Vietnam Journal of Mathematics **32**:2 (2004) 209–218

Vietnam Journal
of
MATHEMATICS

Dual Preference in Leontief Production Problem and Its Extension

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Received May 05, 2003
Revised September 03, 2003

Abstract. It is known that a Leontief production function defines a dual preference in the space of prices of input factors which can be measured via the income by selling out a unit prototype vector of inputs. This primal-dual preference relation will be extended to production problems in which the input factors are used for production in a nonlinear proportion. The obtained result enables us to define a dual problem for a general production problem.

1. Introduction

In a Leontief production process we are given a set X of feasible vectors of inputs which can be used to produce a single commodity, called output

$$X \subseteq \{x = (x_1, x_2, \dots, x_n) : x_i \geq 0,\ i = 1, 2, \dots, n\},$$

where x_i denotes the quantity of the i-th input factor. The feasible set X is assumed full-dimensional, bounded, closed, convex and satisfying a free disposal condition:

$$x \in X \implies y \in X, \quad \forall y : 0 \leq y \leq x.$$

The problem we are dealing with is to maximize the output produced by Leontief technology subject to the feasible set X of inputs. It is well-known that Leontief production process has important applications in production analysis, and it plays a profound role in input-output models of mathematical economics (cf. [1- 3]). In particular, Leontief production functions have been used extensively in studying such problems as analysing behaviors of the firms, finding efficient frontiers in microeconomics, establishing general competitive equilibrium, etc. (cf. [3 - 4]). Along with the concept of Leontief production, other related concepts such as Leontief preference, Leontief utility have a great impact on the theory of consumer and individual preferences (cf. [3]). In this article we are confined in duality studying for Leontief production problems. Namely, we present an alternative duality scheme which derives from a given Leontief production problem a dual problem defined in the space of prices, yielding the duality 0-maximum principle. Furthermore, the alternative duality scheme can be extended to a class of nonconvex Leontief production problems in which the inputs are consumed in a nonlinear proportion.

This article is organized as follows. After the introduction, Sec. 2 recalls a dual for the Leontief production problem with a feasible set X of inputs given by linear constraints. Sec. 3 generalizes the primal-dual preference relation and define a dual for a production problem in which the inputs are used in a nonlinear proportion. Finally, Sec. 4 closes the article by several concluding remarks and discussions.

2. Dual Preference of Leontief Production

Suppose that we are given a vector $c = (c_1, c_2, \dots, c_n)$ of Leontief technology coefficients ($c_i > 0, i = 1, 2, \dots, n$) and the feasible set X of inputs defined by the following linear constraints

$$X = \{x \geq 0 : \; x \leq A^t u, \; \sum_{i=1}^{m} u_i \leq 1, \; u_i \geq 0, \; i = 1, 2, \dots, m\}, \tag{1}$$

where A is a $m \times n$-matrix of nonnegative elements such that in any column of A there is a positive element, and A^t denotes the transpose of A. The representation (1) of X could arise, for example, from the following practical problem. The output is the quantity of steel to be produced. The inputs are the quantities of ores necessary for the steel production, and to be extracted from m different cites of mine with a given one budgetary unit. u_i, $i = 1, 2, \dots, m$, are budgetary proportions allocated to m cites of mine, respectively. For given u_i budgetary proportion, the i-th cite of mine can extract an amount of $u_i a_{ij}$ units of the j-th ore ($j = 1, 2, \dots, n$).

Based on the free disposal condition a linear program which maximizes the output over X can be formulated as follows

$$\max \; \theta, \tag{2}$$

$$\text{s.t.} \;\; \theta c - x \leq 0, \tag{3}$$

$$- A^t u + x \; \leq 0, \tag{4}$$

$$\sum_{i=1}^{m} u_i \; \leq \; 1, \tag{5}$$

$$x \geq 0, \; u \geq 0, \; \theta \geq 0. \tag{6}$$

Since X is bounded, closed and full-dimensional, program (2) - (6) is solvable and it has a positive optimal value. Denote by r, s and α the simplex multipliers of the constraints (3), (4) and (5), respectively. Then the dual of (2) - (6) is

$$\begin{aligned} &\min \alpha, \\ \text{s.t. } & c^t r \geq 1, \\ & \alpha e \geq As, \\ & s \geq r, \\ & r \geq 0,\ s \geq 0,\ \alpha \geq 0, \end{aligned} \tag{7}$$

where $e = (1, 1, \dots, 1) \in R^n$ and $c^t r$ denotes the inner product of vectors c and r. In order to reduce the number of variables of this dual program we use the following transformation. Since the optimal value of the dual is equal to the optimal value of the primal (2) - (6) which is positive, the value of feasible α in the dual must be positive. Let

$$\beta = \frac{1}{\alpha},\ q = \frac{1}{\alpha} r, \text{ and } p = \frac{1}{\alpha} s,$$

then program (7) can be rewritten as follows

$$\begin{aligned} &\max \beta, \\ \text{s.t. } & c^t q \geq \beta, \\ & e \geq Ap, \\ & p \geq q, \\ & q \geq 0,\ p \geq 0,\ \beta \geq 0. \end{aligned}$$

This program is equivalent to

$$\begin{aligned} &\max c^t p, \\ \text{s.t. } & e \geq Ap, \\ & p \geq 0. \end{aligned} \tag{8}$$

Denote by P the feasible set of the dual (8):

$$P = \{p \geq 0 : e \geq Ap\}. \tag{9}$$

A relation between the feasible set X in the primal and the feasible set P in the dual is given in the following proposition.

Proposition 2.1. $P = \{p \geq 0 : p^t x \leq 1\ \forall x \in X\}$.

Proof. Suppose that $p \geq 0$ and $p^t x \leq 1$, for all $x \in X$. Denote by e^i the i-th unit vector in R^m : $e^i = (e^i_1, e^i_2, \dots, e^i_m)$ where $e^i_j = 0\ \forall j \neq i$ and $e^i_i = 1$. Set $x^i = A^t e^i$, then $x^i = A^i$ where A^i is the i-th row of the matrix A. Since $x^i \in X, i = 1, 2, \dots, n$, one has

$$p^t x^i = p^t A^i \leq 1, \quad i = 1, 2, \dots, m,$$

proving $Ap \leq e$. So, $p \in P$.Conversely, suppose that $p \in P$. For any $x \in X$ one has

$$p^t x \ \le \ p^t A^t u \ \text{ for some ; } u \ge 0 : \sum_{i=1}^{m} u_i \le 1.$$

Further,

$$p^t A^t u \ = \ u^t A p \ \le \ u^t e \ \le \ 1.$$

Therefore, $p^t x \le 1$, proving the proposition. ■

From Proposition 2.1 it follows that a vector p of prices is feasible to the dual (8) if it is nonnegative and satisfies the normalized budget constraint

$$p^t x \ \le \ 1$$

for all feasible vectors of inputs $x \in X$. The dual (8) can be interpreted as finding a feasible vector of prices which maximizes the income by selling out vector $c = (c_1, c_2, \ldots, c_n)$ of inputs. The objective in (8) defines a preference in the set of prices $\{p \in R^n : p \ge 0\}$ which says that a price vector p is preferred to a price vector q if

$$p^t c \ > \ q^t c, \tag{10}$$

i.e., by selling vector c of inputs the income according to the price p is greater than the income according to the price q.

In order to see more clearly the relation between the primal preference and the dual one let us define a set of prototype vectors of inputs. The set $\{\theta c : \theta > 0\}$ is called the set of prototype vectors of inputs. A vector in this set is called a prototype vector of inputs. Any prototype vector of inputs can be used for production without waste of inputs. By the primal preference a prototype vector y of inputs is preferred to a prototype vector x of inputs if $y_i > x_i$ for any $i = 1, 2, \ldots, n$ or equivalently by vector notation

$$y > x. \tag{11}$$

This primal preference can be used to define a dual preference.

Suppose that for one unit of monetary income we need to sell a prototype vector x of inputs according to a given price vector p ($p \ne 0$):

$$p^t x \ = \ 1, \ \text{where } x \ = \ \theta c.$$

Then this prototype vector x can be calculated by

$$x \ = \ \frac{1}{p^t c} c \quad (\theta = \frac{1}{p^t c}).$$

Similarly, for one unit of income we need to sell, according to a given price vector q, the prototype vector $y = \dfrac{1}{q^t c} c$ of inputs. The price vector p is preferred to the price vector q if $x < y$, i.e., for one unit of income the prototype vector of inputs we need to sell according to the price p is less than that according to the price q. Since

$$\begin{aligned} x < y &\Leftrightarrow \frac{1}{p^t c} c < \frac{1}{q^t c} c \\ &\Leftrightarrow p^t c \ > \ q^t c, \end{aligned}$$

this dual preference is the same as the preference defined by (10). In the next section the above dual preference will be generalized to a production problem in which the inputs are used in a nonlinear proportion.

3. Extension to Nonlinear Problems

In this section first we relax the linearity assumption of the feasible set X of input vectors: X can be defined by nonlinear constraints. Second, the inputs can be used for production in a nonlinear proportion. Namely, in order to produce θ units of output the production process requires $f_i(\theta)$ units of the i-th input factor, $i = 1, 2, \ldots, n$, where $f_i(\cdot)$ is an increasing continuous function defined on $R_+ = \{\theta \geq 0\}$ such that for any $i = 1, 2, \ldots, n$

$$\begin{aligned} f_i(0) &= 0, \\ f_i(\theta) &\to \infty \text{ as } \theta \ \to \ \infty. \end{aligned}$$

The functions $f_i(\cdot)$, $i = 1, 2, \ldots, n$ can be nonlinear in general.

The production problem can be formulated as follows

$$\begin{aligned} &\max \theta, \\ \text{s.t. } \ x_i \ &\geq \ f_i(\theta), \quad i = 1, 2, \ldots, n, \\ &x \in X. \end{aligned}$$

Denote by $f_i^{-1}(\cdot)$ the inverse of $f_i(\cdot)$ and set

$$F(x) \ = \ \min\{f_i^{-1}(x_i), \ i = 1, 2, \ldots, n\}.$$

Then this program can be rewritten by

$$\begin{aligned} &\max F(x) \\ &\text{s.t. } \ x \in X. \end{aligned} \tag{12}$$

Since $f_i(\cdot)$ is increasing, so $f_i^{-1}(\cdot)$ is, hence $f_i^{-1}(\cdot)$ is quasiconcave ($i = 1, 2, \ldots, n$). The objective function $F(\cdot)$ is a minimum of quasiconcave functions, therefore it is also quasiconcave. So, program (12) is a maximization of a quasiconcave function over a convex set.

Similar as in the last section, we call a vector x of inputs a prototype vector of inputs if there is $\theta > 0$ such that $x_i = f_i(\theta), \ i = 1, 2, \ldots, n$.

Proposition 3.1. *For any price vector p ($p \geq 0, \ p \neq 0$), there is a unique prototype vector $x(p)$ of inputs such that*

$$p^t x(p) = 1, \tag{13}$$

or equivalently, there is a unique value $\theta(p)$ such that $\theta(p) > 0$ and

$$\sum_{i=1}^{n} p_i f_i(\theta(p)) = 1. \tag{14}$$

Proof. Since $p \geq 0,\ p \neq 0$, the function

$$g(\theta) \ := \ \sum_{i=1}^{n} p_i f_i(\theta)$$

is continuous, increasing and satisfying

$$g(0) = 0,\ \ g(\theta) \to \infty \ \text{ as } \ \theta \to \infty.$$

So there is a unique value $\theta(p) > 0$ such that $g(\theta(p)) = 1$, proving (14). By setting

$$x(p) = (f_1(\theta(p)), f_2(\theta(p)), \ldots , f_n(\theta(p)))$$

one has (13). ■

Similar to the linear case, we define a dual preference in the set of prices as follows. A price vector p ($p \geq 0,\ p \neq 0$) is preferred to a price vector q ($q \geq 0,\ q \neq 0$) if

$$x(p) < x(q). \tag{15}$$

In other words, p is preferred to q if for one unit of income the prototype vector of inputs we need to sell according to the price p is less than that according to the price q.

Set $H(0) = -\infty$ and for $p \geq 0,\ p \neq 0$, set

$$H(p) = -\theta(p),$$

where $\theta(p)$ is defined as in Proposition 3.1. Then (15) is equivalent to

$$H(p) \ > \ H(q).$$

Thus, the problem of finding the most preferred feasible price vector is equivalent to the problem of maximizing the function $H(\cdot)$ over the set of feasible prices:

$$\begin{aligned} &\max H(p),\\ &\text{s.t. } \ p \in P, \end{aligned} \tag{16}$$

where the set P of feasible prices is defined by

$$P = \{p \geq 0 : p^t x \leq 1 \ \ \forall x \in X\}.$$

The program (17) is called a dual of the program (12).

Proposition 3.2. *The function $H(\cdot)$ is quasiconcave in $p \geq 0$.*

Proof. Let $p \geq 0$, $q \geq 0$ and $0 \leq \lambda \leq 1$. If either $p = 0$ or $q = 0$ then obviously

$$H(\lambda p + (1 - \lambda)q) \ \geq -\infty = \min\{H(p), H(q)\}.$$

Now suppose that $p \neq 0$ and $q \neq 0$. Setting $\overline{\theta} = \max\{\theta(p), \theta(q)\}$ one has $\overline{\theta} \geq \theta(p)$ and $\overline{\theta} \geq \theta(q)$, hence

$$\sum_{i=1}^{n} p_i f_i(\overline{\theta}) \geq \sum_{i=1}^{n} p_i f_i(\theta(p)) = 1,$$
$$\sum_{i=1}^{n} q_i f_i(\overline{\theta}) \geq \sum_{i=1}^{n} p_i f_i(\theta(q)) = 1.$$

Therefore

$$\sum_{i=1}^{n} (\lambda p_i + (1-\lambda) q_i) f_i(\overline{\theta}) \ \geq \ 1.$$

So

$$\max\{\theta(p), \theta(q)\} = \overline{\theta} \ \geq \ \theta(\lambda p + (1-\lambda) q).$$

Thus

$$H(\lambda p + (1-\lambda) q) \geq \min\{H(p), H(q)\},$$

proving the proposition. ■

By Proposition 3.2 the dual program (16) is a quasiconcave maximization over a convex set. Now the duality theorem can be stated as follows.

Theorem 3.1. *Both the primal program* (12) *and the dual program* (17) *are solvable and one has the following 0-maximum principle*

$$\max\{F(x) : x \in X\} + \max\{H(p) : p \in P\} = 0. \qquad (17)$$

Furthermore, for any optimal solution $\overline{x}$ *to the primal and any optimal solution* $\overline{p}$ *to the dual one has*

$$\overline{p}^t \overline{x} = 1.$$

Proof. For any $x \in X$ and $p \in P$ one has

$$\begin{aligned}
&\sum_{i=1}^{n} p_i x_i \ \leq \ 1 \\
&\Rightarrow \sum_{i=1}^{n} p_i f_i(F(x)) \ \leq \ 1 \quad \text{(because } F(x) \leq f_i^{-1}(x_i) \ \forall i = 1, 2, \ldots, n) \\
&\Rightarrow \theta(p) \ \geq \ F(x) \\
&\Rightarrow H(p) \ \leq \ -F(x).
\end{aligned}$$

Therefore

$$F(x) + H(p) \ \leq \ 0, \quad \forall x \in X, \ \forall p \in P. \qquad (18)$$

Since $F(\cdot)$ is continuous and X is bounded, closed, the primal program (12) is solvable, i.e., $F(\cdot)$ attains its maximum value at some $\overline{x} \in X$. Then the intersection between X and the upper level set $\{x : F(x) > F(\overline{x})\}$ must be empty. Since

$$\{x : F(x) > F(\overline{x})\} = \{x : x_i > f_i\big(F(\overline{x})\big), \ i = 1, 2, \ldots, n\},$$

by the separation theorem there is a vector $\overline{p}$ such that

$$\overline{p}^t x \leq 1, \quad \forall x \in X, \tag{19}$$

$$\sum_{i=1}^{n} \overline{p}_i x_i > 1, \quad \forall x : x_i > f_i\big(F(\overline{x})\big), \ \ i = 1, 2, \dots, n. \tag{20}$$

From (20) it follows that $\overline{p} \geq 0$. This together with (19) implies $\overline{p} \in P$. Again from (20) it follows that

$$\sum_{i=1}^{n} \overline{p}_i f_i\big(F(\overline{x})\big) \ \geq \ 1.$$

Since $\overline{x}_i \geq f_i(F(\overline{x}))$ and $\overline{x} \in X$, from (19) it follows that

$$\sum_{i=1}^{n} \overline{p}_i f_i\big(F(\overline{x})\big) \ \leq \ 1.$$

Thus

$$\sum_{i=1}^{n} \overline{p}_i f_i\big(F(\overline{x})\big) = 1,$$

which says that $\theta(\overline{p}) = F(\overline{x})$, or equivalently $F(\overline{x}) + H(\overline{p}) = 0$. This together with (18) implies that $\overline{p}$ is optimal to (16). Now suppose that $\overline{x}$ solves the primal program (12) and $\overline{p}$ solves the dual program (16). By the above arguments we have shown $H(\overline{p}) = -F(\overline{x})$, or equivalently, $\theta(\overline{p}) = F(\overline{x})$. Therefore, $\sum_{i=1}^{n} \overline{p}_i f_i(F(\overline{x})) = 1$. Since $\sum_{i=1}^{n} \overline{p}_i \overline{x}_i \geq \sum_{i=1}^{n} \overline{p}_i f_i\big(F(\overline{x})\big)$, one has $\overline{p}^t \overline{x} = \sum_{i=1}^{n} \overline{p}_i \overline{x}_i \geq 1$. On the other hand, since $\overline{x} \in X$ and $\overline{p} \in P$, one has $\overline{p}^t \overline{x} \leq 1$. Thus, $\overline{p}^t \overline{x} = 1$. ∎

One of basic properties of Leontief preference concerns a phenomenon of superfluity. The i-th input factor is called superfluous at a vector x of inputs if the decrement of x_i does not lead to the decrement of $F(x)$, or in other words

$$x_i \ > \ f_i(F(x)),$$

It is obvious that there is no superfluous input factors at a vector x if and only if x is a prototype vector, i.e.

$$x_i = f_i(F(x)), \quad \forall i = 1, 2, \dots, n.$$

The following corollary presents a relation between superfluous properties of optimal input vectors and optimal price vectors.

Corollary 3.1. *If the i-th input factor is superfluous at an optimal vector $\overline{x}$ of inputs then the optimal price of the i-th input factor must be 0, or in other words $\overline{p}_i = 0$ for any optimal solution $\overline{p}$ to the dual program.*

Proof. Suppose that $\overline{x}$ is optimal to the primal program (12) and the i-th input factor is superfluous at $\overline{x}$:

$$\overline{x}_i \ > \ f_i(F(\overline{x})).$$

We define an input vector $\hat{x}$ as follows

$$\hat{x}_i = f_i(F(\overline{x})),$$
$$\hat{x}_j = \overline{x}_j, \quad \forall j \neq i, \ j = 1, 2, \dots, n.$$

Since $\overline{x} \geq \hat{x}$, $\hat{x}$ is feasible to (12). Since $\overline{x}$ is optimal to (12) and

$$\begin{aligned} F(\hat{x}) &= \min\{f_j^{-1}(\hat{x}_j), \ j = 1, 2, \dots, n\} \\ &= \min\{F(\overline{x}), \ f_j^{-1}(\overline{x}_j), \ j \neq i, \ j = 1, 2, \dots, n\} \\ &= F(\overline{x}), \end{aligned}$$

$\hat{x}$ is also optimal to the program (12). By Theorem 3.1, for any price vector $\overline{p}$ which is optimal to the dual program (16) one has

$$\overline{p}^t \overline{x} = 1 = \overline{p}^t \hat{x}.$$

So,

$$0 = \overline{p}^t \overline{x} - \overline{p}^t \hat{x} = \sum_{i=1}^{n} \overline{p}_i (\overline{x}_i - \hat{x}_i) = \overline{p}_i (\overline{x}_i - f_i\big(F(\overline{x})\big).$$

This implies $\overline{p}_i = 0$. ■

4. Discussions

As we have presented, a price vector p is preferred to a price vector q if by selling vector c of inputs the income according to the price p is greater than the income according to the price q, or equivalently if for the same level of income the prototype vector of inputs we need to sell according to the price p is less than that according to the price q. Using this dual preference we can generalize the duality of Leontief production problem to a nonlinear production problem. The dual preference can be defined via the function $H(\cdot)$.

For any $p \geq 0, \ p \neq 0$ one has

$$\sum_{i=1}^{n} p_i f_i(\theta(p)) = 1.$$

Then

$$\begin{aligned} \theta(p) &= \max\{\theta : \sum_{i=1}^{n} p_i f_i(\theta) \leq 1\} \quad \text{(because } f_i(\cdot) \text{ is increasing)} \\ &= \max\{\theta : \sum_{i=1}^{n} p_i x_i \leq 1, \ f_i(\theta) \leq x_i, \ i = 1, 2, \dots, n\} \\ &= \max\{\theta : \sum_{i=1}^{n} p_i x_i \leq 1, \ \theta \leq f_i^{-1}(x_i), \ x_i \geq 0, \ i = 1, 2, \dots, n\} \end{aligned}$$

$$= \max\{\min_{i=1,2,\dots,n}\{f_i^{-1}(x_i)\} : \sum_{i=1}^{n} p_i x_i \le 1,\ x_i \ge 0,\ i = 1,2,\dots,n\}$$

$$= \max\{F(x) : \sum_{i=1}^{n} p_i x_i \le 1,\ x_i \ge 0,\ i = 1,2,\dots,n\}.$$

Since $H(p) = -\theta(p)$, one has

$$H(p) \ = \ -\max\{F(x) : p^t x \le 1,\ x \ge 0\}.$$

This relation between $F(\cdot)$ and $H(\cdot)$ has been known as a quasiconcave conjugation (Refs. [4 - 7]). More relations between $F(\cdot)$ and $H(\cdot)$, for instance the involutory relation

$$F(x) \ = \ -\max\{H(p) : p^t x \le 1, p \ge 0\},$$

can be found in [4 - 7]. Since the dual preference is consistent with that defined via the quasiconcave conjugation, the duality result presented in this article can be regarded as a unified duality scheme applied to a class of quasiconcave maximization problems (cf. [8]).

Acknowledgement. The author expresses his thanks to helpful remarks of an anonymous referee for the revision of this article.

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