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Moment Spaces on H^{∞}*

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Dedicated to Samuel Karlin on the occasion
of his 80th birthday with gratitude, esteem and friendship

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Abstract. In this paper, we study the geometry of the space of moments, relative to a weak Chebyshev system, of functions holomorphic in the unit disc which are real-valued on the real axis. We use our conclusion to identify the envelope of all functions which are real-valued on the real axis, whose k-th derivative is holomorphic in the disc and bounded by one there.

1. Introduction

We let $\Delta := \{z : |z| < 1\}$ be the unit disc, $T := \{z : |z| = 1\}$ its boundary, $I := [-1, 1]$ and $H^{\infty}(\Delta)$ the Banach space of all complex-valued analytic functions f with norm

$$\|f\|_{\Delta} := \sup\{|f(z)| : \; z \in \Delta\}.$$

The subspace of functions in $H^{\infty}(\Delta)$ which are *real* on the interval I shall be denoted by $H_r^{\infty}(\Delta)$. That is, by the reflection principle they satisfy the equation

$$f(z) = \overline{f(\overline{z})}, \quad z \in \Delta.$$

For the unit ball in $H_r^{\infty}(\Delta)$, we use

$$U(\Delta) := \{f : \; f \in H_r^{\infty}(\Delta), \; \|f\|_{\Delta} \leq 1\}$$

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and by analogy $U(I)$ stands for the corresponding unit ball in the space of real-valued functions in $L^\infty(I)$.

Let E be a closed subset of $I^0 := (-1, 1)$ and $\mathrm{d}\mu$ a Borel measure on E with $\mu(E) > 0$, having no atoms there. We set $Z_n := \{0, 1, \dots, n-1\}$ and prescribe a set of functions $\{u_j : j \in Z_n\}$ in $C(I)$ which are linearly independent on E. The object of our study in this paper is the moment space $M(\Delta)$ generated by these functions which is defined by

$$M(\Delta) := \{((u_j, h) : j \in Z_n)) : \ h \in U(\Delta)\}$$

where we set

$$(u, h) := \int_E u(t)h(t)\mathrm{d}\mu(t).$$

We also use $M(I)$ for the moment space corresponding to $L^\infty(I)$.

When the functions $\{u_j : j \in Z_n\}$ form a *Chebyshev System* on I, the moment space $M(I)$ is considered in [8, Chapter VIII]. Here, we provide similar results for $M(\Delta)$ when these functions form a *weak* Chebyshev system on I. Specifically, our goal here is two fold: to provide a version of the *Markoff-Krein* theorem as presented in [8, Chapter III], under our circumstances, and to apply this theorem to the solution of the following extremal problems. To describe them, we let, for every positive integer k, $W_k^\infty(I)$ be the Sobolev class of all real-valued functions f such that all of its derivatives $f^{(j)}$, $j \in Z_k$ are *absolutely continuous* on I and that $f^{(k)} \in L^\infty(I)$. We use $S_k(\Delta)$ for the subset of functions defined by

$$S_k(\Delta) := \{f : f \in W_k^\infty(I), \quad f^{(k)} \in U(\Delta)\}.$$

We specify an increasing sequence of points $\pi := \{t_j : j \in Z_{k+n}\}$ in I^0, where n is a nonnegative integer, data $Y := \{y_j : j \in Z_{k+n}\}$ and let $D(Y)$ be the subspace of all functions f in $C(I)$ such that $f(t_j) := y_j, \quad j \in Z_{k+n}$. For any $x \in I$, we consider the extremal problems

$$\underline{f}(x)_\Delta := \min\{f(x) : f \in D(Y) \cap S_k(\Delta)\} \tag{1.1}$$

and

$$\overline{f}(x)_\Delta := \max\{f(x) : f \in D(Y) \cap S_k(\Delta)\}, \tag{1.2}$$

which determine the *envelope* of the function class $S_k(\Delta)$ defined to be the boundary of the points in the planar region $\mathcal{S}_k(\Delta) := \{(x, y) : y \in [\underline{f}(x)_\Delta, \overline{f}(x)_\Delta],$ $x \in I\}$. We shall give a description of this set. Our motivation and guide for this task comes from our early interest concerning the planar region $\mathcal{S}_k(I)$ which was important in [10] for accelerating the convergence of the binary search algorithm for finding a root of a function, see [10] and also [11, 12] for further details. The study of the envelope $\mathcal{S}_k(I)$ was also considered in [4] as well as in [5-6, 13] where its relevance to *optimal* interpolation is discussed.

2. The Moment Space $M(\Delta)$

We begin by reviewing necessary properties of weak Chebyshev systems. To this end, we adopt the following terminology and notation from [9].

Definition 2.1. *A set of functions $\{u_j : j \in Z_n\}$ which are continuous on the interval I is called a weak Chebyshev system provided that for all $-1 \leq t_1 < t_2 < \cdots < t_n \leq 1$ the $n \times n$ determinant*

$$U\begin{pmatrix} t_0, t_1, \ldots, t_{n-1} \\ u_0, u_1, \ldots, u_{n-1} \end{pmatrix} := det(u_j(t_l))_{j,l \in Z_n}$$

is nonnegative. When all these determinants are positive, these functions are called a Chebyshev system.

We use U to denote the subspace spanned by the set of functions $\{u_j : j \in Z_n\}$ and K for the convex cone of all functions $u_n \in C(I)$ such that the set of functions $\{u_j : j \in Z_{n+1}\}$ form a weak Chebyshev system. We say that an $h \in L^\infty(E)$ has m weak sign changes provide that there are points $t_1, t_2, \ldots, t_m \in E$ with $-1 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$ and an integer r such that for $t \in (t_j, t_{j+1}) \cap E$, $j \in Z_{m+1}$ we have that

$$h(t) = (-1)^{j+r}|h(t)|.$$

In addition, we say h is positively oriented, if $r = m$, and negatively oriented, if $r = m - 1$. Note that a positively oriented function is nonnegative on the last interval (t_m, t_{m+1}). When h has a finite number of weak sign changes on E we let $S^-(h)$ denote the maximum number of such sign changes, otherwise we set $S^-(h) = \infty$.

Lemma 2.1. *If the set of functions $\{u_j : j \in Z_n\}$ form a weak Chebyshev system on the interval I, E is a closed subset of I with $\mu(E) > 0$ on which these functions are linearly independent, h a function in $L^\infty(E)$ with the property that $\mu\{\{t : h(t) = 0,\ t \in E\}\} = 0$ and*

$$(u_j, h) = 0, \quad j \in Z_n$$

then $S^-(h) \geq n$.

The proof of this lemma follows from the same line of reasoning used to prove Lemma 1 in [9], which covers the case when $d\mu$ is Lebesgue measure.

The next lemma provides us information about the orientation of the function h in Lemma 2.1 when it has the least number of sign changes.

Lemma 2.2. *Let the functions $\{u_j : j \in Z_n\}$ form a Chebyshev system on I and h satisfy the hypotheses of Lemma 2.1. In addition, if h has exactly n sign changes, is positively oriented, the functions $\{u_j : j \in Z_{n+1}\}$ form a weak Chebyshev system and are linearly independent on E then*

$$(u_n, h) > 0.$$

Proof. By hypothesis there exists points $\{t_j : j \in Z_{n+2}\} \subseteq E$ with $-1 = t_0 < t_1 < \cdots < t_n < t_{n+1} = 1$ so that for $t \in E_j := (t_j, t_{j+1}) \cap E$, $j \in Z_{n+1}$, there holds

$$h(t) = (-1)^{i+n}|h(t)|.$$

We consider the function u defined for $t \in I$ by the equation

$$u(t) = U\begin{pmatrix} t_0, & t_1, \dots, t_{n-1}, t \\ 0, & 1, \dots, n-1, n \end{pmatrix}.$$

This function can be written in the form

$$u = \sum_{j \in Z_{n+1}} a_j u_j$$

for some constants $\{a_j : j \in Z_{n+1}\}$ where

$$a_n = U\begin{pmatrix} t_0, & t_1, \dots, t_{n-1} \\ 0, & 1, \dots, n-1 \end{pmatrix} > 0.$$

Therefore, we have that

$$a_n(u_n, h) = (u, h) = \sum_{i \in Z_{n+1}} \int_{E_i} u(t)h(t)\mathrm{d}\mu(t) = \int_E |u(t)|\ |h(t)|\mathrm{d}\mu(t) > 0$$

and we conclude that $(u_n, h) > 0$. ∎

We now introduce an useful *norm* on R^n induced by $M(\Delta)$. Specifically, for each $y = (y_j : j \in Z_n) \in R^n$ we let

$$\|y\| := \min\{\|h\|_\Delta : (u_j, h) = y_j, \quad j \in Z_n, \quad h \in H_r^\infty(\Delta)\}. \tag{2.1}$$

In the definition of this norm we only require that the function $\{u_j : j \in Z_n\}$ are in $L^1(E, \mathrm{d}\mu)$.

The minimum is achieved because, by a normal family argument, every minimizing sequence contains a subsequence which converges in $H_r^\infty(\Delta)$. The next lemma use this norm to characterize $\partial M(\Delta) :=$ boundary of $M(\Delta)$.

Lemma 2.3. *If the functions $\{u_j : j \in Z_n\}$ in $L^1(E, d\mu)$ are linearly independent on E then $y \in M(\Delta)$ if and only if $\|y\| \le 1$ where equality holds if and only if $y \in \partial M(\Delta)$.*

Proof. This fact is straightforward to prove. Indeed, by definition, $y \in M(\Delta)$ if and only if $\|y\| \le 1$. If $\|y\| < 1$ there is an $h \in H_r^\infty(\Delta)$ with $\|h\|_\Delta < 1$ such that

$$(u_j, h) = y_j, \quad j \in Z_n.$$

Now, choose any functions $\{v_j : j \in Z_n\} \subset H_r^\infty(\Delta)$ such that

$$(u_j, v_k) = \delta_{jk}, \quad j, k \in Z_n.$$

Therefore, for any vector $x = (x_j : j \in Z_n) \in R^n$ for which

$$\|x\|_1 := \sum_{j \in Z_n} |x_j| \le \kappa^{-1}(1 - \|h\|_\Delta)$$

where $\kappa := \max\{\|v_j\|_\Delta : j \in Z_n\}$, the function

$$H := h + \sum_{j \in Z_n} x_j v_j$$

is in $U(\Delta)$ and satisfies the moment conditions

$$(u_j, h) = y_j + x_j, \quad j \in Z_n.$$

This implies that $y \notin \partial M(\Delta)$. Conversely, if $y \in M^0(\Delta)$ then for some $\epsilon > 0$ there is an $h \in U(\Delta)$ such that

$$(u_j, h) = (1+\epsilon) y_j, \quad j \in Z_n.$$

Hence, we conclude that $\|y\| \leq (1+\epsilon)^{-1} < 1$. ∎

Our next goal is to characterize the boundary of $M(\Delta)$ in terms of Blaschke products. Recall that a function $B \in H^\infty(\Delta)$ is a Blaschke product of degree n if for some constant $\lambda \in T$ and points $\{z_j : j \in Z_n\} \subseteq \Delta$, we have for all $z \in \Delta$ that

$$B(z) = \lambda \prod_{j \in Z_n} \frac{z - z_j}{1 - z\overline{z}_j}.$$

The Blaschke product B has a modulus which is identically one on T and B is in $U(\Delta)$ if and only if $\lambda = \pm 1$ and its zeros satisfy the condition that $\{z_j : j \in Z_n\} = \{\bar{z}_j : j \in Z_n\}$. We denote the set of all Blaschke products of degree n or less by $\mathcal{B}^n$ and a finite Blaschke product is any function in

$$\mathcal{B} := \bigcup_{n \in Z_+} \mathcal{B}^n.$$

Likewise, we set $\mathcal{B}_r^n := \mathcal{B}^n \cap H_r^\infty(\Delta)$ and $\mathcal{B}_r := \mathcal{B} \cap H_r^\infty(\Delta)$.

For any $y \in M(\Delta)$ we say that the function h *represents* y provided that $h \in U(\Delta)$ and $(u_j, h) = y_j$, $j \in Z_n$. Our first result characterizes the boundary of $M(\Delta)$ in terms of Blaschke products.

Theorem 2.1. *If the set of functions $\{u_j : j \in Z_n\}$ form a weak Chebyshev system and E is a closed subset of I^0 on which these functions are linearly independent then $y \in \partial M(\Delta)$ if and only if y is represented by a Blaschke product in $\mathcal{B}_r^{n-1}$. Moreover, when this is the case this representation is unique.*

We prove this result by means of a series of lemmas, the first of which is a useful fact obtainable from [3].

Lemma 2.4. *There exists an odd continuous mapping $F : R^n \backslash \{0\} \to \mathcal{B}_r^{n-1}$ where $\mathcal{B}_r^{n-1}$ is given the topology of uniform convergence on compact subsets of Δ.*

From this lemma follows the next result.

Lemma 2.5. *If the set of functions $\{u_j : j \in Z_n\}$ form a weak Chebyshev system and E is a closed subset of I on which these functions are linearly independent then for every $y = (y_j : j \in Z_n) \in R^n \backslash \{0\}$ there is a $B \in \mathcal{B}_r^{n-1}$ which represents $\|y\|^{-1}y$.*

Proof. Let us first prove the existence of $B \in \mathcal{B}_r^{n-1}$ which represents $\|y\|^{-1}y$. To this end, we first extend the vector y to a basis $\{y^j : j \in Z_n\}$ of R^n where we choose $y^{n-1} := y$. Next, we define a map

$$G := (G_j : j \in Z_n) : \ R^n \backslash \{0\} \rightarrow R^{n-1}$$

by setting for each $w \in R^n \backslash \{0\}$ and $j \in Z_{n-1}$

$$G_j(w) := \sum_{k \in Z_n} y_k^j (u_k, F(w)),$$

where the map $F : R^n \backslash \{0\} \rightarrow \mathcal{B}_r^{n-1}$ is given in Lemma 2.4. We observe that $G : \ R^n \backslash \{0\} \rightarrow R^{n-1}$ is an odd continuous map and so by the Borsuk antipodal mapping theorem there is a $w \in R^n \backslash \{0\}$ for which $G(w) = 0$. Thus, there is an $s \in R^n$ and $\hat{B} \in \mathcal{B}_r^{n-1}$ such that

$$(u_j, \hat{B}) = s y_j, \quad j \in Z_{n-1}. \tag{2.2}$$

By Lemma 2.1 we see that $s \neq 0$. Therefore, when we set $t := |s|$ and $B := (sgn\ s)\hat{B}$ so that (2.4) implies that

$$(u_j, B) = t y_j, \quad j \in Z_n,$$

which provides the conclusion that $\|y\| \leq t^{-1}$. We shall show that $\|y\| = t^{-1}$. Suppose to the contrary that $\|y\| < t^{-1}$. We choose an $f \in H_r^\infty(\Delta)$ which represents y such that $\|f\|_\Delta = \|y\|$ (we prove in Lemma 2.6 that any such f is in $\mathcal{B}_r$), let $g = f - t^{-1}B$ and observe for $z \in T$ that

$$|g(z) + t^{-1}B(z)| \leq \|f\|_\Delta < |t^{-1}B(z)|.$$

Since $B \in \mathcal{B}_r^{n-1}$, we conclude, from Rouche's theorem, that g has at most $n-1$ zeros in Δ. However, we have for $j \in Z_n$ that

$$(u_j, g) = (u_j, f) - t^{-1}(u_j, B) = 0.$$

Invoking Lemma 2.1, we conclude that either g is identically zero or has at least n zeros. In either case, we derive a contradiction, thereby proving the result. ∎

Below we shall prove that the Blaschke product appearing in Lemma 2.5 is the *unique* representer of $\|y\|^{-1}y$. We start with Lemma 2.6.

Lemma 2.6. *If the functions $\{u_j : j \in Z_n\} \subset L^1(E, d\mu)$ are linearly independent on E, a closed subset of I^0, then every $y \in \partial M(\Delta)$ has a unique representer and this representer must be in $\mathcal{B}_r$.*

Proof. Let $y = (y_j : j \in Z_n) \in \partial M(\Delta)$ then there is $(x_j : j \in Z_n) \in R^n \backslash \{0\}$ such that for all $f \in U(\Delta)$

$$\sum_{k\in Z_n} x_k y_k \geq \sum_{k\in Z_n} x_k (u_k, f).$$

We choose a function $f_0 \in \partial U(\Delta)$ which represents y. Therefore, by the Cauchy integral formula, we have the inequality

$$\frac{1}{2\pi}\int_T f_0(e^{i\theta})g(e^{i\theta})\mathrm{d}\theta \geq \frac{1}{2\pi}\int_T f(e^{i\theta})g(e^{i\theta})\mathrm{d}\theta. \tag{2.3}$$

where the function g is defined for $z \in \Delta$ by the equation

$$g(z) := z\int_E \frac{\sum_{i\in Z_n} x_i u_i(t)}{z-t}\mathrm{d}\mu(t).$$

This function is analytic in $C\backslash E$ and real-valued on I. Let $H^1(\Delta)$ consist of all those analytic functions h on Δ for which

$$\sup\left\{\frac{1}{2\pi i}\int_T |h(r\zeta)|\frac{\mathrm{d}\zeta}{\zeta} : 0 < r < 1\right\} < \infty$$

and $H_0^1(\Delta)$ those functions in $H^1(\Delta)$ which vanish at the origin. Since the function g is in $L^1(T)$ and satisfies the condition $\overline{g(\zeta)} = g(\overline{\zeta})$, $\zeta \in T$, it has a best approximation $g_0 \in H_0^1(\Delta)$ which is real on I. We learn more about this function from the following result which is a consequence of [14, Proposition 6].

Lemma 2.7. *Let u be a function analytic in an annular region containing T. If v is any best $H_0^1(\Delta)$ approximation to u in $L^1(T)$ then v is analytic in a neighborhood of T.*

We apply this lemma to the above circumstance to conclude that g_0 is analytic in a neighborhood of Δ. Moreover, by a standard extremal argument cf. [14] inequality (2.5) implies that

$$f_0(\zeta)(g(\zeta) + g_0(\zeta)) \geq 0, \text{ a.e. } \zeta \in T \tag{2.4}$$

and

$$|f_0(\zeta)| = 1, \quad \text{a.e. } \zeta \in T. \tag{2.5}$$

We wish to prove from equations (2.4) and (2.5) that f_0 is a Blaschke product. For the proof of this claim we use the following fact [15, Lemma 4.5].

Lemma 2.8. *Let Γ be an open arc in T. If $f \in H^\infty(\Delta)$, $g \in H^1(\Delta)$, fg can be continued analytically across Γ and $|f(\zeta)| = 1$, a.e. $\zeta \in T$ then f and g can be continued analytically across Γ.*

This lemma ensures that f_0 can be extended analytically across T. Therefore, we conclude that $f_0 \in \mathcal{B}$. Also, it follows from (2.4) and (2.5) that f_0 is unique and must be in $\mathcal{B}_r$. This proves Lemma 2.6. ∎

Lemma 2.9. *If the functions* $\{u_j : j \in Z_n\} \in L^1(E, d\mu)$ *are linearly independent on* E*, a closed subset of* I^0*, then for every* $y \in R^n \backslash \{0\}$ *there exists a unique solution to the minimum problem* (2.4) *and this function must be in* $\|y\|^{-1}\mathcal{B}_r$*.*

Proof. For every $y \in R^n \backslash \{0\}$ we have by Lemma 2.3 that $\|y\|^{-1}y \in \partial M(\Delta)$. Moreover, a function $f \in H_r^\infty(\Delta)$ solves the minimum problem (2.4) if and only if the function $\|y\|^{-1}f$ represents the vector $\|y\|^{-1}y$. Thus, the result follows from Lemma 2.6. ∎

If the functions $\{u_j : j \in Z_n\}$ satisfy the hypothesis of Lemma 2.5 the only function $f \in U(\Delta)$ which represents $\|y\|^{-1}y$ is the Blaschke product constructed in Lemma 2.5. In particular, it is the unique representation of the vector $\|y\|^{-1}y$ in $\mathcal{B}_r^{n-1}$.

We are now ready to prove Theorem 2.1.

Proof. (Theorem 2.1). If $y \in \partial M(\Delta)$ then $\|y\| = 1$ and so the result follows from Lemma 2.5 and the above remark. Conversely, if $B \in \mathcal{B}_r^{n-1}$ we define the vector $y = (y_j : j \in Z_n) \in R^n$ by setting $y_j = (u_j, B)$, $j \in Z_n$ then Lemma 2.1 implies that $y \neq 0$. Consequently, $\|y\|^{-1}y \in \partial M(\Delta)$ by Lemma 2.3 and the function $\|y\|^{-1}B$ represents it. But then, by Lemma 2.5, $\|y\|^{-1}B \in \mathcal{B}_r$ and, in particular, $\|y\| = 1$. Thus Lemma 2.3 tells us that $y \in \partial M(\Delta)$. ∎

Our next goal is to construct two distinguished representers for any interior point of $M(\Delta)$.

Theorem 2.2. *If the functions* $\{u_j : j \in Z_n\}$ *form a weak Chebyshev system and* E *is a closed subset of* I^0 *on which these functions are linearly independent then every* $y \in M^0(\Delta)$ *has exactly two representers in* $\mathcal{B}_r^n$*, one is positively oriented* $B_+ \in \mathcal{B}_r^n$ *and the other* $B_- \in \mathcal{B}_r^n$ *is negatively oriented. Moreover, if* u_n *is in the convexity cone* K *of the functions* $\{u_j : j \in Z_n\}$ *then for every* $h \in U(\Delta)$ *which represents* y *we have that*

$$(u_n, B_-) \leq (u_n, h) \leq (u_n, B_+). \tag{2.6}$$

This theorem embodies our version for the moment space $M(\Delta)$ of the Markoff-Krein theorem for $M(I)$, as presented in [8]. We note that not only each of the Blaschke products B_- and B_+ constructed in Theorem 2.2 has n simple zeros in I^0 but also their difference $B_- - B_+$ has n simple zeros in I^0.

For the proof of the above theorem we use the following fact about Blaschke products.

Lemma 2.10 *If* $B_j \in \mathcal{B}^{n_j}$*,* $j = 1, 2$ *are distinct Blaschke products in* $\mathcal{B}^n$ *then* $B_1 - B_2$ *has at most* $n := \min\{n_1, n_2\}$ *zeros in* Δ*. Moreover, if* $n_1 = n_2$ *and* $B_1 - B_2$ *has* n *zeros in* Δ *then it is not zero on* T*.*

Proof. First, we suppose $n = n_2$ and $B_1 - B_2$ has more than n zeros in Δ. Consequently, for $\epsilon > 0$ sufficiently small the function $g := (1 - \epsilon)B_1 - B_2$ likewise has more than n zeros in Δ. But, for $z \in T$, we have that

$$|g(z) + B_2(z)| = (1-\epsilon)|B_1(z)| < |B_2(z)|,$$

and so by Rouche's theorem g can only have n zeros in Δ. Hence, we have established by contradiction that $B_1 - B_2$ has at most n zeros in Δ. Now, suppose $n_1 = n_2$, $B_1 - B_2$ has exactly n zeros at $\{z_j\colon j \in Z_n\}$ in Δ and there is a $\zeta \in T$ so that $B_1(\zeta) = B_2(\zeta)$. We can write each B_j, $j = 1, 2$, for $z \in \Delta$ in the form

$$B_j(z) = \lambda \zeta^n \frac{p_j(z)}{z^n \overline{p_j(\overline{z}^{-1})}},$$

where each p_j, $j = 1, 2$, are polynomials of degree at most n with zeros only in Δ normalized so that $p_j(\zeta) = 1$, $j = 1, 2$ and λ is a complex constant in T. Hence, the polynomial q of degree $2n$ defined for $z \in C$ by

$$q(z) := z^n \Big\{ p_1(z)\overline{p_2\Big(\frac{1}{\overline{z}}\Big)} - p_2(z)\overline{p_1\Big(\frac{1}{\overline{z}}\Big)} \Big\}$$

vanishes in the extended complex plane on the points $\{z_j, \overline{z_j}^{-1} : j \in Z_n\}$ as well as at $\zeta \in T$. This contradiction proves the lemma. ∎

We now turn to the proof of Theorem 2.2.

Proof of Theorem 2.2. Let us first establish the existence of the functions $B_\pm$ and then we will prove inequality (2.6). In fact, (2.6) is the key to our construction of the functions $B_\pm$. To this end, we first transform the functions $\{u_j : j \in Z_n\}$ into a Chebyshev system. The standard way to accomplish this is to choose a $\delta > 0$ and define functions $u_j(\cdot, \delta), j \in Z_n$, at $t \in I$, by the formula

$$u_j(t; \delta) = 1/\sqrt{2\pi\delta} \int_0^1 \exp(-(t-r)^2/2\delta^2) u_j(r) \mathrm{d}r, \quad j \in Z_n. \tag{2.7}$$

The functions $\{u_j(\cdot, \delta) : j \in Z_n\}$ form a Chebyshev system on I and $\lim\limits_{\delta \to 0^+} u_i(\cdot; \delta) = u_i$, $i \in Z_n$, uniformly on compact subsets of I^0.

According to a result from [16] there exists a function $u_n(\cdot, \delta)$ continuous on I such that the functions $\{u_j(\cdot, \delta) : j \in Z_{n+1}\}$ likewise from a Chebyshev system on I. (This result was proved earlier in [8, p. 241-245] under stronger hypotheses on the initial Chebyshev system $\{u_j : j \in Z_n\}$.) We define the quantities

$$y_+(\delta) := \max\{(u_n(\cdot, \delta),\ h) :\ h \in U(\Delta), (u_j(\cdot, \delta),\ h) = y_j,\ j \in Z_n\},$$

and also

$$y_-(\delta) := \min\{(u_n(\cdot, \delta),\ h) :\ h \in U(\Delta), (u_j(\cdot, \delta),\ h) = y_j,\ j \in Z_n\}.$$

Since $y \in M^0(\Delta)$ it follows that $y_+(\delta) > y_-(\delta)$ for δ sufficiently small. Let us restrict ourselves to these small values of δ and also observe that the vectors $(y_1, y_2, \ldots, y_n, y_\pm(\delta))$ are both on the boundary of the moment space

$$M_\delta(\Delta) := \{(u_j(\cdot,\delta) : j \in Z_{n+1}) : \ h \in U(\Delta)\}.$$

Hence, by Theorem 2.1 there are Blaschke products $B_\pm(\cdot,\delta) \in \mathcal{B}_r^n$ which represent the vectors $(y_1, y_2, \dots, y_n, y_\pm(\delta))$, respectively. Therefore, the (nonzero) function

$$g := B_+(\cdot,\delta) - B_-(\cdot,\delta)$$

has the property that

$$(u_j(\cdot,\delta), g) = 0, \quad j \in Z_n. \tag{2.8}$$

This implies, by Lemma 2.1, that g has at least n zeros in I^0. Consequently, by Lemma 2.10 we conclude that g has exactly n zeros in I^0 and that $g(1) \neq 0$. We also know that

$$(u_{n+1}(\cdot,\delta), g) > 0$$

and so by Lemma 2.2, g must be positively oriented which implies that $g(1) > 0$. Consequently, we have established that

$$B_\pm(1;\delta) = \pm 1$$

for δ sufficiently small. Now, we let $\delta \to 0^+$ and by a standard compactness argument we conclude the existence of two Blaschke products $B_\pm \in \mathcal{B}_r^n$ which represent $y \in M^0(\Delta)$ with the property that $B_\pm(1) = \pm 1$. According to (2.7) and Lemma 2.10, they necessarily have exactly n zeros in I^0.

Next, we prove the inequality (2.6). First, we use Lemma 2.1 to conclude for *any* h which represents y that the functions $B_+ - h$ and $B_- - h$ have at least n zeros in I^0. Now, we invoke Rouche's theorem to show that each of these functions has *exactly* n zeros, since for any $\epsilon \in (0,1)$ there is a $\delta \in (0,1)$ such that for all $z \in \delta T$ we have that

$$|B_\pm(z) - (1-\epsilon)h(z) - B_\pm(z)| = (1-\epsilon)|h(z)| \leq 1 - \epsilon < |B_\pm(z)|.$$

We shall apply this fact in the following way. Since $y \in M^0(\Delta)$ there is a function h_0 which represents it for which $\|h_0\|_\Delta < 1$. Therefore, for any $\rho \in (0,1)$ the function $h_\rho := (1-\rho)h + \rho h_0$ also represents y and likewise satisfies $\|h_\rho\|_\Delta < 1$. Moreover, $B_+ - h_\rho$ is positively oriented and so, by our above remark, has n zeros in I^0. We appeal to Lemma 2.2 to obtain that $(u_n, h_\rho) > 0$. Letting $\rho \to 0^+$ proves the upper bound in (2.6). The lower bound is proved in a similar way.

Now, suppose there is another $B \in \mathcal{B}_r^n$ which represents y and is different from both B_+ and B_-. Therefore, for some $\delta > 0$, B is different from $B_+(\cdot,\delta)$ and $B_-(\cdot,\delta)$ as well. By construction, we have that

$$(u_n(\cdot,\delta), B_-(\cdot,\delta)) \leq (u_n(\cdot,\delta), B) \leq (u_n(\cdot,\delta), B_+(\cdot,\delta))$$

and

$$(u_j, B_\pm(\cdot,\delta)) = (u_j, B) = y_j, \quad j \in Z_n.$$

Thus, we may apply the above argument to the function $B_+(\cdot,\delta) - B$ and conclude that it has n zeros in Δ. Therefore, Lemma 2.10 implies that $B(1) = -1$.

Likewise, when we apply this argument to $B-B_-(\cdot,\delta)$ we conclude that $B(1)=1$ which is a contradiction. ∎

As remarked earlier, not only each $B_\pm$ has n zeros in I^0 but also $B_+ - B_-$ has n zeros in I^0 and is positive in some neighborhood of one in I.

Before we apply Theorem 2.2 to the identification of the envelope of the function class $S_k(\Delta)$, we give one additional result for the class $U(\Delta)$ which parallels a result presented in [9] for the class $U(I)$. The formulation of this result uses the vector $e := ((-1)^j : j \in Z_n)$ and for every $y = (y_j : j \in Z_n)$ we set

$$\|y\|_\infty := \max\{|y_j| : j \in Z_n\}.$$

Theorem 2.3. *If for $j \in Z_n$ the functions $\{u_k : k \in Z_n \setminus \{j\}\}$ form a weak Chebyshev system on I^0 and E is a closed subset of I^0 on which these functions are linearly independent then for every $y = (y_j : j \in Z_n) \in R^n \backslash \{0\}$ we have that*

$$\|y\| \leq \|e\| \|y\|_\infty$$

and equality holds if and only if $y = \pm e$.

Proof. We assume that $\|y\|_\infty \leq 1$, let $d := \|y\|^{-1}\|e\|$ and choose $B_1, B_2 \in \mathcal{B}_r^{n-1}$ such that B_1 represents $\|y\|^{-1}y$ and B_2 represents $\|e\|^{-1}e$. Also, we choose a $\sigma = \pm 1$ so that the function $h := \sigma d \|y\| B_1 - \|e\| B_2$ vanishes at one. Let $x := \sigma d y - e$ and observe that

$$(u_j, h) = x_j, \quad j \in Z_n.$$

If $d \leq 1$ then the vector x weakly alternates, that is, $x_j x_{j+1} \leq 0,\ j \in Z_n$. Also, note that if $x = 0$ then h has n zeros in I^0 which is impossible by Lemma 2.10 and so $x \neq 0$. We choose a $j \in Z_n$ so that $x_j \neq 0$ and define the functions $v_k := u_k - x_k x_j^{-1} u_j$, $k \in Z_n \setminus \{j\}$. Hence, we have that $(v_k, h) = 0$, $k \in Z_n \setminus \{j\}$. Since the set of functions $\{v_k : k \in Z_n \setminus \{j\}\}$ form a weak Chebyshev system, see [9, Lemma 5], we conclude by Lemma 2.1 that h has $n-1$ zeros in I^0, and by Lemma 2.10 $h(1) \neq 0$. This is a contradiction and hence $d > 1$ unless $y = \pm e$. ∎

3. Envelopes

In this section we identify the envelope of the function class $S_k(\Delta)$ as described in the introduction. We shall prove the following theorem.

Theorem 3.1. *Let k be a positive integer and n a nonnegative integer. Given any points $\{t_j : j \in Z_{k+n}\}$ such that $-1 < t_0 < \cdots < t_{k+n-1} < 1$ and a data $(y_j : j \in Z_{k+n})$ in the interior of the moment space*

$$\{(f(t_j) : j \in Z_{n+k}) : f \in S_k(\Delta)\},$$

there are two (unique) functions $f_\pm \in D(Y)$ such that $f_\pm^{(k)} \in \mathcal{B}_r^n$ where $f_+^{(k)}$ is positively oriented, $f_-^{(k)}$ is negatively oriented and for all $x \in I$,

$$\overline{f}_\Delta(x) = \max(f_+(x), f_-(x)) \tag{3.1}$$

and

$$\underline{f}_\Delta(x) = \min(f_+(x), f_-(x)). \tag{3.2}$$

Proof. The first step in the proof is to reduce the requirement that a function $f \in D(Y) \cap S_k(\Delta)$ to moment conditions on $f^{(k)}$. Here we use divided differences. We let $[y_j, \ldots, y_{j+k}], j \in Z_n$ denote the k-th divided difference of the data $y_j, \ldots, y_{j+k}$ at $t_j, \ldots, t_{j+k}$, and M_j denote the B-spline of degree $k-1$ with knots at $t_j, \ldots, t_{j+k}$ so that $f \in D(Y)$ implies that

$$\int_E M_j(t) g(t) \mathrm{d}t = [y_j, \ldots, y_{j+k}]$$

where $g := f^{(k)}$ and $E := [t_0, t_{k+n-1}]$. Conversely, when a function g satisfies (3.1) there is an $f \in D(Y)$ with $g = f^{(k)}$. Thus, we are led to consider the moment space $M(\Delta)$ generated by the functions $\{M_j : j \in Z_n\}$. Here we must pause a moment and contemplate that the B-splines in (3.1) are continuous only when $k > 1$. So, let us first impose this restriction and deal with the exceptional case later. It is known that these functions form a weak Chebyshev system on R and are linearly independent on *any* subinterval of the real line, [7]. Our hypothesis on the data Y implies that the vector $d := ([y_j, \ldots, y_{j+k}] : j \in Z_n)$ is in the interior of $M(\Delta)$ and so we can apply Theorem 2.2 and obtain two functions $f_\pm \in D(Y)$ with the property that $f_+^{(k)} \in \mathcal{B}_r^n$ is positively oriented and $f_-^{(k)} \in \mathcal{B}_r^n$ is negatively oriented.

Let us now prove these functions envelope our class $S_k(\Delta)$. To this end, let f be any other function in $S_k(\Delta) \cap D(Y)$. The function $h := f_+ - f$ has at least $n+k$ zeros on the set of points $\{t_j : j \in Z_{n+k}\}$. If it vanishes at another point not in this set then Rolle's theorem implies that $f_+^{(k)} - f^{(k)}$ has at least $n+1$ zeros. But then for ϵ, a small positive number, the function $g := (1-\epsilon) f^{(k)} - f_+^{(k)}$ has at least $n+1$ zeros. However, there is a $\delta \in (0,1)$ such that for any $z \in \delta T$ we have that

$$|g(z) + f_+^{(k)}(z)| = |(1-\epsilon) f^{(k)}(z)| < |f_+^{(k)}(z)|.$$

Thus, by Rolle theorem, f_+ *crosses* f only on the set $\{t_j : j \in Z_{k+n}\}$ and f_+ exceeds f beyond t_{n+k-1}. Analogous facts hold for f_-. This proves (3.1) and (3.2).

There remains the case $k = 1$. A cursory glance at the proof of Theorem 2.1 reveals that the need for continuity of the functions $\{u_j : j \in Z_n\}$ disappears as soon as they were "heated up" by (2.7). We only needed that the functions (2.7) were a Chebyshev system and that they converge in $L^1(I)$. This all works for the B-spline in (3.1) even for $k = 1$. ∎

Theorem 3.1 is also true for the case $k = 0$. The proof of this fact relies on the Pick-Nevanlinna interpolation theorem. Thus we do not require Lemmas 2.1, 2.2 and 2.5, but only need to use Rolle's theorem on I and Rouche's theorem on Δ to prove the result.

We remark in the special case that $y = 0$ we have that $f_+ = -f_-$ and Theorem 3.1 was proved in [2].

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