

A New Characterization of Commutative Artinian Rings*

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Received May 18, 2003

Revised September 10, 2003

Abstract. Let R be a commutative Noetherian ring. It is shown that R is Artinian if and only if every R -module is good, if and only if every R -module is representable. As a result, it follows that every nonzero submodule of any representable R -module is representable if and only if R is Artinian. This provides an answer to a question which is investigated in [1].

1. Introduction

All rings considered in this paper are assumed to be commutative with identity. There are several characterizations of Artinian rings. In particular, it is known that a Noetherian ring R is Artinian if and only if every prime ideal of R is maximal. In this article, we present a new characterization of Artinian rings according to the notions of primary decomposition and (its dual) secondary representation. To do so, we need to introduce a generalization of the notions of Hopficity and co-Hopficity.

In [2] Hiremath introduced the concept of Hopficity for R -modules. The dual notion is defined by Varadarajan [6]. An R -module M is said to be Hopfian (resp. co-Hopfian) if any surjective (resp. injective) R -homomorphism is auto-

*This work was in part supported by a grant from IPM, Tehran, Iran.

matically an isomorphism. We refer the reader to [6] for reviewing the most important properties of Hopfian and co-Hopfian R -modules. We extend these definitions as follows: An R -module M is said to be semi-Hopfian (resp. semi-co-Hopfian) if for any $x \in R$, the endomorphism of M induced by multiplication by x is an isomorphism, provided it is surjective (resp. injective). Clearly any Hopfian (resp. co-Hopfian) R -module is semi-Hopfian (resp. semi-co-Hopfian). Also, it is obvious that R , as an R -module, is Hopfian (resp. co-Hopfian) if and only if it is semi-Hopfian (resp. semi-co-Hopfian).

As the main result of this note, we establish the following characterization of Artinian rings.

Theorem 1.1. *Let R be a commutative Noetherian ring. Then the following are equivalent:*

- i) R is Artinian.
- ii) Every nonzero R -module is good.
- iii) Every R -module is semi-Hopfian.
- iv) Every nonzero R -module is representable.
- iv') Every nonzero Noetherian R -module is representable.
- v) Every R -module is semi-co-Hopfian.
- v') Every Noetherian R -module is semi-co-Hopfian.
- v'') Every Noetherian R -module is co-Hopfian.
- vi) Every nonzero R -module is Laskerian.

In [1] the following question was investigated: When are submodules of representable R -modules representable? In that paper [1, Th. 2.3], it is shown that this is the case, when R is Von Neumann regular. For a Noetherian ring R , we prove that every nonzero submodules of any representable R -modules is representable if and only if R is Artinian (see 2.4).

2. The Proof of the Main Theorem

Recall that a nonzero R -module M is called *good* if its zero submodule possesses a primary decomposition. A nonzero R -module S is said to be *secondary* if for any $x \in R$, the map induced by multiplication by x is either surjective or nilpotent. We say the R -module M is *representable* if there are secondary submodules S_1, S_2, \dots, S_k of M such that $M = S_1 + S_2 + \dots + S_k$. The two notions of primary decomposition and secondary representation are dual concepts. We refer the reader to [3, Appendix to §6], for more details about secondary representation. Also, recall that an R -module M is said to be *Laskerian* if any submodule of M is an intersection of a finite number of primary submodules.

Lemma 2.1.

- i) Every finitely generated R -module is Hopfian.
- ii) Every Artinian R -module is co-Hopfian.
- iii) Every good R -module is semi-Hopfian.
- iv) Every representable R -module is semi-co-Hopfian.

Proof.

i) See [7, Proposition 1.2].

ii) is well known and can be checked easily.

iii) Let $x \in R$ be an M -coregular element of R . Let $0 = \bigcap_{i=1}^n Q_i$ be a primary decomposition of the zero submodule of M . Fix $1 \leq i \leq n$. Since Q_i is a proper submodule of M and $\frac{M}{Q_i} \xrightarrow{x} \frac{M}{Q_i}$ is either injective or nilpotent, it follows that x is $\frac{M}{Q_i}$ -regular. Now, if $xm = 0$ for some element m in M , then for each i , it follows that $xm \in Q_i$ and so $m \in Q_i$. Hence $m = 0$, and so x is M -regular as required.

iv) is similar to (iii). ■

Example 2.2. Let N be a nonzero co-Hopfian R -module. Set $M = \bigoplus_{i \in \mathbb{N}} N$. Then M is semi-co-Hopfian, but not co-Hopfian. To this end, define the R -homomorphism $\psi : M \rightarrow M$ by $\psi(m_1, m_2, \dots) = (0, m_1, m_2, \dots)$ for all $(m_1, m_2, \dots) \in M$. Then ψ is injective, while it is not surjective.

Proof of Theorem 1.1. (i) \Rightarrow (ii) Let M be an R -module. Since R is Artinian, it is representable as an R -module. Hence $M \simeq \text{Hom}_R(R, M)$ is good, by [4, Th. 2.8]. The implications (ii) \Rightarrow (iii), (iv) \Rightarrow (v) and (iv') \Rightarrow (v') follow, by 2.1.

Now we prove (iii) \Rightarrow (v). Let M be an R -module and $D(\cdot) = \text{Hom}_R(\cdot, E)$, where E is an injective cogenerator of R . Let $x \in R$ be such that the map $M \xrightarrow{x} M$ is injective. Then the map $D(M) \xrightarrow{x} D(M)$ is surjective and it is also injective, because $D(M)$ is semi-Hopfian. But this implies that x is M -coregular, as the functor $D(\cdot)$ is faithfully exact. Hence M is semi-co-Hopfian.

(v') \Rightarrow (i) Suppose the contrary and assume that $\mathfrak{p} \subset \mathfrak{m}$ is a strict inclusion of prime ideals of R . Let $x \in \mathfrak{m} \setminus \mathfrak{p}$. Then x is R/\mathfrak{p} -regular, but it is not R/\mathfrak{p} -coregular, which is a contradiction. Therefore every prime ideal of R is maximal and so R is Artinian.

Next, we prove (i) \Rightarrow (iv). Since R is Artinian, $\text{Max}(R)$ is finite. Let $\text{Max}(R) = \{\mathfrak{m}_1, \dots, \mathfrak{m}_k\}$. There are \mathfrak{m}_i -primary ideals \mathfrak{a}_i of R such that $R \simeq \prod_{i=1}^k R/\mathfrak{a}_i$. Let $F = \bigoplus_{j \in J} R$ be an arbitrary free R -module. Set $S_i = \bigoplus_{j \in J} R/\mathfrak{a}_i$ for $i = 1, 2, \dots, k$. Then

$$F \simeq \bigoplus_{j \in J} \left(\prod_{i=1}^k R/\mathfrak{a}_i \right) \simeq \prod_{i=1}^k S_i = \bigoplus_{i=1}^k S_i.$$

It follows that for each $i = 1, 2, \dots, k$, the R -module S_i is \mathfrak{m}_i -secondary and hence F is representable. But any R -module is homeomorphic image of some free R -module and so the conclusion follows. Note that one can check easily that any nonzero quotient of a representable R -module is also representable.

It follows from [8, Theorem] that the statements (i) and (v'') are equivalent. Let N be a proper submodule of an R -module M . Then N possesses a primary decomposition if and only if the R -module M/N is good. Thus (ii) and (vi) are equivalent. Now, because the implications (iv) \Rightarrow (iv') and (v) \Rightarrow (v') are clearly hold, the proof is complete. ■

Corollary 2.3. *Let M be an R -module such that the ring $R/\text{Ann}_R M$ is Artinian. Then M is both good and representable.*

Proof. Set $S = R/\text{Ann}_R M$. Then M possesses the structure of an S -module in a natural way. A subset N of M is an R -submodule of M if and only if it is an S -submodule of M . Thus it is straightforward to see that M is good (resp. representable) as an R -module if and only if it is good (resp. representable) as an S -module. Now the conclusion follows by 1.1. ■

Proposition 2.4. *Let R be a Noetherian ring. The following statements are equivalent:*

- i) *Every nonzero submodule of any representable R -module is representable.*
- ii) *R is Artinian.*

Proof. (ii) \Rightarrow (i) is clear by 1.1.

(i) \Rightarrow (ii) By [5], any nonzero injective module over a commutative Noetherian ring is representable. Since any R -module can be embedded in an injective R -module, it follows that all nonzero R -modules are representable. Therefore by the implication (iv) \Rightarrow (i) of 1.1, it follows that R is Artinian. ■

A commutative ring R is said to be Von Neumann regular if for each element $a \in R$, there exists $b \in R$ such that $a = a^2b$. In [1, Th. 2.3], it is shown that over a commutative Von Neumann regular ring R every nonzero submodule of a representable R -module is representable. Since commutative Artinian rings are Noetherian, we can deduce the following result, by 2.4.

Corollary 2.5. *Let R be a commutative Von Neumann regular ring. Then R is Noetherian if and only if it is Artinian.*

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