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A Geometrical Approach to the Linear Complementarity Problem

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Abstract. In this paper we give some necessary conditions for the continuity of the solution map in linear complementarity problems by considering the geometrical feature of its effective domain. At first, we provide a criterion for a face of a complementarity cone to lie on the boundary of the domain. Basing on this result, we give alternative proofs and, at the same time, extend some well-known necessary conditions for continuity. Especially, we shall prove that the number of solutions to the problem is a constant for each interior point of the domain.

1. Introduction

For a given matrix $M \in \mathbb{R}^{n\times n}$ and a given vector $q \in \mathbb{R}^n$, the linear complementarity problem $\mathrm{LCP}(M, q)$ is that of finding $x \in \mathbb{R}^n$ such that

$$x \geq 0, \quad Mx + q \geq 0 \quad \text{and} \quad \langle x, Mx + q\rangle = 0. \tag{1.1}$$

Let $S_M(q)$ denote the set of all solutions to (1.1). Studying various types of continuous dependence of solution set on M and q forms an interesting research area in the theory of linear complementarity problems. Many results have been achieved in the area (see for example, the book by Cottle, Pang and Stone [1], the papers by Robinson [14-15], Ha [5], Jansen and Tijs [6], Mangasarian and Shian [7], Gowda [2-3], Gowda and Pang [4], Oettli and Yen [12], Murthy, Parthasarathy and Sabatini [8-9]...) The aim of this paper is to give some necessary conditions for the continuity of S_M by considering the geometrical feature of its domain and the number of solutions to the $\mathrm{LCP}(M, q)$. Specifically, in Sec. 3 we develop a characterization for a complementarity face to lie on the boundary of DomS_M in the case where S_M is Lipschitzian. Basing on this result we derive some important necessary conditions for continuity, and especially, in Sec. 4 we shall prove that, if S_M is Lipschitzian, then the cardinalities of solution sets at all interior points of DomS_M are the same.

2. Preliminaries and Notations

From now on, we denote by M the real square matrix with entries m_{ij}, $i,j \in I := \{1,2,\dots,n\}$. The set of all $q \in \mathbb{R}^n$ for which the LCP(M,q) has a solution is denoted by $K(M)$. So $K(M)$ is the effective domain of the set-valued map S_M. M is called a Lipschitzian matrix if S_M is Lipschitzian on $K(M)$. In fact, we have shown in [13] that M is Lipschitzian if and only if S_M is lower semicontinuous. For each $\alpha \subseteq I$ we denote by $M[\alpha]$ the $n \times n-$matrix, whose jth column vector is defined as follows

$$M[\alpha]^j := \begin{cases} -M^j & \text{if } j \in \alpha, \\ E^j & \text{if } j \in I \setminus \alpha, \end{cases} \tag{2.1}$$

where M^j and E^j respectively denote the jth column vector of M and E, the unit matrix of order n. If α is nonempty, M_α will stand for the submatrix of M obtained by omitting the rows and the columns corresponding to indices not belong to α. The determinants of these matrices are called principal minors of M. It is easy to verify that

$$\det(M_\alpha) = (-1)^{|\alpha|}.\det(M[\alpha]) \tag{2.2}$$

for every nonempty subset $\alpha \subseteq I$.

M is said to be a P-matrix (resp. N-matrix, nondegenerate matrix) and denoted by $M \in \mathcal{P}$, $M \in \mathcal{N}$, $M \in \mathcal{N}^d$ if all its principal minors are positive (resp. negative, nonzero) (see for instance [1]). M is called an almost N-matrix if it has positive determinant and all its proper principal minors are negative (see [3]). We also say that M is an N_2-matrix and write $M \in \mathcal{N}_2$ if it has negative diagonal entries and $m_{ii}m_{jj} - m_{ij}m_{ji} < 0$ for every $i,j \in I$, $i \neq j$. Thus, $\mathcal{N} \subseteq \mathcal{N}_2$ and an almost N-matrix whose order is greater than 2 must belong to $\mathcal{N}_2 \setminus \mathcal{N}$. It is easy to check that $M \in \mathcal{P}$ $(M \in \mathcal{N})$ if and only if $m_{rr} > 0$ $(m_{rr} < 0)$ for every $r \in I$ and

$$\det(M_{\alpha\setminus\{r\}}).\det(M_{\alpha\cup\{r\}}) > 0 \tag{2.3}$$

whenever $\alpha \setminus \{r\} \neq \emptyset$. This together with (2.2) yields the following result

Lemma 2.1.

a) *$M \in \mathcal{P}$ if and only if for every $r \in I$ and $\alpha \subseteq I$ the following inequality holds:*

$$\det(M[\alpha \setminus \{r\}]).\det(M[\alpha \cup \{r\}]) < 0. \tag{2.4}$$

b) *$M \in \mathcal{N}$ if and only if $m_{rr} < 0$ for every $r \in I$ and* (2.4) *holds whenever $\alpha \setminus \{r\} \neq \emptyset$.*

It is well known that if M is a P-matrix or a negative N-matrix, then it is Lipschitzian [2, 7] and if M is Lipschitzian, then it is nondegenerate [10, 13]. Besides, it has been shown in [3, 9] that an almost N-matrix can not be Lipschitzian.

The cone generated by the columns A^i, $i \in \alpha$, of a matrix A is denoted by $\mathrm{Pos}\{A^\alpha\}$, i.e. $\mathrm{Pos}\{A^\alpha\} = \{\sum_{i\in\alpha} \lambda_i A^i \mid \lambda_i \geq 0\}$. For abbreviation, we write $\mathrm{Pos}\{A\}$ instead of $\mathrm{Pos}\{A^I\}$. As each $\alpha \subseteq I$, $\mathrm{Pos}\{M[\alpha]\}$ is called the complementarity cone corresponding to α and denoted briefly by K_α. The LCP(M, q) has a solution if and only if q belongs to a certain complementarity cone. Thus, $K(M)$ is the union of all complementarity cones (see [11]).

Now assume that M is nondegenerate. For each $\alpha \subseteq I$, $M[\alpha]$ is nonsingular and the complementarity cone K_α is a polyhedral convex cone, having nonempty interior. For every $q \in \mathbb{R}^n$, by setting $\lambda := M[\alpha]^{-1}q$, we have $q = M[\alpha]\lambda$ and $q \in K_\alpha$ if and only if $\lambda \geq 0$. Also, $q \in \mathrm{Int}(K_\alpha)$ if and only if $\lambda > 0$. By π_α, F_α we denote the mappings of $\mathbb{R}^n$ into $\mathbb{R}^n$ defined by

$$\forall u \in \mathbb{R}^n,\ u \to \pi_\alpha(u) \in \mathbb{R}^n, \ \text{ where } \pi_\alpha(u)_i := \begin{cases} u_i & i \in \alpha, \\ 0 & i \notin \alpha; \end{cases}$$
$$\forall q \in \mathbb{R}^n \to F_\alpha(q) \in \mathbb{R}^n, \ \text{ where } F_\alpha(q) := \pi_\alpha(M[\alpha]^{-1}q).$$

If $q \in K_\alpha$ then $F_\alpha(q) \in S_M(q)$. In fact, for each $q \in K(M)$ one has

$$S_M(q) = \{F_\alpha(q) \mid \alpha \in J(q)\},\ \forall q \in K(M), \tag{2.5}$$

where

$$J(q) := \{\alpha \subseteq I \mid q \in K_\alpha\}. \tag{2.6}$$

Remark 1. It is evident that if M is nondegenerate, then $|S_M(q)| \leq |J(q)| \leq 2^n$ for all $q \in \mathbb{R}^n$. Besides, if p, $q \in \mathbb{R}^n$ such that
(i) $J(p) \subseteq J(q)$,
(ii) $\forall \alpha,\ \beta \in J(p),\ F_\alpha(p)) \neq F_\beta(p) \Rightarrow F_\alpha(q) \neq F_\beta(q)$,
then $|S_M(p)| \leq |S_M(q)|$.

For each $r \in I$ and $\alpha \subseteq I$ we set $H(\alpha, r) := \mathrm{Pos}\{M[\alpha]^{I\setminus\{r\}}\}$. Since $M[\alpha]$ is nonsingular, $H(\alpha, r)$ is a polyhedral convex cone of dimension $(n-1)$ that will be called a *complementarity face*. Clearly, $H(\alpha, r)$ is the common face of two polyhedral convex cones $K_{\alpha\setminus\{r\}}$ and $K_{\alpha\cup\{r\}}$. This face could either belong to the boundary of $K(M)$ or intersect its interior. In the former case, $H(\alpha, r)$ is called a *boundary face* of $K(M)$. The next section will provide a necessary and sufficient condition for a *complementarity face* to be a *boundary face* in case M is Lipschitzian.

Remark 2. If α, $\beta \subseteq I$ such that $(\alpha \cup \beta) \setminus (\alpha \cap \beta) = \{r\}$ then $H(\alpha; r) = H(\beta; r) \subseteq K_\alpha \cap K_\beta$. In such a case, K_α and K_β will be called adjacent. Also, it could be verified that for any α, $\beta \subseteq I$ there exists a sequence of subsets: $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_k = \beta$ such that $K_{\alpha_{i-1}}$ and K_{α_i} are adjacent for all $i \in \{1, \ldots, k\}$.

3. A Characterization for Boundary Faces

The following theorem provides a criterion for a complementarity face to lie on the boundary of $K(M)$.

Theorem 3.1. *Suppose M is Lipschitzian, $r \in I$ and $\alpha \subseteq I$. Then the following conditions are equivalent:*
(a) $H(\alpha, r) \subseteq \partial K(M)$;
(b) $\det(M[\alpha \setminus \{r\}]).\det(M[\alpha \cup \{r\}]) > 0$.

Proof. First we have two remarks:
(i) By considering $\alpha \setminus \{r\}$ instead of α, if necessary, we can assume that $r \notin \alpha$. Then $\alpha \setminus \{r\} = \alpha$.
(ii) Since M is Lipschitzian, $M[\alpha]$ is nonsingular. Setting $t := M[\alpha]^{-1}(-M^r)$ one has

$$-M^r = M[\alpha]t = \sum_{i\in\alpha} t_i(-M^i) + t_r E^r + \sum_{j\notin\alpha\cup\{r\}} t_j E^j. \tag{3.1}$$

Substituting the right-hand side of (3.1) into the $r-$th column of $M[\alpha \cup \{r\}]$ we obtain

$$\det(M[\alpha \cup \{r\}]) = t_r \det(M[\alpha]) = t_r \det(M[\alpha \setminus \{r\}]). \tag{3.2}$$

Since $M[\alpha \cup \{r\}]$ is nonsingular, t_r is nonzero, and (b) now is equivalent to the inequality

$$t_r = \big(M[\alpha]^{-1}(-M^r)\big)_r > 0. \tag{3.3}$$

(a) $\Rightarrow$ (b). By Remark (ii) it suffices to show that $t_r > 0$. Set

$$\bar{q} := \sum_{i\in\alpha}(-M^i) + \sum_{j\notin\alpha\cup\{r\}} E^j = M[\alpha]\bar{s}, \tag{3.4}$$

where $\bar{s} \in \mathbb{R}^n$ with $\bar{s}_i$ being 0 if $i = r$ and 1 otherwise. As $\bar{q} \in H(\alpha, r) \subseteq \partial K(M)$ there is a sequence $(q^m)_{m\in N}$ converging to $\bar{q}$ with $q^m \notin K(M)$ for all m. Since $q^m \to \bar{q}$, we have $s^m := M[\alpha]^{-1}q^m \to M[\alpha]^{-1}\bar{q} = \bar{s}$. Therefore

$$\lim_{m\to\infty} s_i^m = 1;\ i \neq r, \tag{3.5}$$

$$\lim_{m\to\infty} s_r^m = 0. \tag{3.6}$$

Thus, without loss of generality we can assume that $s_i^m > 0$ for all $i \neq r$. If $s_r^m \geq 0$ then $s^m \geq 0$ and $q^m = M[\alpha]s^m \in K_\alpha \subseteq K(M)$, which contradicts the establishment of q^m. So

$$s_r^m < 0. \tag{3.7}$$

Taking account of (3.1), q^m could be rewritten as follows

$$\begin{aligned} q^m &= q^m + \frac{s_r^m}{t_r}M^r - \frac{s_r^m}{t_r}M^r \\ &= \sum_{i\in\alpha}(s_i^m - \frac{s_r^m}{t_r}t_i)(-M^i) + \sum_{j\notin\alpha\cup\{r\}}(s_j^m - \frac{s_r^m}{t_r}t_j)E^j + \frac{s_r^m}{t_r}(-M^r). \end{aligned}$$

By (3.5)–(3.6), with m sufficiently large we have $s_i^m - \frac{s_r^m}{t_r} t_i > 0$ for all $i \neq r$. Since $q^m \notin K(M)$ so that $q^m \notin K_{\alpha\cup\{r\}}$, it follows that $\frac{s_r^m}{t_r} < 0$, which together with (3.7) implies that $t_r > 0$.

(b) $\Rightarrow$ (a). We shall prove that $t_r < 0$ provided $H(\alpha, r) \cap \mathrm{Int}K(M) \neq \emptyset$. Since $H(\alpha, r)$ is convex, there exists $\bar{q} \in \mathrm{ri}H(\alpha, r) \cap \mathrm{Int}K(M)$, i.e.

$$\bar{q} = M[\alpha]\lambda = \sum_{i\in\alpha} \lambda_i(-M^i) + \sum_{j\notin\alpha\cup\{r\}} \lambda_j E^j, \tag{3.8}$$

where $\lambda \in \mathbb{R}^n$ with $\lambda_i > 0$ for all $i \neq r$ and $\lambda_r = 0$. For each $m \in N$, we set $q^m := \bar{q} - \frac{1}{m}E^r$. Clearly, (q^m) converges to $\bar{q}$ as m tends to $+\infty$. Since $\bar{q} \in \mathrm{Int}K(M)$, we may assume that $q^m \in \mathrm{Int}K(M)$ for all $m \in N$. Set

$$\bar{x} = F_\alpha(\bar{q}) = \pi_\alpha(\lambda) \in S_M(\bar{q}).$$

By continuity of S_M, there exists a sequence (x^m), with $x^m \in S_M(q^m)$, satisfying

$$\lim_{m\to\infty} x^m = \bar{x}, \tag{3.9}$$

which implies

$$\lim_{m\to\infty} (Mx^m + q^m) = M\bar{x} + \bar{q} = \sum_{j\notin\alpha\cup\{r\}} \lambda_j E^j. \tag{3.10}$$

As $\lambda_i > 0$ for all $i \neq r$, there exists an m_0 large enough such that

$$\begin{cases} x_i^{m_0} > 0, & \forall i \in \alpha, \\ (Mx^{m_0} + q^{m_0})_j > 0, & \forall j \notin \alpha \cup \{r\}. \end{cases} \tag{3.11}$$

Noting that $x^{m_0} \in S_M(q^{m_0})$ we have $x_i^{m_0}(Mx^{m_0} + q^{m_0})_i = 0$, for all $i \in I$. Thus, (3.11) implies

$$\begin{cases} (Mx^{m_0} + q^{m_0})_i = 0, & \forall i \in \alpha, \\ x_j^{m_0} = 0, & \forall j \notin \alpha \cup \{r\}. \end{cases} \tag{3.12}$$

By setting $z^{m_0} := Mx^{m_0} + q^{m_0}$ and taking (3.11)-(3.12) into account we have

$$\begin{aligned} q^{m_0} &= z^{m_0} - Mx^{m_0} \\ &= \sum_{j=1}^{n} z_j^{m_0} E^j + \sum_{i=1}^{n} x_i^{m_0}(-M^i) \\ &= \sum_{j\notin\alpha\cup\{r\}} z_j^{m_0} E^j + \sum_{i\in\alpha} x_i^{m_0}(-M^i) + z_r^{m_0} E^r + x_r^{m_0}(-M^r). \end{aligned} \tag{3.13}$$

On the other hand, by the definition of q^m, we have

$$q^{m_0} = \bar{q} - \frac{1}{m_0}E^r = \sum_{i\in\alpha} \lambda_i(-M^i) + \sum_{j\notin\alpha\cup\{r\}} \lambda_j E^j - \frac{1}{m_0}E^r. \tag{3.14}$$

A combination of (3.13) and (3.14) gives

$$E^r = m_0\Big[\sum_{i\in\alpha}(\lambda_i - x_i^{m_0})(-M^i) + \sum_{j\notin\alpha\cup\{r\}}(\lambda_j - z_j^{m_0})E^j - x_r^{m_0}(-M^r) - z_r^{m_0}E^r\Big]. \tag{3.15}$$

Substituting the right-hand side of (3.15) into the $r-$th column of $M[\alpha]$ one obtains

$$\det(M[\alpha]) = m_0\left(-x_r^{m_0}\det(M[\alpha\cup\{r\}]) - z_r^{m_0}\det(M[\alpha])\right).$$

This together with (3.2) implies $(1 + m_0x_r^{m_0}t_r + z_r^{m_0}m_0)\det(M[\alpha]) = 0$. Since $\det(M[\alpha]) \neq 0$, $m_0 > 0$, $x_r^{m_0} \geq 0$ and $z_r^{m_0} \geq 0$, t_r must be negative. The proof is complete. ■

The following corollary is a result obtained in [9] which was designed to answer a question by Pang. Recall that M is said to be a Q-matrix if $S_M(q) \neq \emptyset$ for all $q \in \mathbb{R}^n$, or equivalently, if $K(M) = \mathbb{R}^n$.

Corollary 3.1. *If M is a Lipschitzian, Q-matrix then M is a P-matrix.*

Proof. Since $K(M) = \mathbb{R}^n$, $\partial K(M) = \emptyset$. By virtue of Theorem 3.1 it follows that

$$\det(M[\alpha\setminus\{r\}]).\det(M[\alpha\cup\{r\}]) < 0$$

for all $\alpha\subseteq I$ and $r\in I$. Now Lemma 2.1 yields the required conclusion, $M \in \mathcal{P}$. ■

In [3, 10] the authors have shown that a Lipschitzian matrix with negative diagonal entries is nonpositive. As a corollary to the above theorem, we can extend their result as follows

Corollary 3.2. *Let M be a Lipschitzian matrix.*
(a) *If $m_{ii} < 0$ for all $i \in I$, then $M \leq 0$ and $K(M) = \mathbb{R}^n_+$.*
(b) *If $M \in \mathcal{N}_2$, then $M < 0$.*

Proof. We assume that $n \geq 2$, since the theorem is trivial otherwise.
(a) For each $r \in I$, by setting $\alpha = \emptyset$ one has

$$\det(M[\alpha\setminus\{r\}]).\det(M[\alpha\cup\{r\}]) = \det(M[\emptyset]).\det(M[\{r\}]) = -m_{rr} > 0.$$

It follows from Theorem 3.1 that

$$\mathrm{Pos}\{E^j \mid j \neq r\} = H(\alpha, r) \subseteq \partial K(M). \tag{3.16}$$

Suppose that $m_{rs} > 0$ for some $r, s \in I$, $r \neq s$. Setting

$$q = -M^s + \sum_{k\neq r,s}(1 + |m_{ks}|)E^k + m_{rs}E^r \tag{3.17}$$

we have

$$q \in \mathrm{Int}K_{\{s\}} \subseteq \mathrm{Int}K(M). \tag{3.18}$$

On the other hand,

$$q_k = \begin{cases} -m_{ks} + 1 + |m_{ks}| \geq 1 > 0, & k \neq r, s, \\ -m_{ss} > 0, & k = s, \\ -m_{rs} + m_{rs} = 0, & k = r. \end{cases}$$

Therefore, $q \in \mathrm{Pos}\{E^k \mid k \neq r\}$. By (3.16) we have $q \in \partial K(M)$, which contradicts (3.18). So $M \leq 0$ and

$$\mathbb{R}^n_+ = K_\emptyset \subseteq K(M) = \bigcup_{\alpha \subseteq I} K_\alpha = \bigcup_{\alpha \subseteq I} \mathrm{Pos}\{-M^i, E^j \mid \ i \in \alpha, j \in I \setminus \alpha\} \subseteq \mathbb{R}^n_+.$$

Consequently, $K(M) = \mathbb{R}^n_+$.

(b) Now assume, in addition, that all second-order principal minors of M are negative, we shall show that $m_{rs} < 0$ for all $r, s \in I$. On the contrary, suppose that $m_{rs} = 0$ for some $s \neq r$. By setting $\alpha = \{s\}$ one has $\alpha \setminus \{r\} = \{s\}$, $\alpha \cup \{r\} = \{s, r\}$ and

$$\det(M[\alpha \setminus \{r\}]). \det(M[\alpha \cup \{r\}]) = -m_{ss} \det M_{\{s,r\}} < 0.$$

By virtue of Theorem 3.1, $H(\alpha, r) \cap \mathrm{Int}K(M) \neq \emptyset$. Besides, since $M^s \leq 0$ and $m_{rs} = 0$, we have

$$H(\alpha, r) = \mathrm{Pos}\{-M^s,\ E^j \mid \ j \neq r, s\} \subseteq \mathrm{Pos}\{E^j \mid j \neq r\},$$

which, from (3.16), implies $H(\alpha, r) \subseteq \partial K(M)$, a contradiction. So $M < 0$ and the proof is complete. ■

The next result can be found in [3, 8]. We give here an alternative proof.

Corollary 3.3. *Suppose M is a negative Lipschitzian matrix. Then M is an $N-$matrix.*

Proof. By virtue of Corollary 3.2 we have

$$K(M) = \mathbb{R}^n_+.$$

Since $M < 0$, $-M^i \in \mathrm{Int}K(M)$ for all $i \in I$. Now, take any $\alpha \subseteq I$ and $r \in I$ such that $\alpha \setminus \{r\} \neq \emptyset$. By choosing $i \in \alpha \setminus \{r\}$ we have $-M^i \in H(\alpha, r)$. Thus, $H(\alpha, r) \not\subseteq \partial(\mathrm{Dom}S_M)$ and, from Theorem 3.1, it follows that

$$\det(M[\alpha \setminus \{r\}]). \det(M[\alpha \cup \{r\}]) < 0.$$

Since the inequality holds whenever $\alpha \setminus \{r\} \neq \emptyset$, $M \in \mathcal{N}$ by virtue of Lemma 2.1. This completes the proof. ■

By Corollaries 3.2 and 3.3 we immediately obtain the following result which is a strong extension of [3, Lemma 1] and [9, Corollary 5].

Corollary 3.4. *Suppose $M \in \mathcal{N}_2 \setminus \mathcal{N}$. Then M is not Lipschitzian.* ■

This result provide us a lightly sufficient condition for a given matrix to be not Lipschitzian. Specifically, if $m_{ii} < 0$ for all i, $m_{ii}m_{jj} - m_{ij}m_{ji} < 0$ for all

$i \neq j$, and if there exists a nonnegative principal minor of M, then M cannot be Lipschitzian.

4. Cardinalities of Solution Sets in Lipschitzian Case

It is well known that, if M is a Lipschitzian, $Q-$matrix then $M \in \mathcal{P}$, and hence the number of solutions to the LCP(M, q) is one for every $q \in K(M) = \mathbb{R}^n$. In this section, we shall extend the fact to the general case where M need not be a $Q-$matrix. Specifically, we shall prove that if M is Lipschitzian, then the number of solutions to the LCP(M, q) is a constant for all $q \in \mathrm{Int}K(M)$. First we have to prove some lemmas.

In the sequel, we denote by $U = [U^1, U^2, ..., U^n]$ the $n \times n-$matrix with column vectors U^i; $i \in I$. For a pair of such matrices (U, V), we denote by $\mathcal{H}(U, V)$ the set of all matrix $A = [A^1, A^2, ..., A^n]$ with A^i belonging to $\{U^i, V^i\}$ for each $i \in I$.

Lemma 4.1. *Let U, V be $n \times n-$matrices. Then two following properties are equivalent:*

(P1) *For any $A \in \mathcal{H}(U, V)$ and any $r \in I$,*

$$\det([A^1, ..., A^{r-1}, U^r, A^{r+1}, ..., A^n]).\det([A^1, ..., A^{r-1}, V^r, A^{r+1}, ..., A^n]) < 0. \tag{4.1}$$

(P2) *For every $q \in \mathbb{R}^n$ there exist uniquely vectors λ, $\mu \in \mathbb{R}^n$ such that*

$$\lambda \geq 0,\ \mu \geq 0,\ \langle \lambda, \mu \rangle = 0, \tag{4.2}$$

and

$$q = U\lambda + V\mu = \sum_{i=1}^{n} \lambda_i U^i + \sum_{i=1}^{n} \mu_i V^i. \tag{4.3}$$

For the convenience of the reader, the proof of this lemma will be given in the appendix of the paper.

Lemma 4.2. *Assume that M is Lipschitzian. Then for every $q \in \mathrm{Int}K(M)$ there exists an $\epsilon > 0$ such that $B(q; \epsilon) \subseteq K(M)$ and*

$$|S_M(p)| \leq |S_M(q)|, \quad \text{for all } p \in B(q; \epsilon). \tag{4.4}$$

Proof. Since $q \in \mathrm{Int}K(M)$ and $\{K_\alpha \,|\, \alpha \subseteq I\}$ is a finite class of closed convex cones, we can choose $\epsilon > 0$ so small that $B(q; \epsilon) \subseteq K(M)$ and

$$B(q; \epsilon) \cap K_\alpha = \emptyset,\ \forall \alpha \notin J(q). \tag{4.5}$$

We next claim that, with such an ϵ, (4.4) holds. Indeed, suppose by contrary, that there exists $\bar{p} \in B(q; \epsilon)$ satisfying

$$|S_M(\bar{p})| > |S_M(q)|. \tag{4.6}$$

By the choice of ϵ we have $J(\bar{p}) \subseteq J(q)$. This together with (4.6) and Remark 1 implies that there exist two subsets

$$\alpha_1,\ \alpha_2 \in J(\bar{p})$$

such that

$$v^1 := F_{\alpha_1}(\bar{p}) \neq F_{\alpha_2}(\bar{p}) =: v^2 \tag{4.7}$$

while

$$F_{\alpha_1}(q) = F_{\alpha_2}(q) =: u. \tag{4.8}$$

We define $\alpha := \alpha_1 \cap \alpha_2$, $\beta := I \setminus (\alpha_1 \cup \alpha_2)$, $\gamma := I \setminus (\alpha \cup \beta) = (\alpha_1 \cup \alpha_2) \setminus (\alpha_1 \cap \alpha_2)$, $y := Mu + q$, $y^1 := Mv^1 + \bar{p}$ and $y^2 := Mv^2 + \bar{p}$. Then $y \geq 0$ and

$$q = \sum_{i=1}^{n} u_i(-M^i) + \sum_{j=1}^{n} y_j E^j. \tag{4.9}$$

Since $u = F_{\alpha_1}(q) = F_{\alpha_2}(q)$, we have $u_i = 0$ for all $i \notin \alpha$ and $y_j = 0$ for all $j \notin \beta$. (4.9) is now rewritten as follows:

$$q = \sum_{i \in \alpha} u_i(-M^i) + \sum_{j \in \beta} y_j E^j. \tag{4.10}$$

By definition, it follows from (4.7) that $y^1 \geq 0$, $y^2 \geq 0$, $v^1 \geq 0$, $v^2 \geq 0$ and

$$\begin{cases} \bar{p} = \sum_{i=1}^{n} v_i^1(-M^i) + \sum_{j=1}^{n} y_j^1 E^j, \\ \bar{p} = \sum_{i=1}^{n} v_i^2(-M^i) + \sum_{j=1}^{n} y_j^2 E^j, \\ v_i^1 = v_i^2 = 0;\ \forall i \in \beta, \\ y_j^1 = y_j^2 = 0;\ \forall j \in \alpha, \\ \langle v^1, y^1 \rangle = \langle v^2, y^2 \rangle = 0, \\ v_i^1 \neq v_i^2 \text{ for some } i \in I. \end{cases} \tag{4.11}$$

Now consider two matrices U, V defined by

$$U^i := \begin{cases} -M^i & \text{if } i \in \alpha, \\ -E^i & \text{if } i \in \beta, \\ -M^i & \text{if } i \in \gamma. \end{cases} \quad V^i := \begin{cases} M^i & \text{if } i \in \alpha, \\ E^i & \text{if } i \in \beta, \\ E^i & \text{if } i \in \gamma. \end{cases}$$

From (4.11) we obtain

$$\begin{cases} \bar{p} = \sum_{i=1}^{n} v_i^1 U^i + \sum_{j=1}^{n} y_j^1 V^j, \\ \bar{p} = \sum_{i=1}^{n} v_i^2 U^i + \sum_{j=1}^{n} y_j^2 V^j, \\ v^1 \geq 0,\ v^2 \geq 0,\ y^1 \geq 0,\ y^2 \geq 0, \\ \langle v^1, y^1 \rangle = \langle v^2, y^2 \rangle = 0, \\ v^1 \neq v^2. \end{cases} \tag{4.12}$$

That means, Property (P2) stated in Lemma 4.1 does not hold. Nevertheless, we shall show that (P1) does, a contradiction and, from this, the lemma follows.

What is left is to prove that

$$\det([A^1,...,A^{r-1},U^r,A^{r+1},...,A^n]).\det([A^1,...,A^{r-1},V^r,A^{r+1},...,A^n])<0, \tag{4.13}$$

for every $A \in \mathcal{H}(U,V)$ and $r \in I$.

The fact is trivial if $r \in \alpha \cup \beta$. Now, for the case $r \in \gamma$, we have $U^r = -M^r$, $V^r = E^r$. For each $i \neq r$, set

$$\epsilon_i := \begin{cases} -1 & \text{if } i \in \alpha \text{ and } A^i = M^i, \\ -1 & \text{if } i \in \beta \text{ and } A^i = -E^i, \\ 1 & \text{otherwise.} \end{cases}$$

Then

$$\epsilon_i.A^i \begin{cases} = -M^i & \text{if } i \in \alpha, \\ = E^i & \text{if } i \in \beta, \\ \in \{-M^i, E^i\} & \text{if } i \in \gamma \setminus \{r\}. \end{cases}$$

By setting

$$\bar{\alpha} := \{i \neq r \mid \epsilon_i.A^i = -M^i\},$$
$$\bar{\beta} := \{i \neq r \mid \epsilon_i.A^i = E^i\},$$

we have $\alpha \subseteq \bar{\alpha}$, $\beta \subseteq \bar{\beta}$, $\bar{\alpha} \setminus \{r\} = \bar{\alpha}$, $\bar{\alpha} \cup \{r\} = I \setminus \bar{\beta}$ and

$$\prod_{i\neq r} \epsilon_i.\det([A^1,.,A^{r-1},U^r,A^{r+1},.,A^n])$$
$$= \det([\epsilon_1 A^1,.,\epsilon_{r-1}A^{r-1},-M^r,\epsilon_{r+1}A^{r+1},.,\epsilon_n A^n])$$
$$= \det(M[\bar{\alpha} \cup \{r\}]),$$
$$\prod_{i\neq r} \epsilon_i.\det([A^1,.,A^{r-1},V^r,A^{r+1},.,A^n])$$
$$= \det([\epsilon_1 A^1,.,\epsilon_{r-1}A^{r-1},E^r,\epsilon_{r+1}A^{r+1},.,\epsilon_n A^n])$$
$$= \det(M[\bar{\alpha} \setminus \{r\}]).$$

From (4.10) we have $q \in H(\bar{\alpha}, r)$ and this complementarity face is not a boundary one. (4.13) now is derived by virtue of Theorem 3.1. ■

Lemma 4.3. *Suppose that M is Lipschitzian. Then* Int$K(M)$ *is a connected subset in* $\mathbb{R}^n$.

Proof. On the contrary, suppose that Int$K(M)$ is not connected. Then there are two nonempty open sets $\mathcal{U}$ and $\mathcal{V}$ such that $\mathcal{U} \cup \mathcal{V} = \mathrm{Int}K(M)$ and $\mathcal{U} \cap \mathcal{V} = \emptyset$. We should note that, for any $\alpha \subseteq I$, $\emptyset \neq \mathrm{Int}K_\alpha \subset \mathcal{U} \cup \mathcal{V}$. Since Int$K_\alpha$ is connected, it follows that either Int$K_\alpha \subset \mathcal{U}$ or Int$K_\alpha \subset \mathcal{V}$.

Now take $p \in \mathcal{U}$, $q \in \mathcal{V}$ and suppose that $p \in K_\alpha$, $q \in K_\beta$ for some $\alpha, \beta \subseteq I$. Since $\mathcal{U}$ is open, Int$K_\alpha \cap \mathcal{U} \neq \emptyset$, and hence Int$K_\alpha \subset \mathcal{U}$. Analogously, Int$K_\beta \subset \mathcal{V}$. By virtue of Remark 2 there is a sequence of subsets $\alpha = \alpha_0, \alpha_1, \ldots, \alpha_k = \beta$ such that $K_{\alpha_{i-1}}$ and K_{α_i} are adjacent for all $i \in \{1,...,k\}$. Since Int$K_{\alpha_0} \subset \mathcal{U}$

and $\mathrm{Int}K_{\alpha_k} \subset \mathcal{V}$, there is an m such that

$$\mathrm{Int}K_{\alpha_m} \subset \mathcal{U}, \quad \mathrm{Int}K_{\alpha_{m+1}} \subset \mathcal{V}, \tag{4.14}$$

where K_{α_m} and $K_{\alpha_{m+1}}$ are adjacent. We may assume without loss of generality that

$$\alpha_m = \{1, 2, \ldots, r-1\}, \quad \alpha_{m+1} = \{1, 2, \ldots, r-1, r\}.$$

Thus, $H(\alpha_m; r) \subset K_{\alpha_m} \cap K_{\alpha_{m+1}}$. Since $\mathcal{V}$ is open and $\mathcal{V} \cap \mathrm{Int}K_{\alpha_m} = \emptyset$, one has $\mathcal{V} \cap K_{\alpha_m} = \emptyset$, and hence $\mathcal{V} \cap H(\alpha_m; r) = \emptyset$. Similarly, $\mathcal{U} \cap H(\alpha_m; r) = \emptyset$, and therefore, $H(\alpha_m; r)$ is a boundary face. By the proof of Theorem 3.1, it follows that

$$-M^r = \sum_{i=1}^{r-1} t_i(-M^i) + t_r E^r + \sum_{i=r+1}^{n} t_i E^i \tag{4.15}$$

with $t_r > 0$. Setting

$$q := \sum_{i=1}^{r-1} (|t_i| + 1)(-M^i) + t_r E^r + \sum_{i=r+1}^{n} (|t_i| + 1) E^i$$

we have $q \in \mathrm{Int}K_{\alpha_m}$. On the other hand, by (4.15), q can be rewritten as follows

$$q = \sum_{i=1}^{r-1} (|t_i| + 1 - t_i)(-M^i) + (-M^r) + \sum_{i=r+1}^{n} (|t_i| + 1 - t_i) E^i \in \mathrm{Int}K_{\alpha_{m+1}},$$

which contradicts the fact that $\mathrm{Int}K_{\alpha_m} \cap \mathrm{Int}K_{\alpha_{m+1}} = \emptyset$. The proof is complete.

The next theorem is the main result in this section.

Theorem 4.1. *Assume that M is Lipschitzian. Then there exists a natural number k such that*

$$|S_M(q)| = k, \ \textit{for all } q \in \mathrm{Int}K(M) \tag{4.16}$$

and

$$|S_M(q)| \le k, \ \textit{for all } q \in \partial K(M). \tag{4.17}$$

Proof. For each $q \in \mathrm{Int}K(M)$, we set

$$\Omega(q) := \{p \in K(M) \mid |S_M(p)| \le |S_M(q)|\}.$$

Since M is Lipschitzian, $|S_M(p)| < \infty$ for all $p \in \mathbb{R}^n$ and it is not difficult to verify that $\Omega(q)$ is a closed subset of $\mathbb{R}^n$. Besides, by Lemma 4.2, it follows that $\Omega(q) \cap \mathrm{Int}K(M)$ is an open subset. Since $\mathrm{Int}K(M)$ is connected and $q \in \Omega(q) \cap \mathrm{Int}K(M)$, it implies that $\Omega(q) \supset \mathrm{Int}K(M)$. Now, for every $\alpha \subseteq I$, we have $\Omega(q) \supset \mathrm{Int}K_\alpha$ and hence, by noting that $\Omega(q)$ is closed and K_α is convex this implies that $\Omega(q) \supset K_\alpha$. Therefore, $K(M) = \Omega(q)$. Since q is arbitrarily chosen in $\mathrm{Int}K(M)$, the theorem follows. ∎

Appendix: Proof of Lemma 4.1

The proof of the lemma will be divided into four steps:

Firstly, *If either* (P1) *or* (P2) *holds then V is nonsingular.* Indeed, if (P1) holds, then

$$\det([V^1, ..., V^{r-1}, U^r, V^{r+1}, ..., V^n]).\det([V^1, ..., V^{r-1}, V^r, V^{r+1}, ..., V^n]) < 0,$$

which implies $\det(V) \neq 0$. Now assume that (P2) holds but V is singular. Then there exists a nonzero vector $z \in \mathbb{R}^n$ such that $Vz = 0$. Setting

$$\rho := \max\{|z_1|, ..., |z_n|\} > 0,$$

$$\mu := (\rho, ..., \rho)^T \in \mathbb{R}^n, \quad \mu' := (\rho + z_1, ..., \rho + z_n)^T \in \mathbb{R}^n, \quad \lambda := (0, ..., 0)^T \in \mathbb{R}^n,$$

we have $\lambda, \mu, \mu' \geq 0$, $\langle \lambda, \mu \rangle = 0$, $\langle \lambda, \mu' \rangle = 0$, $\mu \neq \mu'$ and

$$U\lambda + V\mu = U\lambda + V\mu',$$

which contradicts (P2). So V is nonsingular.

Secondly, *Given a nonsingular $n \times n$-matrix W. Then (WU, WV) has Property* (P1) *(Property* (P2)*) if and only if so does (U, V).* Since $U = W^{-1}WU$ and $V = W^{-1}WV$, it is sufficient to prove the "if" part. It is worth noticing that $(WU)^i = WU^i$ and $(WV)^i = WV^i$ for every $i \in I$. Therefore, $A \in \mathcal{H}(WU, WV)$ if $A = WB$ with $B \in \mathcal{H}(U, V)$. So

$$\det([A^1, ..., A^{r-1}, (WU)^r, A^{r+1}, ..., A^n]).\det([A^1, ..., A^{r-1}, (WV)^r, A^{r+1}, ..., A^n])$$
$$= \det([B^1, ..., B^{r-1}, U^r, B^{r+1}, ..., B^n]).\det([B^1, ..., B^{r-1}, V^r, B^{r+1}, ..., B^n]).$$

It follows that if (U, V) has Property (P1), then so does (WU, WV).

Now assume (U, V) has Property (P2). For each $q \in \mathbb{R}^n$, there exist uniquely vectors $\lambda, \mu \in \mathbb{R}^n$ satisfying (4.2) and $W^{-1}q = U\lambda + V\mu$, or,

$$q = WU\lambda + WV\mu,$$

which implies (WU, WV) has Property (P_2).

Thirdly, *M is a $\mathcal{P}$-matrix if and only if (-M, E) has Property* (P1). Indeed, $(-M, E)$ has Property (P1) if and only if for every $\alpha \subseteq I$ and $r \in I$ the following inequality holds:

$$\det(M[\alpha \setminus \{r\}]).\det(M[\alpha \cup \{r\}]) < 0,$$

which, by Lemma 2.1, is equivalent to $M \in \mathcal{P}$.

Fourthly, *M is a $\mathcal{P}-$matrix if and only if (-M, E) has Property* (P2). Indeed, by definition, $(-M, E)$ has Property (P2) if and only if for every $q \in \mathbb{R}^n$ there exist uniquely vectors $x, y \in \mathbb{R}^n_+$ satisfying $\langle x, y \rangle = 0$ and $q = (-M)x + Ey$. That means, for every $q \in \mathbb{R}^n$ the problem $\mathrm{LCP}(M, q)$ has a unique solution. The assertion now follows from [11, Theorem 4.2].

Finally, let U, V be $n \times n-$matrices. We have

$$\begin{aligned}(U, V) \text{ has Property (P1)} &\Leftrightarrow (V^{-1}U, E) \text{ has Property (P1)}\\ &\Leftrightarrow V^{-1}U \in \mathcal{P}\\ &\Leftrightarrow (V^{-1}U, E) \text{ has Property (P2)}\\ &\Leftrightarrow (U, V) \text{ has Property (P2).}\end{aligned}$$

This completes the proof of the lemma. ■

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