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Kernels of Restriction Maps in the mod-p Cohomology of p-Groups

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Dedicated to Professor Huỳnh Mùi on the occasion of his 60th birthday

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Abstract. Let p be a prime number. The purpose of this paper is to investigate kernels of restriction maps in the mod-p cohomology of p-groups. This result is applied to prove that, if K is a 2-group which is not elementary abelian, and if $\xi, \xi_1, \dots, \xi_r, \dots$ is a sequence of mod-2 cohomology classes of K which restrict trivially to all proper subgroups, then $Sq^{n-r}(\xi) \cdot \xi_0 \cdot \xi_1 \cdot \dots \cdot \xi_r = 0$ where $n = \deg(\xi)$ and $\xi_0 = 1$.

1. Statement of Results

Let p be a prime number and let G be a p-group. Denote by $H^*(G) = H^*(G, \mathbb{Z}_p)$ the cohomology ring of G. Suppose that H is a maximal subgroup of G. We can consider a decomposition $G = H \coprod Ht \coprod \dots \coprod Ht^{p-1}$ into disjoint right cosets of H with $t \in G$. Let $\sigma = (1\ p\ p-1\ \dots\ 2)$ be the p-cycle of the symmetric group $\mathfrak{S}_p$. The elementary abelian group $\mathbb{Z}_p^p$ becomes a G-module under the action defined by

$$t^k(a_1, \dots, a_p) = (a_{\sigma^k(1)}, \dots, a_{\sigma^k(p)}), \quad 1 \le k \le p-1,$$

and H acts trivially on $\mathbb{Z}_p^p$. Consider the (non-split) short exact sequence of G-modules

$$0 \longrightarrow \mathbb{Z}_p \xrightarrow{i} \mathbb{Z}_p^p \xrightarrow{j} \mathbb{Z}_p^{p-1} \longrightarrow 0 \tag{1}$$

with diagonal homomorphism

$$i(a) = (a, \dots, a), \quad a \in \mathbb{Z}_p$$

and

$$j(a_1, \dots, a_p) = (a_1 - a_p, \dots, a_{p-1} - a_p), \quad a_k \in \mathbb{Z}_p, \quad 1 \le k \le p.$$

The cohomology long exact sequence associated to (1) is the following (see e. g. [1])

$$\cdots \longrightarrow H^n(G) \xrightarrow{\mathrm{Res}_H^G} H^n(H) \xrightarrow{\mathrm{j}^*} H^n(G, \mathbb{Z}_p^{p-1}) \xrightarrow{\bar{\theta}} H^{n+1}(G) \longrightarrow \cdots \qquad (2)$$

with $\bar{\theta}$ the connecting homomorphism.

We have the following result.

Theorem A. *There exist a G-pairing* $\theta : \mathbb{Z}_p^{p-1} \otimes \mathbb{Z}_p^{p-1} \to \mathbb{Z}_p$ *and an element* $[u_H] \in H^1(G, \mathbb{Z}_p^{p-1})$ *such that, for every* $\xi \in \mathrm{Ker}\ \mathrm{Res}_H^G$,

$$\xi = \theta_*([u_H] \times [\eta]),$$

where $[\eta] \in H^*(G, \mathbb{Z}_p^{p-1})$.

For $p = 2$, we obtain the following exact sequence (which is known to be the Gysin sequence of the O-sphere bundle $BH \to BG$)

$$\dots \longrightarrow H^n(G) \xrightarrow{\mathrm{Res}_H^G} H^n(H) \xrightarrow{\mathrm{cor}_G^H} H^n(G) \xrightarrow{[u_H]\cup} H^{n+1}(G) \longrightarrow \dots . \qquad (3)$$

We now assume that K *is a* 2*-group which is not elementary abelian.* An approach to study $H^*(K)$ is to investigate the restrictions of its elements to cohomology of all proper subgroups of K. Let $Ess(K)$ be the essential cohomology of K consisting of all mod-p cohomology classes of K which restrict trivially to all proper subgroups of K. The following conjecture, stated by Mui, claims that $Ess(K)$ is small in some sence. In other words, it predicts that the discussed approach is somehow essential.

Conjecture. (Mui [3]) *$Ess(K)$ is of nilpotency degree ≤ 2, that is,*

$$Ess(K)^2 = \{0\}.$$

Our work is also an attempt to contribute to a solution of Mui's conjecture. *For convenience, from now on we denote* $\xi_0 = 1 \in H^0(K)$. It was shown by Minh in [2] that $\xi^2 = 0$ for every $\xi \in Ess(K)$. In order to generalize the result of Minh, we prove the following

Theorem B. *Let* $\xi, \xi_1, \dots, \xi_r, \dots$ *be a sequence of elements of* $Ess(K)$ *with* $\deg(\xi) = n$ *then*

$$Sq^{n-r}(\xi) \cdot \xi_0 \cdot \xi_1 \cdot \dots \cdot \xi_r = 0.$$

Especially, for $r = 0$, as $Sq^{n-r}(\xi) \cdot \xi_0 = Sq^n(\xi) = \xi^2$, we obtain the mentioned result of Minh. The proof of Theorem B, which is given in Sec. 3, is based on the Gysin exact sequence in mod-2 cohomology of 2-groups.

2. Proof of Theorem A

Let $\theta : \mathbb{Z}_p^{p-1} \otimes \mathbb{Z}_p^{p-1} \to \mathbb{Z}_p$ be the G-pairing represented by the following $(p-1) \times (p-1)$ matrix

$$\begin{pmatrix} 0 & 1 & \dots & (-1)^{p-3} \\ -1 & 0 & \dots & (-1)^{p-4} \\ \vdots & \vdots & & \vdots \\ -(-1)^{p-3} & -(-1)^{p-4} & \dots & 0 \end{pmatrix} \quad \text{for } p \text{ odd}$$

and $\theta(a \otimes b) = a \cdot b, \ \ a, b \in \mathbb{Z}_2$, for $p = 2$.

Let $X \longrightarrow \mathbb{Z}_p$ be the Bar resolution of G. We define $u_H \in \operatorname{Hom}_G(X_1, \mathbb{Z}_p^{p-1})$ by

$$u_H(g) = \begin{cases} (\underbrace{0, \dots, 0}_{p-1 \text{ times}}) & \text{if } g \in H, \\ (\underbrace{0, 1, \dots, 1}_{p-1 \text{ times}}) & \text{if } g \in Ht, \\ t u_H(t^{k-1}) + u_H(t) & \text{if } g \in Ht^k, \ 2 \le k \le p-1, \end{cases} \qquad \text{for } p \text{ odd}$$

and

$$u_H(g) = \begin{cases} 0 & \text{if } g \in H, \\ 1 & \text{otherwise,} \end{cases} \quad \text{for } p = 2,$$

and extend u_H linearly to X_1.

Lemma A. *u_H is a 1-cocycle.*

Proof. For $p = 2$, as $t^2 \in H$, so we have

$$u_H(g_1 g_2) = \begin{cases} 0 & \text{if } g_1, g_2 \in H \ \textit{or} \ g_1, g_2 \in Ht, \\ 1 & \text{otherwise.} \end{cases}$$

Furthermore, we have

$$u_H(g_1) + u_H(g_2) = \begin{cases} 0 & \text{if } g_1, g_2 \in H \text{ or } g_1, g_2 \in Ht, \\ 1 & \text{otherwise.} \end{cases}$$

So,

$$\begin{aligned} \partial u_H(g_1, g_2) &= g_1 u_H(g_2) - u_H(g_1 g_2) + u_H(g_1) \\ &= u_H(g_2) + u_H(g_1) - u_H(g_1 g_2) = 0, \end{aligned}$$

for every $g_1, g_2 \in G$. Hence, u_H is a 1-*cocycle.*

We will prove the lemma for the case $p > 2$. For every element $g \in G$, we write $g = ht^p$, with $h = gt^{-p} \in H$ if $g \in H$ and $g = ht^k, k > 0$ in otherwise. By the definition above, it may be concluded that

$$u_H(g) = t u_H(t^{k-1}) + u_H(t) = (t^{k-1} + \dots + t + 1) u_H(t)$$

by noting that $u_H(h) = (t^{p-1} + \ldots + t + 1)u_H(t) = (0, \ldots, 0) \in \mathbb{Z}_p^{p-1}$. So, for every $g_1, g_2 \in G, g_1 \in Ht^k, g_2 \in Ht^\ell, 1 \leq k, \ell \leq p$,

$$\begin{aligned}\partial u_H(g_1, g_2) = g_1 u_H(g_2) - u_H(g_1 g_2) + u_H(g_1) &= (t^{k+\ell-1} + \ldots + t^k)u_H(t) \\ &- (t^{k+\ell-1} + \ldots + t + 1)u_H(t) + (t^{k-1} + \ldots + t + 1)u_H(t) \\ = (0, \ldots, 0) \in \mathbb{Z}_p^{p-1}.\end{aligned}$$

Hence, u_H is a 1-*cocycle*. ∎

The case $p = 2$ in the following proposition was due to Rusin [4, Theorem 5] (see also Serre [5, p.12, Ex.2]).

Proposition A. $\bar{\theta}[\eta] = \theta_*([u_H] \times [\eta])$ *for every* $[\eta] \in H^*(G, \mathbb{Z}_p^{p-1})$.

Proof. We are now interested in the case $p > 2$. Recall that $X \longrightarrow \mathbb{Z}_p$ is the Bar resolution of G, so the components X_n of degree n ($n \geq 0$) are free G-modules. Therefore, the sequence

$$0 \longrightarrow \mathrm{Hom}_G(X, \mathbb{Z}_p) \xrightarrow{i^*} \mathrm{Hom}_G(X, \mathbb{Z}_p^p) \xrightarrow{j^*} \mathrm{Hom}_G(X, \mathbb{Z}_p^{p-1}) \longrightarrow 0$$

is exact. The exact sequence (2) is nothing but the cohomology long exact sequence associated to the above short exact sequence.
We define a map $\Omega : \mathrm{Hom}_G(X, \mathbb{Z}_p^{p-1}) \longrightarrow \mathrm{Hom}_G(X, \mathbb{Z}_p^p)$ as follows. For each $\eta \in \mathrm{Hom}_G(X_n, \mathbb{Z}_p^{p-1})$, write $\eta(x) = (\eta_1(x), \ldots, \eta_{p-1}(x))$, $x \in X_n$, with η_i a map from X_n to $\mathbb{Z}_p$. Define a homomorphism $\Omega(\eta) \in \mathrm{Hom}_G(X_n, \mathbb{Z}_p^p)$ by

$$\Omega(\eta)(x) = (\eta_1(x), \ldots, \eta_{p-1}(x), 0), \forall x = (g_1, \ldots, g_n) \in X_n, g_i \in G,$$

and extend $\Omega(\eta)$ linearly to X_n. Then

$$j^*(\Omega(\eta))(x) = j(\Omega(\eta)(x)) = \eta(x), \forall x \in X_n.$$

So

$$j^*(\Omega(\eta)) = \eta.$$

For each $[\eta] \in H^n(G, \mathbb{Z}_p^{p-1})$ with $\eta(x) = (\eta_1(x), \ldots, \eta_{p-1}(x)), x \in X_n$, we have

$$\begin{aligned}0 &= \partial(\eta)(g_1, ..., g_{n+1}) \\ &= g_1\eta(g_2, ..., g_{n+1}) + \sum_{i=1}^{n}(-1)^i\eta(g_1, ..., g_i.g_{i+1}, ..., g_{n+1}) + (-1)^{n+1}\eta(g_1, ..., g_n) \\ &= g_1(\eta_1(g_2, ..., g_{n+1}), ..., \eta_{p-1}(g_2 ..., g_{n+1})) \\ &\quad + \sum_{i=1}^{n}(-1)^i(\eta_1(g_1, ..., g_i.g_{i+1}, ..., g_{n+1}), ..., \eta_{p-1}(g_1, ..., g_i.g_{i+1}, ..., g_{n+1})) \\ &\quad + (-1)^{n+1}(\eta_1(g_1, ..., g_n), ..., \eta_{p-1}(g_1, ..., g_n)).\end{aligned}$$

Besides, $\bar\theta[\eta]$ can be represented by the $(n{+}1)$-cocycle $\bar\theta\eta$ given by

$$(\bar\theta\eta)(g_1,\dots,g_{n+1}) = i^{-1}\left(\partial(\Omega(\eta))(g_1,\dots,g_{n+1})\right)$$

where

$$\begin{aligned}
&\partial(\Omega(\eta))(g_1,\dots,g_{n+1}) = g_1.\Omega(\eta)(g_2,\dots,g_{n+1})\\
&\quad+\sum_{i=1}^{n}(-1)^i\Omega(\eta)(g_1,\dots,g_i.g_{i+1},\dots,g_{n+1}) + (-1)^{n+1}\Omega(\eta)(g_1,\dots,g_n)\\
&= g_1(\eta_1(g_2,\dots,g_{n+1}),\dots,\eta_{p-1}(g_2,\dots,g_{n+1}),0)\\
&\quad+\sum_{i=1}^{n}(-1)^i(\eta_1(g_1,...,g_i.g_{i+1},...,g_{n+1}),...,\eta_{p-1}(g_1,...,g_i.g_{i+1},...,g_{n+1}),0)\\
&\quad+(-1)^{n+1}(\eta_1(g_1,...,g_n),...,\eta_{p-1}(g_1,...,g_n),0)\\
&=\begin{cases}(\underbrace{0,\dots,0}_{\text{p times}}) & \text{if } g_1\in H,\\ (\underbrace{\eta_{p-k}(g_2,...,g_{n+1}),...,\eta_{p-k}(g_2,...,g_{n+1})}_{\text{p times}}) & \text{if } g_1\in Ht^k,\ 1\le k\le p-1.\end{cases}
\end{aligned}$$

So

$$(\bar\theta\eta)(g_1,\dots,g_{n+1}) = \begin{cases}0 & \text{if } g_1\in H,\\ \eta_{p-k}(g_2,\dots,g_{n+1}) & \text{if } g_1\in Ht^k,\ 1\le k\le p-1.\end{cases}$$

Furthermore, we also have

$$\begin{aligned}
&\theta_*(u_H\times\eta)(g_1,\dots,g_{n+1}) = \theta(u_H(g_1)\otimes g_1\eta(g_2,\dots,g_{n+1}))\\
&=\theta(u_H(g_1)\otimes g_1(\eta_1(g_2,\dots,g_{n+1}),\dots,\eta_{p-1}(g_2,\dots,g_{n+1})))\\
&=\begin{cases}0 & \text{if } g_1\in H,\\ \eta_{p-k}(g_2,\dots,g_{n+1}) & \text{if } g_1\in Ht^k,\ 1\le k\le p-1.\end{cases}
\end{aligned}$$

So $\bar\theta\eta = \theta_*(u_H\times\eta)$. Therefore, $\bar\theta[\eta] = \theta_*([u_H]\times[\eta])$. The proposition follows. ∎

Lemma B. $0\ne[u_H]\in H^1(G,\mathbb{Z}_p^{p-1})$.

Proof. Consider the connecting homomorphism $\bar\theta : H^0(G,\mathbb{Z}_p^{p-1}) \longrightarrow H^1(G)$. Proposition A shows that $\bar\theta(a) = \theta_*([u_H]\times a)$ for every $a\in H^0(G,\mathbb{Z}_p^{p-1})$. Choose $0\ne a=(1,2,\dots,p-1)\in(\mathbb{Z}_p^{p-1})^G = H^0(G,\mathbb{Z}_p^{p-1})$. As in the proof of Proposition A, we have

$$\bar\theta(a)(g) = \begin{cases}0 & \text{if } g\in H,\\ p-k & \text{if } g\in Ht^k,\ 1\le k\le p-1.\end{cases}$$

So $\theta_*([u_H]\times a) = \bar\theta(a)\ne 0$. Therefore, $[u_H]\ne 0$. The lemma follows. ∎

Proof of Theorem A. The theorem is proved by combining Proposition A, Lemma A, Lemma B and the exactness of the sequence (2). ∎

3. Proof of Theorem B

We need some lemmas. For $i > 0$, set $S_i = \sum_{0\neq u\in H^1(K)} u^i$. The following was given in [2].

Lemma C. $S_i = 0$. ■

Remark that Lemma C is not true if K is an elementary abelian group.

The following lemma was proved in [2] by using the Evens norm map. We will provide here a simple proof of the first part of it by means of the Gysin sequence.

Lemma D. (Minh [2]) *If* $0 \neq u \in H^1(K)$ *and* $\mathrm{Res}^K_{\mathrm{Ker\ u}}(\xi) = 0$ *with* $\deg(\xi) = n$, *then*

$$\xi^2 = \sum_{j>0} Sq^{n-j}(\xi)\cdot u^j. \tag{4}$$

Hence, if $\xi \in Ess(K)$, *then* $\xi^2 = 0$.

Proof. From the exactness of Gysin sequence (3), we have $\xi = x\cdot u$ where $x \in H^{n-1}(K)$. For all $j, 1 \leq j < n$, we have

$$\begin{aligned} Sq^{n-j}(\xi)\cdot u^j &= \big(\sum_{i\geq 0} Sq^{(n-j)-i}(x)\cdot Sq^i(u)\big)\cdot u^j \\ &= Sq^{n-j}(x)\cdot u^{j+1} + Sq^{n-j-1}(x)\cdot u^{j+2}. \end{aligned}$$

So

$$\begin{aligned} \sum_{j>0} Sq^{n-j}(\xi)\cdot u^j &= \sum_{j=1}^{n-1}\big(Sq^{n-j}(x)\cdot u^{j+1} + Sq^{n-j-1}(x)\cdot u^{j+2}\big) + Sq^0(\xi)\cdot u^n \\ &= Sq^{n-1}(x)\cdot u^2 = \xi^2. \end{aligned}$$

Finally, if $\xi \in Ess(K)$, it follows from what we just proved that

$$\xi^2 = \sum_{0\neq u\in H^1(K)} \xi^2 = \sum_{j>0} Sq^{n-j}(\xi)\cdot S_j.$$

So, by Lemma C, $\xi^2 = 0$. The lemma follows. ■

Let $\mathcal{T}$ be the ideal of $H^*(K)$ generated by the images of corestrictions from proper subgroups of K. We have the following (recall that $\xi_0 = 1 \in H^0(K)$)

Proposition B. *Let* $\xi, \xi_1, \ldots, \xi_r, \ldots$ *be a sequence of elements of* $Ess(K)$ *with* $\deg(\xi) = n$. *For* $r \geq 1$, *we have*

$$Sq^{n-r}(\xi)\cdot\xi_0\cdot\xi_1\cdot\ldots\cdot\xi_{r-1} \in \mathcal{T}.$$

Proof. By induction on r. For any element $0 \neq u \in H^1(K)$, set

$$\xi_u^{(r)} = \sum_{i=r}^{n} Sq^{n-i}(\xi) \cdot \xi_0 \cdot \xi_1 \cdot \ldots \cdot \xi_{r-1} \cdot u^{i-r}.$$

For $r = 1$, by Lemma D, we have $\xi_u^{(1)} \cdot u = \xi^2 = 0$. Following the exactness of the Gysin sequence (3), we have $\xi_u^{(1)} \in \mathcal{T}$. Hence, we have

$$\begin{aligned}\sum_{0\neq u\in H^1(K)} \xi_u^{(1)} &= \sum_{0\neq u\in H^1(K)} Sq^{n-1}(\xi)u^0 + \sum_{i=2} Sq^{n-i}(\xi)(\sum_{0\neq u\in H^1(K)} u^{i-1}) \\ &= Sq^{n-1}(\xi) \in \mathcal{T}.\end{aligned}$$

Suppose that $r > 1$ and the result holds for all $k, 1 \leq k \leq r-1$, that is

$$Sq^{n-k}(\xi) \cdot \xi_0 \cdot \xi_1 \cdot \ldots \cdot \xi_{k-1} \in \mathcal{T}.$$

By the Gysin exact sequence (3), we can decompose $\xi_k = u \cdot x_k$ with $x_k \in H^{\deg(\xi_k)-1}(K)$, $1 \leq k \leq r-1$. Note that, if $\alpha \in \mathcal{T}$ and $\beta \in Ess(K)$ then $\alpha \cdot \beta = 0$ by the Frobenius formula (see [3, Remark]). Therefore, by multiplying (4) with $x_1 \cdot \ldots \cdot x_{r-1}$, we have $\xi_u^{(r)} \cdot u = 0$. It follows from the Gysin exact sequence that there exist $\eta_u \in H^*(\mathrm{Ker}\, u)$, $\xi_u^{(r)} = \mathrm{cor}_K^{\mathrm{Ker\ u}}(\eta_u)$. Hence,

$$\begin{aligned}Sq^{n-r}(\xi) \cdot \xi_0 \cdot \xi_1 \cdot \ldots \cdot \xi_{r-1} &= \sum_{0\neq u\in H^1(K)} \xi_u^{(r)} \quad \text{(by Lemma C)} \\ &= \sum_{0\neq u\in H^1(K)} \mathrm{cor}_K^{\mathrm{Ker\ u}}(\eta_u) \in \mathcal{T}.\end{aligned} \tag{5}$$

The proposition follows. ■

Proof of Theorem B. By Lemma D, the theorem holds for $r = 0$. For $r > 1$, from (5), we have

$$\begin{aligned}Sq^{n-r}(\xi) \cdot \xi_0 \cdot \xi_1 \cdot \ldots \cdot \xi_{r-1} \cdot \xi_r &= \sum_{0\neq u\in H^1(K)} \mathrm{cor}_K^{\mathrm{Ker\ u}}(\eta_u) \cdot \xi_r \\ &= \sum_{0\neq u\in H^1(K)} \mathrm{cor}_K^{\mathrm{Ker\ u}}(\eta_u \cdot \mathrm{Res}_{\mathrm{Ker\ u}}^K(\xi_r)) \\ &= 0 \quad \text{since} \quad \mathrm{Res}_{\mathrm{Ker\ u}}^K(\xi_r) = 0.\end{aligned}$$

This completes the proof of the theorem. ■

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