{S,\leq}](*)} \mathrm{Hom}_{[[R^{S,\leq}]]}([M^{S,\leq}],[E_1^{S,\leq}])$$
$$\xrightarrow{[\delta_1^{S,\leq}](*)} Hom_{[[R^{S,\leq}]]}([M^{S,\leq}],[E_2^{S,\leq}]) \xrightarrow{[\delta_2^{S,\leq}](*)} \cdots,$$

where $[\delta_i^{S,\leq}](*) = \mathrm{Hom}_{[[R^{S,\leq}]]}([M^{S,\leq}],[\delta_i^{S,\leq}])$ for every $i = 0, 1, \ldots$. On the other hand, we have the complex

$$0 \longrightarrow Hom_R(M, E_0) \xrightarrow{\delta_0(*)} \mathrm{Hom}_R(M, E_1) \xrightarrow{\delta_1(*)} \mathrm{Hom}_R(M, E_2) \xrightarrow{\delta_2(*)} \cdots,$$

where $\delta_i(*) = \mathrm{Hom}_R(M, \delta_i)$ for every $i = 0, 1, \cdots$. Thus, by Lemma 2.2, we have the complex

$$\begin{aligned} 0 \longrightarrow [[\mathrm{Hom}_R(M, E_0)^{S,\leq}]] &\xrightarrow{[[\delta_0(*)^{S,\leq}]]} [[\mathrm{Hom}_R(M, E_1)^{S,\leq}]] \\ &\xrightarrow{[[\delta_1(*)^{S,\leq}]]} [[\mathrm{Hom}_R(M, E_2)^{S,\leq}]] \xrightarrow{[[\delta_2(*)^{S,\leq}]]} \cdots. \end{aligned}$$

By Lemma 3.3, there exists a natural isomorphism

$$\mathrm{Hom}_{[[R^{S,\leq}]]}([M^{S,\leq}], [E^{S,\leq}]) \cong [[\mathrm{Hom}_R(M, E)^{S,\leq}]].$$

Thus, by Lemmas 2.4, 2.3 and 3.3, we have

$$\begin{aligned} \mathrm{Ext}^i_{[[R^{S,\leq}]]}([M^{S,\leq}], [N^{S,\leq}]) &= \mathrm{Ker}\,([\delta_i^{S,\leq}](*))/\mathrm{Im}\,([\delta_{i-1}^{S,\leq}](*)) \\ &\cong \mathrm{Ker}\,([[\delta_i(*)^{S,\leq}]])/\mathrm{Im}\,([[\delta_{i-1}(*)^{S,\leq}]]) \\ &\cong [[\mathrm{Ker}\,(\delta_i(*))^{S,\leq}]]/[[\mathrm{Im}\,(\delta_{i-1}(*))^{S,\leq}]] \\ &\cong [[(\mathrm{Ker}\,(\delta_i(*))/\mathrm{Im}\,(\delta_{i-1}(*)))^{S,\leq}]] \\ &= [[Ext^i_R(M, N)^{S,\leq}]] \\ &\cong \prod_{i \in S} Ext^i_R(M, N). \end{aligned}$$

Corollary 4.2. *If R is a left noetherian ring and M and N are left R-modules, then there exists an isomorphism of abelian groups*

$$\mathrm{Ext}^i_{R[[x]]}(M[x^{-1}], N[x^{-1}]) \cong \prod_{i=0}^{\infty} \mathrm{Ext}^i_R(M, N) \cong \mathrm{Ext}^i_R(M, N)[[x]].$$

Corollary 4.3. *Let S be a finitely generated torsion-free and cancellative monoid, and $(S, \leq)$ be artinian and narrow. If R is a left noetherian ring and M and N are left R-modules, then*

$$\mathrm{Ext}^i_{[[R^{S,\leq}]]}([M^{S,\leq}], [N^{S,\leq}]) \cong \prod_S \mathrm{Ext}^i_R(M, N).$$

Proof. If $(S, \leq)$ is torsion-free and cancellative, then by [1, 3.3], there exists a compatible strict total order $\leq'$ on S, which is finer than $\leq$, that is, for any $s, t \in S$, $s \leq t$ implies $s \leq' t$. Since $(S, \leq)$ is artinian and narrow, by [1, 2.5] it follows that $(S, \leq')$ is artinian and narrow. Thus, by Theorem 4.1, $\mathrm{Ext}^i_{[[R^{S,\leq'}]]}([M^{S,\leq'}], [N^{S,\leq'}]) \cong \prod_S \mathrm{Ext}^i_R(M, N)$.

On the other hand, since $(S, \leq)$ is narrow, by [1, 4.4], $[[R^{S,\leq}]] = [[R^{S,\leq'}]]$. Clearly $[M^{S,\leq}] = [M^{S,\leq'}]$ and $[N^{S,\leq}] = [N^{S,\leq'}]$. Now the result follows. ∎

Any submonoid of the additive monoid $\mathbb{N} \cup \{0\}$ is called a numerical monoid. It is well-known that any numerical monoid is finitely generated (see [1, 1.3]). Thus we have

Corollary 4.4. *Let S be a numerical monoid and $\leq$ the usual natural order of $\mathbb{N}\cup\{0\}$. If R is a left noetherian ring and, M and N are left R-modules, then*

$$\mathrm{Ext}^i_{[[R^{S,\leq}]]}([M^{S,\leq}],[N^{S,\leq}]) \cong \prod_S \mathrm{Ext}^i_R(M,N).$$

Corollary 4.5. *Suppose that $(S_1,\leq_1),\ldots,(S_n,\leq_n)$ are strictly totally ordered monoids which are finitely generated and artinian. Denote by $(lex \leq)$ and $(revlex \leq)$ the lexicographic order, the reverse lexicographic order, respectively, on the monoid $S_1\times\cdots\times S_n$. If R is a noetherian ring and M,N are left R-modules, then there exist isomorphisms of abelian groups*

$$\begin{aligned}
&\mathrm{Ext}^i_{[[R^{S_1\times\cdots\times S_n,(lex\leq)}]]}([M^{S_1\times\cdots\times S_n,(lex\leq)}],[N^{S_1\times\cdots\times S_n,(lex\leq)}])\\
&\cong \mathrm{Ext}^i_{[[R^{S_1\times\cdots\times S_n,(revlex\leq)}]]}([M^{S_1\times\cdots\times S_n,(revlex\leq)}],[N^{S_1\times\cdots\times S_n,(revlex\leq)}])\\
&\cong \prod_{S_1\times\cdots\times S_n} \mathrm{Ext}^i_R(M,N).
\end{aligned}$$

Proof. It is easy to see that $(S_1\times\cdots\times S_n,(lex \leq))$ and $(S_1\times\cdots\times S_n,(revlex \leq))$ are strictly totally ordered monoids which are finitely generated and artinian. Thus the result follows from Theorem 4.1. ∎

Let $p_1,\ldots,p_n$ be prime numbers. Set

$$N(p_1,\ldots,p_n)=\{p_1^{m_1}p_2^{m_2}\ldots p_n^{m_n} | m_1,m_2,\ldots,m_n\in\mathbb{N}\cup\{0\}\}.$$

Then $N(p_1,\ldots,p_n)$ is a submonoid of $(\mathbb{N},\cdot)$. Let $\leq$ be the usual natural order.

Corollary 4.6. *Let R be a left noetherian ring and M and N be left R-modules. Then*

$$\mathrm{Ext}^i_{[[R^{N(p_1,\ldots,p_n),\leq}]]}([M^{N(p_1,\ldots,p_n),\leq}],[N^{N(p_1,\ldots,p_n),\leq}]) \cong \prod_{i=0}^{\infty} \mathrm{Ext}^i_R(M,N).$$

References

1. P. Ribenboim, Noetherian rings of generalized power series, *J. Pure Appl. Algebra* **79** (1992) 293–312.
2. P. Ribenboim, Rings of generalized power series II: Units and zero-divisors, *J. Algebra* **168** (1994) 71–89.
3. P. Ribenboim, Special properties of generalized power series, *J. Algebra* **173** (1995) 566–586.
4. P. Ribenboim, Semisimple rings and von Neumann regular rings of generalized power series, *J. Algebra* **198** (1997) 327–338.
5. K. Varadarajan, Noetherian generalized power series rings and modules, *Comm. Algebra* **29** (2001) 245–251.
6. Z. Liu and H. Cheng, Quasi-duality for the rings of generalized power series *Comm. Algebra* **28**(2000) 1175–1188.

7. Z. Liu, Morita duality for the rings of generalized power series *Acta Mathematica Sinica, English Series* **18** (2002) 245–252.
8. Z. Liu, A Note on Hopfian Modules, *Comm. Algebra* **28** (2000) 3031–3040.
9. Z. Liu, Injectivity of modules of generalized inverse polynomials, *Comm. Algebra* **29** (2001) 583–592.
10. A. S. McKerrow, On the injective dimension of modules of power series, *Quart. J. Math. Oxford* **25** (1974) 359–368.
11. D. G. Northcott, Injective envelopes and inverse polynomials, *London Math. Soc.* **3** (1974) 290–296.
12. S. Park, The Macaulay-Northcott functor, *Arch. Math.* **63** (1994) 225–230.
13. S. Park, Inverse polynomials and injective covers, *Comm. Algebra* **21**(1993) 4599–4613.