Vietnam Journal of Mathematics **32**:3 (2004) 303–317

Vietnam Journal of MATHEMATICS

Modules Whose Certain Submodules Are Prime*

M. Behboodi, O. A. S. Karamzadeh, and H. Koohy

Department of Mathematics, The University, Ahvaz, Iran

Received May 5, 2003
Revised October 15, 2003

Abstract. Modules in which every proper submodule (resp. proper nonzero submodule) is prime (called fully prime (almost fully prime)) and with some other related notions are fully investigated. It is shown that over a commutative ring R, an R-module M is fully prime (fully semiprime) if and only if M is a homogeneous semisimple (co-semisimple) module. This in particular shows that a f.g. R-module M is co-semisimple if and only if $\frac{R}{\mathrm{Ann}\,(M)}$ is a regular (von-Neumann) ring. Modules in which nonzero direct summands are prime are also characterized. When R is a one-dimensional Noetherian domain we determine all modules in which the zero submodule is the only prime (semiprime) submodule. Finally, we observe that R is a Max-ring if and only if every R-module contains a prime (semiprime) submodule.

Introduction

All rings in this article are commutative with identity and modules are unital. Let R be a ring. Then an R-module $M \neq 0$ is called a *prime module* if its zero submodule is prime, i.e., the relation $rx = 0$ for $x \in M, r \in R$ implies that $x = 0$ or $rM = (0)$ ($i.e., r \in \mathrm{Ann}\,(M)$). We call a proper submodule N of an R-module M a *prime submodule* of M if M/N is a prime module, i.e., whenever $rm \in N$, then either $m \in N$ or $rM \subseteq N$ for any $r \in R$, $m \in M$. Thus N is a prime submodule of M if and only if $P = \mathrm{Ann}(M/N)$ is a prime ideal of R and M/N is a torsion free $\dfrac{R}{P}$-module. This notion of prime submodule was

*This work was supported by the Research Institute for Fundamental Sciences, Tabriz, Iran.

first introduced and systematically studied in [10, 6] and recently has received a good deal of attention from several authors; see for example [1, 17 - 23, 25, 27, 28], and many others. Clearly, the set of prime submodules of a ring R and its set of prime ideals coincide. Therefore, naturally when dealing with this notion one tries to extend the theory of prime ideals to these objects. Unfortunately, unlike the rings with unity, not every R-module contains a prime submodule; for example $\mathbb{Z}_{p^\infty}$ does not contain a prime submodule; see [18] or [22]. More generally, we know that if R is a domain and not a field, then no divisible R-module M (i.e., $rM = M$ for all $0 \neq r \in R$) has a maximal submodule and we also note that each prime submodule of M is divisible. This simple observation shows that no non-torsion free divisible R-module (R is still a domain) whose proper non-zero submodules have maximal submodules (such as $\mathbb{Z}_{p^\infty}$) can have prime submodules. We also note that if R is a domain and M is a divisible torsion free R-module, then a submodule N of M is a prime submodule if and only if it is a direct summand of M. This simple observation generalizes the fact that (which is Theorem 1 in [18]), if R is a domain and $K \neq R$ is its field of fractions, then K has no maximal R-submodule and the zero submodule of K is the only prime submodule; see also [1].

One can easily see that for any R-module M, if $\mathrm{Hom}(M, R/rad(R)) \neq 0$, where $rad(R)$ denote the nil radical of R, then M contains a prime submodule; see also [22, Corollary1.3]. It is also easy to show that whenever M is an R-module and P is a maximal ideal of R with $M \neq PM$, then each proper submodule of M containing PM is a prime submodule. Thus, we can naturally provide nontrivial rings over which every module has a prime submodule, simply by taking a maximal ideal P in any ring R, then the rings $\dfrac{R}{P^n}$, $n = 1, 2, 3, ...$, give us some natural examples.

We note that if each principal ideal is prime, then R is a field and also R is a regular ring (von-Neumann) if and only if each principal ideal is a semiprime ideal. Motivated by all these simple observations and the fact that the development of the theory of prime submodules is still at its early stage, it becomes of interest to ask: what are the R-modules M such that each proper submodule is prime (semiprime)? And also, what are the modules M such that the zero submodule is the only prime (semiprime) submodule? What can we say about the modules in which nonzero proper direct summands are prime? Finally, what are the rings R such that each R-module has a prime submodule? Our main purpose of this article is to settle all these and some other related questions.

We call an R-module M *fully prime* if every proper submodule is a prime submodule and call it *almost fully prime* if every proper nonzero submodule is a prime submodule. We give several equivalent conditions for an R-module M to be fully prime (almost fully prime). Modules in which each nonzero proper direct summand is prime are determined. We show that over a one-dimensional Noetherian domain R, the simple R-modules and the field of fractions of R are the only modules in which the zero submodule is the only prime submodule. It is also shown that every R-module contains a prime submodule if and only if R is a Max-ring (i.e., every R-module contains a maximal submodule). Finally, we try to extend some of our results to semiprime modules. By a semiprime submodule N of M we mean a proper submodule $N \neq M$ such that $r^2m \in N$ implies $rm \in N$ for all $r \in R$, $m \in M$ and $M \neq 0$ is called semiprime R-module if its zero submodule is a semiprime submodule. We show that each proper submodule of M is a semiprime submodule if and only if M is co-semisimple (see [2, 26]) and give some other equivalent conditions.

This article consists of three sections. In Sec. 1, we study fully (almost fully) prime modules. Sec. 2, is devoted to fully (almost fully) semiprime modules. In Sec. 3, modules with the zero submodule as the only prime submodule are studied. We also give a new characterization of Max-rings. For undefined terms and terminology, the reader is referred to [2, 26].

1. Fully Prime Modules

Clearly, homogeneous semisimple modules are fully prime and each nonzero cyclic submodule of an R-module M is a simple R-module if and only if M is a homogeneous semisimple module. Let us also call an R-module M to be almost prime if each nonzero proper direct summand of M is a prime submodule of M.

We begin with the following useful and evident results.

Lemma 1.1. *Let M be an R-module and $M = M_1 \oplus M_2$. Then each M_i, $i = 1, 2$ is a prime submodule of M if and only if each M_i, $i = 1, 2$ is a prime module by itself.*

Proposition 1.2. *Let M be an R-module. Then the following statements are equivalent.*
1. *M is a prime module.*
2. *Each proper direct summand of M is a prime submodule (i.e., each nonzero summand becomes a prime module by itself).*
3. *All nonzero cyclic R-submodules of M are isomorphic.*
4. *For all $0 \neq m \in M$, $Ann(m) = Ann(M)$.*

Proposition 1.3. *An R-module M is homogeneous semisimple if and only if M is a prime R-module with $soc(M) \neq 0$.*

Corollary 1.4. *Let each nonzero prime ideal in a ring R be maximal. Then an R-module M is prime if and only if M is either a torsion free R-module or a homogeneous semisimple module.*

Next, we learn that an almost prime R-module which is not prime cannot have many summands.

Theorem 1.5. *M is an almost prime R-module if and only if one of the following statements hold.*
1. *M is a prime module.*
2. *M is an indecomposable module.*

3. $M = M_1 \oplus M_2$, *where* $M_1 \neq 0 \neq M_2$ *are unique and each* M_i, $i = 1, 2$ *is an indecomposable prime module.*

Proof. Let M be an almost prime R-module which is neither a prime module nor an indecomposable module, i.e., $M = M_1 \oplus M_2$, where $M_1 \neq 0 \neq M_2$. First, we note that M_i, $i = 1, 2$ are prime submodules of M if and only if they are prime module by themselves, see Lemma 1.1. Hence $Ann(x) = Ann(M_1)$ and $\mathrm{Ann}\,(y) = \mathrm{Ann}\,(M_2)$ for all $0 \neq x \in M_1$, $0 \neq y \in M_2$. Now if $\mathrm{Ann}\,(M_1) = \mathrm{Ann}\,(M_2)$, then $\mathrm{Ann}\,(m) = \mathrm{Ann}\,(M)$ for all $0 \neq m \in M$ and M becomes a prime module which is absurd. Therefore $\mathrm{Ann}\,(M_1) \neq \mathrm{Ann}\,(M_2)$, and we claim that both M_1 and M_2 are indecomposable. To see this, let us assume that one of them, say M_1, is decomposable and $M_1 = P \oplus Q$, $P \neq 0 \neq Q$. Then $M = P \oplus Q \oplus M_2$, i.e., $Q \oplus M_2$ is a prime module which means that $\mathrm{Ann}\,(Q) = \mathrm{Ann}\,(M_2)$. But $\mathrm{Ann}\,(Q) = \mathrm{Ann}\,(M_1)$, i.e., $\mathrm{Ann}\,(M_1) = \mathrm{Ann}\,(M_2)$, which is impossible. Thus we have already shown that both M_1 and M_2 are indecomposable.

Finally, in order to prove the uniqueness of M_1 and M_2, let $M = M_1' \oplus M_2'$ and we show that $\{M_1, M_2\} = \{M_1', M_2'\}$. Now by the first part we note that M_i', $i = 1, 2$ are indecomposable and $\mathrm{Ann}\,(x_i) = \mathrm{Ann}\,(M_i')$, $i = 1, 2$ for all $0 \neq x_i \in M_i'$. We also note that $\mathrm{Ann}\,(M_1') \neq \mathrm{Ann}\,(M_2')$. Therefore without losing generality we may assume that $\mathrm{Ann}\,(M_1) \not\subseteq \mathrm{Ann}\,(M_2)$ and $\mathrm{Ann}\,(M_1') \not\subseteq \mathrm{Ann}\,(M_2')$. Thus there exist $r, s \in R$ with $rM_1 = 0$, $rM_2 \neq 0$ and $sM_1' = 0$, $sM_2' \neq 0$. Now it is clear that either $M_1' \cap M_1 = 0$ or $M_1' \cap M_2 = 0$, for otherwise $M_1' \cap M_1 \neq 0 \neq M_1' \cap M_2$ implies that $\mathrm{Ann}\,(M_1) = \mathrm{Ann}\,(M_1') = \mathrm{Ann}\,(M_2)$, which is absurd. We also claim that $M_1' \cap M_1 = 0 = M_1' \cap M_2$ does not occur. To this end, for each $0 \neq x \in M_1'$ we have $x = y + z$, where $0 \neq y \in M_1$, $0 \neq z \in M_2$, i.e., $rx = rz$, for $ry = 0$. Hence $rx = rz \in M_1' \cap M_2 = 0$ and since $\mathrm{Ann}\,(z) = \mathrm{Ann}\,(M_2)$ we have $rM_2 = 0$, which is absurd. Thus we may assume that $M_1' \cap M_1 = (0) \neq M_1' \cap M_2$, i.e., $\mathrm{Ann}\,(M_2) = \mathrm{Ann}\,(M_1')$. This implies $M_2 \cap M_2' = (0)$, for otherwise we have $\mathrm{Ann}\,(M_1') = \mathrm{Ann}\,(M_2')$, which is impossible. Now for each $x \in M_2'$ we have $x = y + z$, $y \in M_1$, $z \in M_2$ and $rx = rz$ for $rM_1 = 0$. But $rx = rz \in M_2 \cap M_2' = (0)$, i.e., $rz = 0$ which implies that $z = 0$, for otherwise we must have $rM_2 = (0)$, which is impossible. This means that $M_2' \subseteq M_1$. Similarly, for each $0 \neq m \in M_1$, we have $m = m_1' + m_2'$, $m_1' \in M_1'$, $0 \neq m_2' \in M_2'$. Now $rm = 0$ implies that $rm_1' = rm_2' \in M_1' \cap M_2' = (0)$, i.e., $rm_1' = 0$ must imply that $m_1' = 0$, for otherwise $rM_1' = (0)$, i.e., $rM_2 = (0)$, which is not possible. Thus $m_1' = 0$ and $m = m_2' \in M_2'$, i.e., $M_1 = M_2'$. Applying this method and replacing r by s, we see that $M_2 = M_1'$. The converse is evident. ∎

Remark 1. The previous theorem reveals that the only decomposable non-prime $\mathbb{Z}$-modules(i.e., of the form $A \oplus B$, $A \neq 0 \neq B$) which are almost prime $\mathbb{Z}$-module are: $\mathbb{Z}_p \oplus \mathbb{Z}_q$, where $p \neq q$ are arbitrary prime numbers, $\mathbb{Z}_p = \frac{\mathbb{Z}}{(p)}$ and $\mathbb{Z}_p \oplus A$ where A is a torsion free indecomposable $\mathbb{Z}$-module.

The following is now immediate.

Corollary 1.6. *Let M be an R-module with the nonzero socle. Then M is* *almost prime if and only if it is of one of the following forms.*

1. *M is a homogeneous semisimple module.*
2. *M is an indecomposable module.*
3. *$M = M_1 \oplus M_2$, where $M_1 \neq 0 \neq M_2$ are unique and one of them is simple and the other is an indecomposable prime module.*

The following evident lemma is needed.

Lemma 1.7. *An R-module M has only two nonzero proper submodules if and only if it has a unique composition series of length three or is of the form $M = M_1 \oplus M_2$, where M_1 and M_2 are the only simple submodules of M (or, equivalently M_1 and M_2 are non-isomorphic simple submodules).*

Corollary 1.8. *If $M = M_1 \oplus M_2$, where M_1 and M_2 are simple R-modules, then M is either prime or almost prime.*

Proof. If $M_1 \cong M_2$, then M becomes homogeneous semisimple, i.e., it is prime. Otherwise, by the above lemma, M_1, M_2 are the only nonzero proper submodules of M and since they are prime and indecomposable, we are through by Theorem 1.5. ∎

The following gives more information about fully prime modules.

Corollary 1.9. *If M is an R-module, then the following are equivalent.*

1. *M is a fully prime module.*
2. *Each cyclic submodule of M is a prime submodule.*
3. *M is prime and each cyclic submodule of M is semiprime.*
4. *M is a homogeneous semisimple module.*
5. *M is prime and each submodule of M is an intersection of maximal submodules of M.*
6. *M is prime and semisimple.*
7. *M is prime and M has dcc on its f.g. submodules.*
8. *For any two ideals A, B in R and each $m \in M$, Am, Bm are comparable and $A^2m = Am$.*
9. *M is a prime module and $soc(M) \neq 0$.*
10. *M is prime and regular (i.e., each cyclic submodule of M is a summand, see* [15]).

Proof. Evident. ∎

Next, we aim to characterize almost fully prime modules (i.e., each proper nonzero submodule is prime). We have seen that not even a semisimple module can be fully prime unless it is homogeneous semisimple (i.e., homogeneous semisimple). Can a semisimple module be almost fully prime? The following is the answer.

Lemma 1.10. *A semisimple R-module M which is not homogeneous semisimple is almost fully prime if and only if it is a direct sum of two non-isomorphic simple* *modules.*

Proof. Let $M = \sum_{i \in I} \oplus M_i$, where each M_i is simple, be almost fully prime but not homogeneous semisimple. We claim that $\mid I \mid = 2$, for if not, then $\mid I \mid \geq 3$ and consider $M = M_i \oplus P$, where $P = \sum_{i \neq j \in I} \oplus M_j$, i.e., P is fully prime by Lemma 1.1 and therefore P is homogeneous semisimple, i.e., $\mathrm{Ann}\,(P) = \mathrm{Ann}\,(M_j) = \mathrm{Ann}\,(M_k)$, for any $k, j \neq i$. Now we can consider $M = M_j \oplus Q$, where $Q = \sum_{j \neq i \in I} \oplus M_i$ and similarly Q becomes fully prime, i.e., $\mathrm{Ann}\,(Q) = \mathrm{Ann}\,(M_i) = \mathrm{Ann}\,(M_k)$. This means that $\mathrm{Ann}\,(M_i) = \mathrm{Ann}\,(M_j)$ for all $i \neq j$, i.e., M is homogeneous semisimple which is absurd. Thus $\mid I \mid = 2$, and $M = M_1 \oplus M_2$, $M_1 \ncong M_2$. The converse is evident. ∎

The next lemma, although very easy, plays a crucial part in our investigation.

Lemma 1.11. *Let M be an R-module. Then the following statements are equivalent.*

1. *M is a almost fully prime module.*
2. *For each proper nonzero submodule N, M/N is a homogeneous semisimple module.*
3. *For each $0 \neq m \in M$, M/Rm is a homogeneous semisimple module.*

Now we are in a position to characterize almost fully prime modules.

Theorem 1.12. *An R-module M is almost fully prime if and only if it is of one of the following forms.*

1. *M is a homogeneous semisimple module.*
2. *M is a direct sum of two non-isomorphic simple modules.*
3. *Each nonzero cyclic submodule of M contains at most one nonzero proper submodule which is the unique simple submodule of M.*

Proof. Let M be almost fully prime R-module. If M is semisimple, then we are through by Lemmas 1.10, 1.11. Therefore we may assume that for some $0 \neq m \in M$, Rm is not semisimple. Now we claim that Rm contains only one nonzero proper submodule. For let $(0) \neq K$ be a proper submodule of Rm, i.e., by Lemma 1.11, Rm/K is homogeneous semisimple and since it is cyclic it must be simple. This shows that each nonzero proper submodule of Rm is maximal and therefore is minimal in Rm. But Rm with this latter property is either a direct sum of two simple modules, which is not possible or it contains only one nonzero proper submodule, say N_0. We claim that N_0 is contained in any cyclic submodule $(0) \neq Rx$ of M. To see this, it suffices to prove that $Rx \cap Rm = (0)$ leads us to a contradiction. Now we note that in $Rx \oplus Rm$ the submodules Rx and Rm are prime submodules, i.e., they are prime by themselves by Lemma 1.1, i.e., Rm becomes fully prime and therefore is homogeneous semisimple by Proposition 1.9, which is the desired contradiction. Conversely, it suffices to show that condition (3) implies the almost fully primeness of M. In fact, let N_0 be the intersection of all nonzero submodules of M. Now let $0 \neq m \in M$, i.e., either $Rm = N_0$ or Rm/N_0 is simple which means that each nonzero cyclic submodule of M/N_0 is simple and M/N_0 becomes homogeneous semisimple. But for each nonzero proper submodule N of M we have $N_0 \subseteq N$, i.e., N/N_0 is a prime submodule in M/N_0 which implies that N is a prime submodule of M and we are through. ∎

2. Fully Semiprime Modules

Let us recall that a proper submodule N of an R-module M is called semiprime if whenever $r^2m \in N$, then $rm \in N$, where $r \in R$, $m \in M$ and M is called semiprime if its zero submodule is a semiprime submodule. Clearly, each intersection of prime submodules is a semiprime submodule, but not conversely; see [19], where Noetherian rings over which the converse holds are characterized. We also call an R-module M to be fully semiprime if each proper submodule of M is semiprime and we call it almost fully semiprime if each nonzero proper submodule is semiprime. Finally, we recall that if U, M are R-modules, then, following Azumaya, U is called M-injective if for any submodule N of M, each homomorphism $N \longrightarrow U$ can be extended to $M \longrightarrow U$; see [26], where it is shown that if U is P-injective for each cyclic submodule P of M, then U is M-injective. In this section we characterize both fully and almost fully semiprime modules.

Let us begin in this section with the following lemmas which are similar to their counterpart in prime submodules.

Lemma 2.1. *If $M = M_1 \oplus M_2$, then each M_i, $i = 1, 2$ is a semiprime submodule of M if and only if each M_i, $i = 1, 2$ is a semiprime module by itself.*

Lemma 2.2. *Let M be an R-module. Then the following are equivalent.*
1. *M is a semiprime module.*
2. *Each direct summand of M is a semiprime submodule.*
3. *For each $0 \neq m \in M$, $Ann(m)$ is a semiprime ideal.*

The next result shows that over regular rings all modules are fully semiprime.

Theorem 2.3. *Let M be an R-module. Then the following statements are equivalent.*
1. *M is a fully semiprime module.*
2. *Each cyclic submodule of M is a semiprime submodule.*
3. *For each ideal I of R and $m \in M$ we have $I^2m = Im$.*
4. *For any two ideals I, J of R and $m \in M$ we have $IJm = (I \cap J)m = Im \cap Jm$.*
5. *For each $0 \neq m \in M$, $\dfrac{R}{Ann(m)}$ is a regular (von Neumann) ring.*
6. *Each cyclic (f.g) submodule of M is a regular module.*
7. *Each proper submodule of M is an intersection of maximal submodules (i.e., M is co-semisimple module).*
8. *Each cyclic (f.g) submodule of M is co-semisimple.*
9. *Each proper submodule of M is an intersection of prime submodules.*

Proof (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is evident.
(3) $\Rightarrow$ (4). Clearly, $IJm \subseteq (I \cap J)^2 m \subseteq IJm$. Finally, let $am = bm \in Im \cap Jm$, $a \in I$, $b \in J$. Then $a^2 m = abm \in IJm$, i.e., $am \in IJm$. Thus $IJm = (I \cap J)m = Im \cap Jm$.
(4) $\Rightarrow$ (5). In asmuch as $Rm \simeq \dfrac{R}{\mathrm{Ann}\,(m)}$ we infer by our hypothesis that each ideal in $\dfrac{R}{\mathrm{Ann}\,(m)}$ is idempotent, i.e., $\dfrac{R}{\mathrm{Ann}\,(m)}$ is regular.
(5) $\Rightarrow$ (6) is evident.
(6) $\Rightarrow$ (7). Since each nonzero factor module of M has the property in (6), i.e., it suffices to show that the zero submodule is an intersection of maximal submodules of M. In fact, we show that each $0 \neq m \in M$ is excluded by some maximal submodule. But the cyclic submodule Rm certainly contains a maximal submodule, therefore there exists an epimorphism $f : Rm \longrightarrow S$, where S is a simple R-module. We claim that S is M-injective. For, it suffices to show that S is N-injective for each cyclic submodule N of M; (see [26, 16.3(b)]). Thus if $N = Rx$ and K is a submodule of N with a homomorphism $g : K \longrightarrow S$, then we have to show that there exists $h : N \longrightarrow S$ with $h|_K = g$. We note that $\mathrm{Ann}\,(x) \subseteq \mathrm{Ann}\,(K) \subseteq \mathrm{Ann}\,(S)$, i.e., K, S, and N can be considered as an $\dfrac{R}{\mathrm{Ann}\,(x)}$-module. But $\dfrac{R}{\mathrm{Ann}\,(x)}$ is a regular ring and it is well-known that each simple module over a regular ring is injective, i.e., there exists a homomorphism $h : N \longrightarrow S$ over the ring $\dfrac{R}{\mathrm{Ann}\,(x)}$ such that $h|_K = g$. Thus S is M-injective, i.e., the map $f : Rm \longrightarrow S$ can be extended to $f' : M \longrightarrow S$, i.e., $m \notin \ker f'$ and we are through (see [2, Ex 18.23] and [26, 23.1]).
(7) $\Leftrightarrow$ (8). Follows by the easy fact that direct sums and factor modules (and submodules) of co-semisimple modules are again co-semisimple.
(8) $\Rightarrow$ (9) $\Rightarrow$ (1) is evident. ■

Remark 2. Modules with the property in (7) of the above theorem were first called co-semisimple by Fuller in [11]. It was shown in [11] that the class of co-semisimple modules is closed under taking submodules, homomorphic images and direct sums (see also [2, Ex. 18.23] and [26, 23.4]). We observe that these results over commutative rings are evident by condition (3) of our Theorem 2.3, i.e., each R-module contains largest fully semiprime submodule (it might be (0)), see also [11].

In [26, 37.11], it is proved that if M is a finitely generated, self-projective R-module, then M is co-semisimple if and only if $\dfrac{R}{\mathrm{Ann}\,(M)}$ is a regular ring. In view of part (5) of the previous theorem we have the following.

Corollary 2.4. *Let M be a finitely generated R-module. Then M is co-semisimple if and only if $\overline{R} = \dfrac{R}{\mathrm{Ann}\,(M)}$ is a regular ring.*

Corollary 2.5. *M is a f.g. co-semisimple R-module and N is an R-module with* $\mathrm{Ann}\,(M) \subseteq \mathrm{Ann}\,(N)$, *then N is also co-semisimple.*

Let A be an intersection of essential submodules of an R-module M and $\mathcal{A}$ be a nonempty collection of nonzero proper submodules of M such that whenever $B \in \mathcal{A}$, then $B \supseteq A$ and also if $C \supseteq B$ with C a proper submodule of M, then $C \in \mathcal{A}$. Now $\mathcal{F}$ is called an A-filter on M if $\mathcal{F} = \mathcal{A} \cup \Gamma$ where Γ consists of all proper essential submodules of M containing $\bigcap \mathcal{A}$. Clearly, $\mathcal{F}$ can be merely the set of proper essential submodules of M containing A. We also note that there is the largest A-filter containing every A-filter on an R-module M.

In the next two results we may assume that M is not a semisimple R-module.

Proposition 2.6. *Let M be an R-module and $\mathcal{F}$ be an A-filter on it. Then the following are equivalent.*
1. *Each member of $\mathcal{F}$ is an intersection of prime submodules of M.*
2. *For each $N \in \mathcal{F}$, M/N is a co-semisimple module.*
3. *Each member of $\mathcal{F}$ is an intersection of maximal submodules of M.*
4. $\dfrac{M}{\cap\mathcal{F}}$ *is a co-semisimple module.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is evident by Theorem 2.3.
(3) $\Rightarrow$ (4). It suffices to show that the Jacobson radical of each factor module of $\overline{M} = \dfrac{M}{\cap\mathcal{F}}$ is zero. Let M/N, where $N \supseteq \bigcap \mathcal{F}$ is a factor module of $\bar{M}$. We show that each nonzero element $\bar{x} = x + N$ of M/N is excluded by some maximal submodule of M/N. Clearly, $\bar{x}R$ has maximal submodules, i.e., there exists a nonzero homomorphism $f : \bar{x}R \longrightarrow S$, where S is a simple R-module. We know that $\bar{x}R$ is a direct summand of an essential submodule, say $\bar{E} = E/N$ of M/N. Clearly, f can be extended to $\bar{f} : \bar{E} \longrightarrow S$. If $\bar{E} = M/N$, then we are through, for $\bar{x} \notin \ker \bar{f}$. Thus we may assume that $\bar{E} \neq M/N$. We note that E is essential in M and $\ker \bar{f} = P/N = \bar{P}$ is a maximal submodule of $\bar{E}$. We claim that P is essential in E, for if not, then $E = P \oplus Q$, where Q is a simple submodule of E, i.e., $Q \subseteq A \subseteq \cap\mathcal{F} \subseteq N \subseteq P$, which is absurd. This shows that P is essential in M and $P \in \mathcal{F}$. Hence by our hypothesis $P = \bigcap_{i \in I} U_i$, where each U_i is a maximal submodule of M. Clearly, $x + N = \bar{x} \notin \ker \bar{f} = P/N$, i.e., $x \notin P$. Thus there exists some U_i such that $x \notin U_i$, i.e., $\bar{x} \notin U_i/N$ and the proof is complete.
(4) $\Rightarrow$ (1) is evident. ■

The following interesting result (which is in [7] except its part (1)) is now immediate.

Corollary 2.7. *Let M be an R-module. Then the following statements are equivalent.*
1. *Each proper essential submodule of M is an intersection of prime submodules of M.*
2. *For each proper essential submodule N of M, M/N is a co-semisimple module.*

3. *Each proper essential submodule of M is an intersection of maximal submodules of M.*
4. $\dfrac{M}{soc(M)}$ *is a co-semisimple module.*

Clearly, not every co-semisimple R-module is semisimple, e. g. take R to be a non-Artinian regular ring; see also ([2, page 123]). But we have the following.

Proposition 2.8. *A co-semisimple R-module M has a simple prime submodule if and only if it is a direct sum of a simple module and a homogeneous semisimple module.*

Proof. Let N be a simple submodule of M which is also a prime submodule, i.e., M/N is co-semisimple which is a prime module. Then M/N is homogeneous semisimple by Theorem 2.3. Now for any $x \in M \setminus N$, $\dfrac{Rx+N}{N}$ is a simple module and either $\dfrac{Rx+N}{N} \cong Rx$ or $N \subseteq Rx$. Therefore if Rx is not simple, then N is a maximal submodule of Rx and since Rx is also co-semisimple module, it must contain another maximal submodule, say K. Thus $Rx = N + K$ and $N \cap K = (0)$, which implies that K is also a simple module, i.e., Rx is semisimple. But $M = N + \sum_{x \notin N} Rx$ is semisimple and clearly $M = N \oplus L$, where L is homogeneous semisimple module. The converse is evident. ■

The following is now immediate.

Corollary 2.9. *If M is a co-semisimple module with at least two simple prime submodules, then it is either homogeneous semisimple or a direct sum of two non-isomorphic simple modules.*

Next we aim to characterize almost fully semiprime modules. First, we need the following lemma.

Lemma 2.10. *If M is an almost fully semiprime module, then M is either fully semiprime or the intersection of all nonzero submodules of M is equal to $J(M) \neq (0)$, where $J(M)$ denote the Jacobson radical of M.*

Proof. Clearly, each nonzero submodule N of M is an intersection of maximal submodules of M. Now if M is not fully semiprime (i.e., not co-semisimple), then $J(M) \neq (0)$, i.e., $J(M) \subseteq N$ and we are through. ■

Theorem 2.11. *M is an almost fully semiprime module which is not fully semiprime if and only if one of the following statements hold.*

1. $M = \sum_{i=1}^{n} Rm_i$*, where* $(0) \neq Rm_i$ *has at most one nonzero submodule N and N is contained in all nonzero submodules of M.*
2. *There exists a nonzero submodule N contained in each nonzero submodule and M/N is a fully semiprime module which has not finite uniform dimension.*

Proof. Let M be almost fully semiprime and not fully semiprime. Then by the above lemma, $J(M) = N$ is contained in every nonzero submodule of M. Now we consider two cases. First, let M/N have finite uniform dimension. Then since M/N is a fully semiprime module, it must be semisimple, for each of its nonzero cyclic submodules is a regular module with finite uniform dimension, i.e., is semisimple, see [15, Corollary 1]. Now we have $M/N = \sum_{i=1}^{k} \oplus \dfrac{Rm_i}{N}$, where each $\dfrac{Rm_i}{N}$ is simple, i.e., $M = \sum_{i=1}^{k+1} Rm_i$, where $Rm_{k+1} = N$. Finally, if M/N has no finite uniform dimension, then the second statement holds with $N = J(M)$. The converse is evident. ∎

3. P - Rings

As we have observed earlier some modules M have no prime submodules (for example $\mathbb{Z}_{p^\infty}$) and we call them primeless. One can easily see, that in fact any torsion divisible module over a domain is primeless. Conversely, in [21, Proposition 1.4], it is shown that if R is a one-dimensional Noetherian domain, then primeless modules are torsion divisible R-modules. In this section we observe in that we can abandon the Noetherianness and also replace the divisibility by a weaker condition, namely, having no maximal submodule. We prove that if R is a one-dimensional Noetherian domain, then simple R-modules and the field of fractions of R are the only R-modules in which the zero submodule is the only prime (semiprime) submodule. Finally, we characterize the rings with the title of this section (i.e., rings over which every nonzero module has a prime submodule). It is shown that P-rings coincide with Max-rings (i.e., rings over which every module has a maximal submodule). Max-rings were first characterized in [12] as rings R such that R/J is regular and J is T-nilpotent, where J is the Jacobson radical of R. Max- rings which are also called B-rings, were later studied by various authors, see [5, 8, 9, 13, 14, 16].

Let us begin with the following observation which extends Proposition 1.4, in [21].

Proposition 3.1. *Let R be a one-dimensional domain. Then an R-module M is primeless if and only if M is a torsion module with no maximal submodule.*

Proof. We note that if the torsion submodule $T(M)$ of M is proper, then $T(M)$ is a prime submodule, i.e., if M is primeless then we are through. Conversely, let N be a prime submodule of M and we seek a contradiction. We have $M = T(M)$, i.e., we may take $m \in T(M) \setminus N$, with $rm = 0$ for some $0 \neq r \in R$. Thus $rM \subseteq N$, i.e., $(0) \neq P = \mathrm{Ann}\,(M/N)$ is a prime ideal. This shows that M/N is a vector space as an R/P-module and therefore as an R-module it contains maximal submodules, which is not possible. ∎

Now the above result and Proposition 1.4 in [21] immediately yield the following interesting known results.

Corollary 3.2. *Let R be a one-dimensional Noetherian domain and M be an R-module, and suppose that for some nonzero ideal I of R, $IM \neq M$. Then IM is contained in a maximal submodule of M.*

Corollary 3.3. *If R is a one-dimensional Noetherian domain, then an R-module M is divisible if and only if it has no maximal submodule.*

The following observation is useful.

Lemma 3.4. *Let the zero submodule of an R-module $M \neq (0)$ be the only semiprime submodule of M. Then it is the only prime submodule of M.*

Proof. It suffices to show that M is a prime module. To see this, let $rm = 0$, $0 \neq m \in M$, $r \in R$ and $rM \neq (0)$. Set $(0) \neq N = \{m \in M\ :\ rm = 0\} \neq M$, and we claim that N is semiprime. Let $a^2m \in N$, $a \in R$, $m \in M$, i.e., $ra^2m = 0$. Then $(ra)^2m = 0$ and since M is semiprime, $ram = 0$, i.e., $am \in N$, which means that N is a semiprime submodule, a contradiction. ∎

We have already mentioned in the introduction of this article that if K is the field of fractions of a domain $R \neq K$, then the zero submodule of K is the only prime R-submodule of K; see also [18, Theorem 1]. More generally, we have the following.

Proposition 3.5. *Let R be a domain and $K \neq R$ be its field of fractions. Then the zero submodule of K is the only semiprime submodule.*

Proof. Let $(0) \neq N \subset K$ be a semiprime R-submodule of K and we seek a contradiction. Now take any $0 \neq r/s \in N$, $r, s \neq 0$, i.e., $r \in N$ and $rK = K$. Let $x/y \in K \setminus N$, then there exists $z/t \in K$ with $x/y = rz/t$, i.e., $rtz/t^2 \notin N$. But $t^2r(z/t^2) = rz \in N$ and $tr(z/t^2) \notin N$, i.e., N is not a semiprime submodule, which is absurd. ∎

Remark 3. If R is a domain and K is the field of fractions of R, then every prime R-submodule of $M = \sum_{i\in I} \oplus K_i$, $K_i \cong K$ for all $i \in I$, is of the form $\sum_{i\in J\subset I} \oplus K_i$ (up to isomorphism). In particular if $M = K$, then the zero submodule is the only prime (semiprime) submodule.

Next, over one-dimensional Noetherian domains we determine all modules with the zero submodule as their only prime submodule.

Theorem 3.6. *Let R be a one-dimensional Noetherian domain and M be an R-module. Then the following statements are equivalent.*

1. *The zero submodule of M is the only semiprime submodule.*
2. *The zero submodule of M is the only prime submodule.*
3. *M is either a simple R-module, or $M \cong K$, where K is the field of fractions of R.*

Proof. (1) $\Rightarrow$ (2) is evident by Lemma 3.4.

(2) $\Rightarrow$ (3). If the torsion submodule $T(M)$ is a proper submodule, then it is clearly a prime submodule. Thus we must have either $T(M) = (0)$ or $T(M) = M$. First, let us assume that $T(M) = (0)$, i.e., M is a torsion free R-module. Now we claim that M is divisible. To see this, let $0 \neq rM \neq M$ for some $r \in R$ and we obtain a contradiction. By our hypothesis, M/rM is primeless, therefore in view of Proposition 3.1 and Corollary 3.2, we infer that M/rM is a torsion and divisible R-module. But $\frac{M}{rM} = r\frac{M}{rM}$ implies that $M = rM$, which is absurd. Therefore we may assume that M is a torsion free divisible R-module. Then M becomes a vector space over K by defining $\frac{1}{b}m = x$, where $0 \neq b \in R$ and $m = bx$ for some $x \in M$ and $\frac{a}{b}m = ax$ for all $0 \neq b \in R$, $a \in R$. Hence $M = \sum_{i \in I} \oplus K_i$, $K_i \cong K$ for all $i \in I$. But it is clear that if $| I | \geq 2$, then M has nonzero prime submodules, which is absurd, (see Remark 3), therefore $M = K$ in this case. Finally, if $T(M) = M$, then by Proposition 3.1 and Corollary 3.2, M is not divisible. This means that $rM \neq M$ for some $0 \neq r \in R$. Now we claim that M is simple, for otherwise there exists $0 \neq m \in M$, $Rm \neq M$, i.e., $\frac{M}{Rm}$ is primeless. Therefore in view of Proposition 3.1 and its corollary we infer that $\frac{M}{Rm}$ is divisible, i.e., we have $\frac{M}{Rm} = r\frac{M}{Rm}$ which means that $M = rM + Rm$, i.e., $rM \neq (0)$. Now $\frac{M}{rM}$ is primeless, i.e., $\frac{M}{rM}$ is divisible by Corollary 3.2. Thus $\frac{M}{rM} = r\frac{M}{rM}$ implies that $M = rM$, which is absurd. This means that M must be a simple R-module in this case.
(3) $\Rightarrow$ (1) is clear. ■

We need the following lemma.

Lemma 3.7. *Let R be a ring with a non-maximal prime ideal P. Then there exists an R-module which is primeless (even, semiprimeless).*

Proof. Set $R' = R/P$ and let K be the field of fractions of R'. Now by Remark 3, the only prime (semiprime) R'-submodule of K is its zero submodule. This implies that for each $(0) \neq N \subset K$, the R'-module K/N is primeless (semiprimeless). Thus K/N as an R-module is also primeless (semiprimeless). ■

We conclude this section with the following characterization of Max-rings and its interesting corollary.

Theorem 3.8. *The following statements are equivalent.*
1. *R is a P-ring.*
2. *Every R-module has a semiprime submodule.*
3. *R is a Max-ring.*

Proof. (1) $\Rightarrow$ (2) is evident.
(2) $\Rightarrow$ (3). By Lemma 3.7 each prime ideal is maximal. Now let M be an R-module and $N \subset M$ be a semiprime submodule of M, i.e., $I = \mathrm{Ann}\,(M/N)$ is

a semiprime ideal. We note that R/I is a semiprime ring whose prime ideals are maximal, therefore by Kaplansky's result (see the proof of Corollary 2.4) R/I becomes a regular ring. Thus M/N is a fully semiprime R/I-module by Theorem 2.3, i.e., M/N and therefore M, has maximal submodules and the proof is complete.
(3) $\Rightarrow$ (1) is evident. ∎

The proof of the previous theorem and Lemma 3.8 immediately yield the following which shows that in Max-rings each element is either a zero divisor or a unit; see [12, Lemma 2] and see also [9].

Corollary 3.9. *If R is a Max-ring, then each prime ideal is maximal (i.e., is a minimal prime).*

Acknowledgement. We would like to thank Professor R. Wisbauer for his very useful comments on an earlier version of this article.

References

1. S. Abu-Saymeh, On Dimensions of Finitely Generated Modules, *Comm. Algebra* **23** (1995) 1131–1144.
2. F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Springer-Verlag, New York, 1973.
3. M. Behboodi and H. Koohy, On minimal prime submodules, *Far East J. Mathematical Sciences* **6** (2002) 83–90.
4. K. Beidar and R. Wisbauer, Properly Semiprime Self-pp-modules, *Comm. Algebra* **23** (1995) 841–861.
5. V. P. Camillo, On some rings whose modules have maximal submodules, *Proc. Amer. Math. Soc.* **55** (1975) 97–100.
6. J. Dauns, Prime modules, *J. Reine Angew. Math.* **298** (1978) 156–181.
7. N. V. Dung, D. V. Huynh, P. F. Smith, and R. Wisbauer, *Extending Modules*, Pitman Research Notes in Math, Ser. 313, Longman Sci. and Tech., 1994.
8. C. Faith, *Algebra II, Ring Theory*, Springer-Verlag, 1973.
9. C. Faith, Locally perfect commutative rings are those whose modules have maximal submodules, *Comm. Algebra* **23** (1995) 4885–4886.
10. E. H. Feller and E. W. Swokowski, Prime modules, *Canad. J. Math.* **17** (1965) 1041–1052.
11. K. R. Fuller, Relative projectivity and injectivity class determined by simple modules, *J. London Math. Soc.* **5** (1972) 423–431.
12. R. Hamsher, Commutative rings over which every module has a maximal submodule, *Proc. Amer. Math. Soc.* **18** (1967) 1133–1137.
13. Y. Hirano, On rings over which each module has a maximal submodule, *Comm. Algebra* **26** (1998) 3435–3445.
14. O. A. S. Karamzadeh and A. R. Sajedi Nejad, On the loewy length and noetherian dimension of Artinian modules, *Comm. Algebra* **30** (2002) 1077–1084.

15. O. A. S. Karamzadeh, *Modules whose countably generated submodules are epimorhpic images*, *Colloq. Math.* **46** (1982) 143–146.
16. L. A. Koifmann, Rings over which every module has a maximal submodule, *Math. Zametki* (1970) 350–367.
17 C. Lu, Prime submodules of modules, *Math. Univ. Sancti. Pauli* **33** (1984) 61–69.
18. C. P. Lu, Spectra of modules, *Comm. Algebra* **23** (1995) 3741–3752.
19. S. H. Man, On Commutative Noetherian Rings which have the s.p.a.r. property, *Arch. Math.* **70** (1998) 31–40.
20. R. L. McCasland and M. E. Moore, Prime submodules, *Comm. Algebra* **20** (1992) 1803–1817.
21. R. L. McCasland, M. E. Moore, and P. F. Smith, On the spectrum of a module over a commutative ring, *Comm. Algebra* **25** (1997) 79–103.
22. R. L. McCasland and P. F. Smith, Prime submodules of Noetherian modules, *Rocky Mountain J. Math.* **23** (1993) 1041–1062.
23. Y. Tiras, A. Harmanci, and P. F. Smith, A characterization of prime submodules, *J. Algebra* **212** (1999) 743–752.
24. H. Tsutsui, Fully prime rings, *Comm. Algebra* **24** (1996) 2981–2989.
25. R. Wisbauer, Co-semisimple modules and nonassociative V-rings, *Comm. Algebra* **5** (1977) 1193–1209.
26. R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Reading, 1991.
27. R. Wisbauer, *Modules and Algebras*, Pitman Monographs 81, Longman, Harlow, 1996.
28. R. Wisbauer, On prime modules and rings, *Comm. Algebra* **11** (1983) 2249–2265.