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Siu-Yeung's Lemma in the p-Adic Case

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Abstract. In this paper, by using p-adic Nevanlinna-Cartan Theorem, we prove a p-adic version of Siu-Yeung's Lemma and its application in the study of the unique range sets for meromorphic functions.

1. Introduction

Let us start by recalling Borel's Lemma in the complex case:

Borel's Lemma. *Let $f_1, \ldots, f_n$, $n \geq 3$ be non-zero holomorphic functions on $\mathbb{C}$ such that*

$$f_1 + \cdots + f_n = 0.$$

Then the functions $\{f_1, \ldots, f_{n-1}\}$ are linearly dependent.

It is well-known that Borel's Lemma plays an important role in the study of hyperbolic spaces. For different purposes some generalizations of the Lemma are given. We mention here a recent result of Siu and Yeung.

Siu-Yeung's Lemma. [9] *Let $g_j(x_0, \ldots, x_n)$ be a homogeneous polynomial of degree δ_j for $0 \leq j \leq n$. Suppose there exists a holomorphic map $f : \mathbb{C} \to \mathbb{P}^n(\mathbb{C})$ so that its image lies in*

$$\sum_{j=0}^{n} x_j^{k-\delta_j} g_j(x_0, \ldots, x_n) = 0,$$

and $k > (n+1)(n-1) + \sum_{j=0}^{n} \delta_j$. Then there is a nontrivial linear relation among $x_1^{k-\delta_1} g_1(x_0, \ldots, x_n), \ldots, x_n^{k-\delta_n} g_n(x_0, \ldots, x_n)$ on the image of f.

In this paper, by using p-adic Nevanlinna-Cartan Theorem, we prove a p-adic version of Siu-Yeung's Lemma. We then apply the result to give a unique range sets for p-adic meromorphic functions.

2. Siu-Yeung's Lemma in the p-Adic Case

Let p be a prime number, $\mathbb{Q}_p$ the field of p-adic number, and $\mathbb{C}_p$ be the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. The absolute value in $\mathbb{Q}_p$ is normalized so that $|p|_p = p^{-1}$. We use the notation $h(f,t) = -h^+(f,t)$ for the height function and $N(f,t)$ for the counting function of a p-adic entire function f (for details, see [4, 5]). ■

Lemma 2.1. [5] *Let ϕ, ϕ_1, ϕ_2 be p-adic holomorphic functions. Then we have*
1) $h^+(\phi,t) = N(\phi,t) + 0(1)$,
2) $N(\phi_1 + \phi_2, t) \leq \max\{N(\phi_1,t), N(\phi_2,t)\} + 0(1)$,
3) $N(\phi_1\phi_2, t) = N(\phi_1,t) + N(\phi_2,t)$.

Lemma 2.2. *Let $g(x_0,\dots,x_n)$ be a homogeneous polynomial of degree d and $f_0, f_1, \dots, f_n$ be p-adic holomorphic functions. Then*

$$N(g(f_0,\dots,f_n),t) \leq d \max_{0\leq j\leq n} N(f_j,t) + 0(1).$$

Proof. We first prove if $x_0^{\alpha_0}\cdots x_n^{\alpha_n}$ is a monomial of degree d then

$$N(f_0^{\alpha_0}\cdots f_n^{\alpha_n},t) \leq d \max_{0\leq j\leq n} N(f_j,t).$$

We have

$$\begin{aligned} N(f_0^{\alpha_0}\cdots f_n^{\alpha_n},t) &= N(f_0^{\alpha_0},t) + \cdots + N(f_n^{\alpha_n},t) \\ &= \alpha_0 N(f_0,t) + \cdots + \alpha_n N(f_n,t) \\ &\leq (\alpha_0 + \cdots + \alpha_n) \max_{0\leq j\leq n} N(f_j,t) = d \max_{0\leq j\leq n} N(f_j,t). \end{aligned}$$

On the other hand, $g(x_0,\dots,x_n)$ is the sum of the monomials of degree d, therefore from Lemma 2.1 we obtain the proof of Lemma 2.2. ■

Theorem 2.1. (p-adic Nevanlinna–Cartan Theorem [5]) *Let $H_1, \dots, H_q$ be q hyperplanes in general position, and let f be a non-degenerate holomorphic curve in $\mathbb{P}^n(\mathbb{C}_p)$. Then we have*

$$(q-n-1)h^+(f,t) \leq \sum_{j=1}^{q} N_n(foH_j,t) + \frac{n(n+1)}{2}t + 0(1),$$

where $0(1)$ is bounded when $t \to -\infty$.

Theorem 2.2. (Siu-Yeung's Lemma in the p-adic case) *Let $g_j(x_0,\dots,x_n)$ be a homogeneous polynomial of degree $\delta_j$for $0 \leq j \leq n$. Suppose there exists a* *holomorphic map* $f : \mathbb{C}_p \to \mathbb{P}^n(\mathbb{C}_p)$ *so that its image lies in*

$$\sum_{j=0}^{n} x_j^{k-\delta_j} g_j(x_0, \ldots, x_n) = 0,$$

and

$$k \geq (n+1)(n-1) + \sum_{j=0}^{n} \delta_j.$$

Then the following functions are linearly dependent on $\mathbb{C}_p$ *if they have no common zeros:*

$$f_1^{k-\delta_1} g_1(f_0, \ldots, f_n), \ldots, f_n^{k-\delta_n} g_n(f_0, \ldots, f_n).$$

Proof. By the hypothesis we have

$$\sum_{j=0}^{n} f_j^{k-\delta_j} g_j(f_0, \ldots, f_n) = 0.$$

Assume to the contrary that the functions

$$f_1^{k-\delta_1} g_1(f_0, \ldots, f_n), \ldots, f_n^{k-\delta_n} g_n(f_0, \ldots, f_n)$$

are linearly independent, we define a holomorphic curve g in $\mathbb{P}^{n-1}(\mathbb{C}_p)$ by setting

$$g = (f_1^{k-\delta_1} g_1(f_0, \ldots, f_n), \ldots, f_n^{k-\delta_n} g_n(f_0, \ldots, f_n)).$$

Then g is linearly non-degenerate. Consider the following hyperplanes in general position in $\mathbb{P}^{n-1}(\mathbb{C}_p)$:

$$H_1 = \{x_1 = 0\}, \ldots, H_n = \{x_n = 0\}, H_{n+1} = \{x_1 + \cdots + x_n = 0\}.$$

It follows from Theorem 2.1 that

$$h^+(g,t) \leq \sum_{j=1}^{n+1} N_{n-1}(goH_j, t) + \frac{(n-1)n}{2} t + 0(1).$$

By Lemma 2.1 we have

$$\begin{aligned} &h^+(g,t) \\ &= \max\{h^+(f_1^{k-\delta_1} g_1(f_0, \ldots, f_n), t), \ldots, h^+(f_n^{k-\delta_n} g_n(f_0, \ldots, f_n), t)\} \\ &= \max\{N(f_1^{k-\delta_1} g_1(f_0, \ldots, f_n), t), \ldots, N(f_n^{k-\delta_n} g_n(f_0, \ldots, f_n), t)\} + 0(1). \end{aligned}$$

Therefore

$$\begin{aligned} &\max\{N(f_1^{k-\delta_1} g_1(f_0, \ldots, f_n), t), \ldots, N(f_n^{k-\delta_n} g_n(f_0, \ldots, f_n), t)\} \\ &\leq \sum_{j=1}^{n+1} N_{n-1}(goH_j, t) + \frac{(n-1)n}{2} t + 0(1). \end{aligned} \quad (1)$$

For $j = 1, \dots, n$ we have

$$\begin{aligned} N_{n-1}(goH_j, t) &= N_{n-1}(f_j^{\delta_j} g_j(f_0, \dots, f_n), t) \\ &\le N_{n-1}(f_j^{k-\delta_j}, t) + N_{n-1}(g_j(f_0, \dots, f_n), t) \\ &\le (n-1)N_1(f_j^{k-\delta_j}, t) + N_{n-1}(g_j(f_0, \dots, f_n), t) \\ &= (n-1)N_1(f_j, t) + N_{n-1}(g_j(f_0, \dots, f_n), t) \\ &\le (n-1)N(f_j, t) + N_{n-1}(g_j(f_0, \dots, f_n), t) \\ &\le (n-1)\max_{0\le j\le n} N(f_j, t) + N_{n-1}(g_j(f_0, \dots, f_n), t) \\ &\le (n-1)\max_{0\le j\le n} N(f_j, t) + N(g_j(f_0, \dots, f_n), t). \end{aligned}$$

For $j = n+1$ we still have

$$\begin{aligned} N_{n-1}(goH_{n+1}, t) &= N_{n-1}(-f_0^{k-\delta_0} g_0(f_0, \dots, f_n), t) \\ &= N_{n-1}(f_0^{k-\delta_0} g_0(f_0, \dots, f_n), t) \\ &\le N_{n-1}(f_0^{k-\delta_0}, t) + N_{n-1}(g_0(f_0, \dots, f_n), t) \\ &\le (n-1)N_1(f_0^{k-\delta_0}, t) + N_{n-1}(g_0(f_0, \dots, f_n), t) \\ &= (n-1)N_1(f_0, t) + N_{n-1}(g_0(f_0, \dots, f_n), t) \\ &\le (n-1)N(f_0, t) + N_{n-1}(g_0(f_0, \dots, f_n), t) \\ &\le (n-1)\max_{0\le j\le n} N(f_0, t) + N_{n-1}(g_0(f_0, \dots, f_n), t) \\ &\le (n-1)\max_{0\le j\le n} N(f_j, t) + N(g_0(f_0, \dots, f_n), t). \end{aligned}$$

We set for simplicity

$$\max_{0\le j\le n} N(f_j, t) = N(f_{i_0}, t).$$

Then we have

$$\begin{aligned} N(f_{i_0}^{k-\delta_{i_0}} g_{i_0}(f_0, \dots, f_n), t) &= N(f_{i_0}^{k-\delta_{i_0}}, t) + N(g_{i_0}(f_0, \dots, f_n), t) \\ &= (k-\delta_{i_0})N(f_{i_0}, t) + N(g_{i_0}(f_0, \dots, f_n), t). \end{aligned}$$

From this and (1) we obtain

$$\begin{aligned} (k-\delta_{i_0})N(f_{i_0}, t) + N(g_{i_0}(f_0, \dots, f_n), t) &\le (n-1)(n+1)N(f_{i_0}, t) \\ &+ \sum_{j=0}^{n} N(g_j(f_0, \dots, f_n), t) + \frac{(n-1)n}{2}t + 0(1). \end{aligned}$$

Thus

$$\begin{aligned} (k-\delta_{i_0})N(f_{i_0}, t) \le (n-1)(n+1)N(f_{i_0}, t) &+ \sum_{\substack{j=0 \\ j\ne i_0}}^{n} N(g_j(f_0, \dots, f_n), t) \\ &+ \frac{(n-1)n}{2}t + 0(1). \end{aligned}$$

On the other hand, by Lemma 2.2 we have

$$N(g_j(f_0, \dots, f_n), t) \le \delta_j \max_{0 \le j \le n} N(f_j, t).$$

Hence

$$\begin{aligned}&(k - \delta_{i_0})N(f_{i_0}, t)\\ &\le (n-1)(n+1)N(f_{i_0}, t) + \sum_{\substack{j=0\\ j \ne i_0}}^{n} \delta_j N(f_{i_0}, t) + \frac{(n-1)n}{2} t + 0(1).\end{aligned}$$

So

$$\Big(k - \sum_{j=0}^{n} \delta_j - (n-1)(n+1)\Big) N(f_{i_0}, t) \le \frac{(n-1)n}{2} t + 0(1).$$

By the hypothesis $k \ge (n+1)(n-1) + \sum_{j=0}^{n} \delta_j$ we have a contradiction when $t \to -\infty$. ■

3. Applications

For a nonconstant meromorphic function f on $\mathbb{C}_p$ and a set $S \subset \mathbb{C}_p \cup \{\infty\}$, we define

$$E_f(S) = \bigcup_{a \in S} \{(m, z) | f(z) - a = 0, \text{ with multiplicity } m\}.$$

A set S is called a *unique range set for p-adic meromorphic functions* (URSM) if for any pair of nonconstant meromorphic functions f and g, the condition $E_f(S) = E_g(S)$ implies $f \equiv g$. Recently, URSM with finitely many elements have been found by Hu-Yang ([2, 3]). In this section we give a new class of unique range sets for p-adic meromorphic functions. For the proof of the result, we need the following lemma.

Lemma 3.1. [2] *Let f be a nonconstant meromorphic function on $\mathbb{C}_p$ and $a_1, \dots, a_q$ be distinct numbers in $\mathbb{C}_p$. Then*

$$(q-1)T(r, f) \le \overline{N}(r, f) + \sum_{j=1}^{q} \overline{N}\left(r, \frac{1}{f - a_j}\right) - \log r + 0(1).$$

Furthermore,

$$\sum_{a \in \mathbb{C}_p \cup \{\infty\}} \delta_f(a) < 2,$$

where

$$\delta_f(a) = 1 - \lim_{r \to \infty} \sup \frac{\overline{N}\left(r, \frac{1}{f-a}\right)}{T(r, f)}.$$

Theorem 3.1. *Suppose that n and m are two positive integers such that $n \geq 8+4m$. Let*

$$S = \{z \in \mathbb{C}_p \,|\, z^n + az^{n-m} + bz^{n-2m} + c = 0\},$$

where $a, b, c \in \mathbb{C}_p^*$ such that $a^2 - 4b \neq 0$ and the algebraic equation

$$z^n + az^{n-m} + bz^{n-2m} + c = 0,$$

has no mutiple roots. Then S is a unique range set for p-adic meromorphic functions.

Proof. Let $a_1, a_2, \ldots, a_n$ be the distinct roots of the equation $z^n + az^{n-m} + bz^{n-2m} + c = 0$. Let f, g be nonconstant meromorphic functions such that $E_f(S) = E_g(S)$. Represent $f = \frac{f_1}{f_2}$ and $g = \frac{l_1}{l_2}$, where (f_1, f_2) and (l_1, l_2) are some pairs of entire functions without common factors. Then there exists a constant $\beta \neq 0$ such that

$$(f_1 - a_1 f_2)(f_1 - a_2 f_2)\cdots(f_1 - a_n f_2) = \beta(l_1 - a_1 l_2)(l_1 - a_2 l_2)\cdots(l_1 - a_n l_2).$$

Put $g_1 = \lambda l_1, g_2 = \lambda l_2$ (where $\lambda^n = \beta$). We have

$$f_1^{n-2m}(f_1^{2m} + af_1^m f_2^m + bf_2^{2m}) + cf_2^n - g_1^{n-2m}(g_1^{2m} + ag_1^m g_2^m + bg_2^{2m}) - cg_2^n = 0. \tag{2}$$

Since $n \geq 8 + 4m$, the hypothesis of Theorem 2.2 is satisfied (with $n = 3, \delta_0 = \delta_2 = 2m, \delta_1 = \delta_3 = 0$). Without loss of generality we can suppose that there are numbers $\alpha_1, \alpha_2, \alpha_3$, not all are zero, such that

$$\alpha_1 f_1^{n-2m}(f_1^{2m} + af_1^m f_2^m + bf_2^{2m}) + \alpha_2 f_2^n - \alpha_3 g_2^n = 0.$$

We consider the possible cases:

Case 1. $\alpha_1\alpha_2\alpha_3 \neq 0$.

Using again Theorem 2.2 (with $n = 2$, $\delta_0 = 2m$, $\delta_1 = \delta_2 = 0$), we obtain

$$\alpha_1' f_1^{n-2m}(f_1^{2m} + af_1^m f_2^m + bf_2^{2m}) + \alpha_2' f_2^n = 0,$$

where not all α_i' are zeros. This implies that f is constant.

Case 2. $\alpha_3 = 0$. It is clear that f is constant.

Case 3. $\alpha_2 = 0$. Clearly, $\alpha_1\alpha_3 \neq 0$. Then

$$f_1^n + af_1^{n-m} f_2^m + bf_1^{n-2m} f_2^{2m} = \gamma g_2^n,$$

where $\gamma = \frac{\alpha_3}{\alpha_1}$. Therefore

$$1 + a\left(\frac{f_2}{f_1}\right)^m + b\left(\frac{f_2}{f_1}\right)^{2m} = \gamma\left(\frac{g_2}{f_1}\right)^n.$$

Put $f_3 = \left(\frac{f_2}{f_1}\right)^m + \frac{a}{2b}$, and $g_3 = \frac{g_2}{f_1}$, we have

$$bf_3^2 - \frac{a^2 - 4b}{4b} = \gamma g_3^n. \tag{3}$$

If g_3 is constant then by the equation (3), we obtain f_3 is also constant. From this it follows that the function $f = \frac{f_1}{f_2}$ is constant and we have a contradiction. So g_3 is a non-constant meromorphic function.

Now from (3), we have

$$2T(r, f_3) = nT(r, g_3) + 0(1).$$

By Lemma 3.1 we have

$$\begin{aligned} nT(r, g_3) &= T(r, \gamma g_3^n) + 0(1) \\ &\leq \overline{N}(r, g_3^n) + \overline{N}\Big(r, \frac{1}{g_3^n}\Big) + \overline{N}\Big(r, \frac{1}{\gamma g_3^n + \frac{a^2-4b}{4b}}\Big) - \log r + 0(1) \\ &= \overline{N}(r, g_3) + \overline{N}\Big(r, \frac{1}{g_3}\Big) + \overline{N}\Big(r, \frac{1}{f_3^2}\Big) - \log r + 0(1) \\ &\leq T(r, g_3) + T(r, g_3) + T(r, f_3) - \log r + 0(1) \\ &= \Big(2 + \frac{n}{2}\Big) T(r, g_3) - \log r + 0(1). \end{aligned}$$

Hence

$$\frac{n-4}{2} T(r, g_3) \leq -\log r + 0(1).$$

This is a contradiction.

Case 4. $\alpha_1 = 0$. It is clear that $\alpha_2 \alpha_3 \neq 0$. Furthermore,

$$\alpha f_2^n = g_2^n,$$

where $\alpha = \frac{\alpha_1}{\alpha_2}$. From (2) we obtain

$$\begin{aligned} & f_1^{n-2m}(f_1^{2m} + a f_1^m f_2^m + b f_2^{2m}) \\ & + c(1-\alpha) f_2^n - g_1^{n-2m}(g_1^{2m} + a\epsilon^m g_1^m f_2^m + b\epsilon^{2m} f_2^{2m}) = 0, \end{aligned}$$

where $\epsilon^n = \alpha$.

We claim that $\alpha = 1$. Indeed, if $\alpha \neq 1$, then using Theorem 2.1 (with $n = 2, \delta_0 = \delta_2 = 2m, \delta_1 = 0$) we obtain that $f_1^{n-2m}(f_1^{2m} + \alpha f_1^m f_2^m + b f_2^{2m})$ and f_2^n are linearly dependent, and then $\frac{f_1}{f_2} =$ constant. This is a contradiction. Hence $f_2^n = g_2^n$.

Putting $h = \dfrac{f}{g}$, we conclude from (2) that

$$(h^n - 1) g^{2m} + a(h^{n-m} - 1) g^m + b(h^{n-2m} - 1) = 0. \tag{4}$$

We prove that h is constant. In fact, if it is not the case, we write (4) in the form

$$\Big[(h^n - 1) g^m + \frac{a}{2}(h^{n-m} - 1)\Big]^2 = \Psi(h), \tag{5}$$

where $\Psi(z)$ is defined by

$$\Psi(z) = -b(z^n - 1)(z^{n-2m} - 1) + \frac{a^2}{4}(z^{n-m} - 1)^2.$$

Since

$$\Psi'(z) = z^{n-2m-1}\Big[(n-m)\frac{a^2-4b}{2}z^n + bnz^{2m} - \frac{a^2(n-m)}{2}z^m + b(n-2m)\Big],$$

and $\Psi(0) \neq 0$, the polynomial Ψ has at least $(2n-2m) - n = n - 2m$ distinct zeros. From (5), we obtain that the roots of $\Psi(h) = 0$ have multiplicity at least 2. From Lemma 3.1, we have $\frac{n-2m}{2} < 2$, which contracdicts the condition $n \geq 8 + 4m$. Hence h is constant.

On the other hand, since g is not constant, equation (3) give $h^n - 1 = 0$ and $h^{n-1} - 1 = 0$. This implies $h = 1$ and hence $f \equiv g$. So S is a unique range set for p-adic meromorphic functions. ■

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