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Existence of Solution for Multi-Valued Integral Equations

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Abstract. In this paper we shall prove a theorem on existence of global solutions for multi-valued intergral equations in Banach spaces. We shall consider also integral equations containing parameters and prove a continuity result for the set of global solutions.

1. Introduction

The purpose of this paper is to prove an existence theorem for global solutions of multi-valued intergral equations in Banach spaces. Note that the results for local solutions have been established in [7]. Moreover, for the case of integral equations containing parameters we shall establish a continuity property of the set of global solutions with respect to parameters.

Throughout this paper $(\Omega, \mathcal{A})$ is an arbitrary measurable space, $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ are separable Banach spaces and X', Y' stand for their strong duals : the spaces of all linear continuous functionals of X and Y, respectively. The following notation will be used:

$L(Y, X)$: the space of all linear continuous mappings from Y to X;

X_σ: the space X endowed with the weak topology;

$\mathcal{B}(X)$: the Borel σ-field in X;

$\mathcal{L}$: the σ -field of all Lebesgue measurable set in the real line $\mathbb{R}$;

$\mathcal{L}^1_Y(I)$ (resp.,$L^1_Y(I)$): the space of all ds-integrable (resp.,all equivalence classes of ds-integrable) functions from the interval I, endowed with the Lebesgue measure ds, into Y; $\mathcal{L}^1(I) = \mathcal{L}^1_{\mathbb{R}}(I)$ and $L^1(I) = L^1_{\mathbb{R}}(I)$;

$L^{\infty}_{Y'}(I)$: the space of all classes of essentially bounded ds-measurable functions from I into Y'_{σ} (the space Y' endowed with the weak topology $\sigma(Y',Y)$);

$C_X(I)$ (resp., $C_{X_\sigma}(I)$): the space of all continuous functions from I into X (resp., X_σ).

Finally, for a set-valued mapping Γ from a measurable space $(\Sigma,\mathcal{B})$ into a separable metric space E, GraphΓ denotes its graph, i.e., the set $\{(s,a)\in\Sigma\times E : a\in\Gamma(s)\}$; $\Gamma^{-}(A)=\{s\in\Sigma:\Gamma(s)\cap A\neq\emptyset\}$, where $A\subset E$. $\mathcal{S}_\Gamma$ will denote the set of all $\mathcal{B}$-measurable selections of Γ and when $\mathcal{B}$ is equipped with a measure $\mu\geq 0$, the factor of $\mathcal{S}_\Gamma$ by the equivalence relation "equality μ -almost everywhere" is denoted by S_Γ.

2. Main Results

Consider the multi-valued intergral equation

$$x(t)\in a(t)+\int_{T_0}^{t} G(s,t)F(s,x(s))ds, \tag{1}$$

where a,G,F are the following mappings

$$\begin{gathered} a: I=[T_0,T)\to X\\ G:\{(s,t)\in I\times I: t\geq s\}\to L(Y,X)\\ F: I\times X\to 2^{Y}. \end{gathered}$$

A global solution of (1) is, by definition, a solution of (1) defined on the whole interval I, i.e., a function $x: I\to X$ of the form

$$x(t)=a(t)+\int_{T_0}^{t} G(s,t)f(s)ds$$

where $f: I\to Y$ is any measurable function satisfying

$$f(s)\in F(s,x(s)),\quad \text{a.e. on}\quad I$$

Theorem 2.1. *Suppose* $a(\cdot)\in C_X(I)$ *with* $\sup\{\|a(t)\|: t\in I\}<+\infty$ *and the following conditions hold*

(G.1) *For each* $t\in I$ *and* $y\in Y, G(.,t)y$ *is* $(\mathcal{L},\mathcal{B}(X))$*- measurable and* $\sup_{t\in I}\operatorname{ess}\sup_{s\in[T_0,t]}\|G(s,t)\|_{L(Y,X)}<+\infty$;

(G.2) *For each* $t\in I$

$$\int_{T_0}^{\min(t,t')}\|G(s,t')-G(s,t)\|_{L(Y,X)}ds\to 0,\quad \text{as}\quad |t-t'|\to 0.$$

Moreover, assume

(F.1) *There exist a function* $\alpha(\cdot) \geq 0$ *locally integrable on* I *and a relatively weakly compact set* $K \subset Y$ *such that for almost all* s *in* I *and for all* x *in* X, $F(s,x)$ *is closed, convex and contained in* $\alpha(s)(1+\|x\|)K$:

$$F(s,x) \subset \alpha(s)(1+\|x\|)K, \quad \text{a.e. } s \in I. \quad \forall x \in X.$$

(F.2) *For almost* $s \in I, F(s,\cdot)$ *is upper semicontinuous (u.s.c.) for the* $\sigma(X,X')$*-topology in* X *and the* $\sigma(Y,Y')$*-topology in* Y.

(F.3) *For each* $x \in X, F(.,x)$ *admists a ds-measurable selection.*

Then the equation (1) has at least one strongly continuous global solution and any locally bounded global solution of (1) is strongly continuous.

Proof. Set $c = 1+\sup_{t\in I}\|a(t)\|$, $c_1 = \sup_{t\in I} ess.\sup_{s\in[T_0,t]} \|G(s,t)\|_{L(Y,X)}, k = \sup\{\|y\|_Y : y \in K\}; z(t) = -1 + (1+c)\exp(c_1 k \int_{T_0}^t \alpha(s)ds)$. Then $z(t)$ is the (unique) solution of the equation

$$z(t) = c + kc_1 \int_{T_0}^{t} \alpha(s)[1+z(s)]ds$$

(the uniqueness follows from the Gronwall's lemma, although this plays no role in the proof). Set $\Sigma(s) = \alpha(s)[1+z(s)]K$. We can suppose, without loss of generality, that $\alpha(s) \geq 1$ for all $s \in I$ and K is convex, weakly compact and containing the origin in Y. Since $\alpha(\cdot) \in L^1(I)$ one has $S_\Sigma \subset L^1_Y(I)$. For any function $f : I \to Y$ we set formally $i_f(t) = \int_{T_0}^t G(s,t)f(s)ds$ and $x_f(t) = a(t)+i_f(t)$. It follows from (G.1) that if $f \in L^1_Y(I_{loc})$ then $i_f(t)$ and hence $x_f(t)$ exist for all $t \in I$. Moreover, for any $t,t' \in [T_0,T)$ and $T_1 \in (\max(t,t'),T)$, one has

$$\|i_f(t) - i_f(t')\| \leq \int_{T_0}^{\min(t,t')} \|G(s,t') - G(s,t)\|_{L(Y,X)}\|f(s)\|_Y ds + c_1 \int_t^{t'} \|f(s)\|_Y ds$$

$$\leq p \int_{T_0}^{\min(t,t')} \|G(s,t') - G(s,t)\|_{L(Y,X)} ds + 2c_1 \int_{\{s\in[T_0,T_1]:\|f(s)\|_Y>p\}} \|f(s)\|_Y ds$$

$$+ c_1 \int_t^{t'} \|f(s)\|_Y ds$$

for all integer $p \in \mathbb{N}$. This yields the strong continuity of $i_f(\cdot)$. Indeed, for any $t \in I$ and $\varepsilon > 0$, we fix first $T_1 \in (t,T)$ and $p \in \mathbb{N}$ such that

$$2c_1 \int_{\{s\in[T_0,T_1]:\|f(s)\|_Y>p\}} \|f(s)\|_Y ds < \frac{\varepsilon}{3},$$

then we take $\delta > 0$ such that $|t' - t| < \delta$ implies

$$p \int_{T_0}^{\min(t,t')} \|G(s,t') - G(s,t)\|_{L(Y,X)} ds < \frac{\varepsilon}{3} \quad \text{and} \quad c_1 \int_t^{t'} \|f(s)\|_Y ds < \frac{\varepsilon}{3}.$$

We have therefore $\|i_f(t') - i_f(t)\| < \varepsilon$ for all $t' \in (t-\delta, t+\delta)$, showing the strong continuity of $i_f(\cdot)$ and hence $x_f(\cdot)$. For a further purpose, we note here that this argument shows also that for any positive function $\beta(\cdot) \in L^1(I)$, the family $\{x_f(\cdot) : \|f(s)\|_Y \le \beta(s) \text{ a.e. on } I\}$ is strongly equi-continuous.

Set now

$$\Gamma(f) = \{g : I \to Y \ \text{ measurable}: \ g(s) \in F(s, x_f(s)) \ \text{ a.e. on } I\}$$

for every measurable function $f : I \to X$ such that $i_f(t)$ exists for all $t \in I$. We claim that $x(\cdot)$ is a locally bounded solution on I for the equation (2.1) if and only if $x(t) = x_f(t)$, $(\forall t \in I)$ with $f \in S_\Sigma$ satisfying: $f \in \Gamma(f)$. The claim "if" is obvious.

To show the converse, let $x(\cdot)$ be a locally bounded global solution of (1). By definition, there exists, a measurable function $f : I \to Y$ such that $x(t) = x_f(t)$ on I and $f(t) \in F(t, x(t))$ a.e. on I. In view of (F.1) one has $\|f(s)\|_Y < k\alpha(s)[1 + \|x(s)\|]$. Consequently, $f \in L^1_Y(I)$. Hence, as shown above, $x(\cdot) = x_f(\cdot) \in C_X(I)$, proving the second assertion of the theorem.

Furthermore, one has

$$\|x(t)\| \le -1 + c + c_1 \int_{T_0}^{t} \|f(s)\|_Y ds \le -1 + c + c_1 k \int_{T_0}^{t} \alpha(s)[1 + \|x(s)\|] ds$$

hence,

$$\|x(t)\| - z(t) \le -1 + \int_{T_0}^{t} kc_1 \alpha(s)[\|x(s)\| - z(s)] ds$$

for all $t \in I$. Taking into account that $x(s), z(\cdot)$ are continuous and $\alpha(s) \ge 1$, an elementary argument shows that $\|x(t)\| < z(t)$ for all t in I. Therefore, $f(s) \in F(s, x(s)) \subset \alpha(s)[1 + \|x(s)\|]K \subset \alpha(s)[1 + z(s)]K = \Sigma(s)$ a.e. on I. Consequently, $f \in \varphi_\Sigma$. Hence, $f \in \Gamma(f)$ follows immediately from the definition of this last set-valued mapping. Thus to prove that (1) has a strongly continuous solution it suffices to show that Γ admits a fixed point. Suppose for a while that $\alpha(\cdot)$ is integrable on the whole interval I. Clearly, $S_\Sigma \subset L^1_Y(I)$. We assert that $\Gamma(S_\Sigma) \subset S_\Sigma$. Indeed, let $f \in S_\Sigma$. One has

$$\|x_f(t)\| \le c + c_1 \int_{T_0}^{t} \|f(s)\|_Y ds < c + c_1 k \int_{T_0}^{t} \alpha(s)[1 + z(s)] ds = z(t)$$

Hence for any $g \in \Gamma(f)$ one has $g(s) \in F(s, x_f(s)) \subset \alpha(s)[1 + \|x_f(s)\|]K \subset \alpha(s)[1 + z(s)]K = \Sigma(s)$ a.e. on I, so that $g \in S_\Sigma$, proving our assertion.

Let us endow $L^1_Y(I)$ with the weak topology. By [3, Cor. V. 4], S_Σ is a nonempty convex (weakly) compact set in $L^1_Y(I)$ and $\Gamma(f)$ is nonempty (and convex). We will show that Γ is u.s.c., or equivalently [2], Graph Γ is closed in S^2_Γ. We observe that S_Σ is metrizable since $L^1_Y(I)$ is (strongly) separable. Let $(f_n, g_n) \to (f, g)$ in S^2_Σ, as $n \to +\infty$, with $g_n \in \Gamma(f_n)(\forall n \in \mathbb{N})$. We note that for each $h(\cdot) \in L^1_Y(I), t \in I$ and $x' \in X'$ the function $s \to \langle G(s,t)h(s), x'\rangle$ is ds-measurable on I. This implies that $x_{f_n}(\cdot)$ pointwisely converges to $x_f(\cdot)$ in the $\sigma(X, X')$-topology. On the other hand, since $g_n(s) \in F(s, x_{f_n}(s))$ a.e. on I, it follows, by [3, Th. VI. 4], that $g(s) \in F(s, x_f(s))$ a.e. on I, i.e., $g \in \Gamma(f)$, and hence Graph Γ is closed in S^2_Σ. According to Kakutani -Ky Fan's theorem, Γ has a fixed point in S_Σ. As stated above, this completes the proof for the case of $\alpha(\cdot)$ being integrable.

Consider the general case. Take a sequence $\{T_i\}_{i=1}^\infty$ in I such that $T_i \uparrow T$ and set $I_i = [T_0, T_i]$. By the above, for each $i \in \mathbb{N}, \Gamma_i$ admits a point $f_i \in S_{\Sigma_{I_i}}$, where Γ_i is defined exactly as Γ but only of functions defined on I_i and $S_{\Sigma_{I_i}}$ stands for the restriction of Σ to I_i. Since for each $j \in \mathbb{N}, f_i|I_j$ belongs to $S_{\Sigma_{I_i}}$ for all $i \geq j$ and $S_{\Sigma_{I_j}}$ is a compact and metrizable subspace of $L^1_Y(I_j)$ (endowed with the week topology), by using a diagonal process we obtain a subsequence $\{f_{i_k}\}_{k=1}^\infty$ of $\{f_i\}_{i=1}^\infty$ and an element $\bar{f} \in S_\Sigma$ such that for each $j, f_{i_k} \to \bar{f}$ (weakly in $L^1_Y(I_j)$). One has: $\bar{f} \in \Gamma(\bar{f})$. To see this, it suffices to show that $\bar{f}|I_j \in \Gamma_j(\bar{f}|I_j)$ for every j in $\mathbb{N}$. But this inclusion follows immediately from the u.s.c. of Γ_j j and the fact that $f_{i_k} \in \Gamma_j(f_{i_k})$ for all k large enough. This completes the proof.

Let M be a compact metric space. Consider the equation

$$x(\xi, t) \in a(\xi, t) + \int_{T_0}^{t} G(\xi, s, t)F(\xi, s, x(\xi, s)ds \tag{2}$$

where ξ is a parameter taken in M, a, G, F be the functions defined as $a : M \times I \to X, G : \{(\xi, s, t) \in M \times I^2 : s \leq t\} \to L(Y, X)$ and F is a multi-valued function from $M \times I \times X$ to Y. For each $\xi \in M$, let $S(\xi)$ denote the set of all strongly continuous global solutions of (2).

Theorem 2.2. *Suppose that*

$$a(\cdot, \cdot) \in C_X(M \times I) \quad and \quad \sup\{\|a(\xi, t)\| : (\xi, t) \in M \times I\} < \infty.$$

and the following conditions hold:

(G.1) *For each* $\xi \in M, t \in I$ *and* $y \in Y, G(., t)y$ *is* $(\mathcal{L}, \mathcal{B}(X))$*- measurable on* $I_t = [T_0, t]$ *and* $\sup_{(\xi,s)\in M\times[T_0,t]} esssup_{s\in I} t\|G(\xi, s, t)\|_{L(Y,X)} < +\infty$;

(G.2) *For each* $t \in I$

$$\int_{T_0}^{\min(t,t')} \|G(\xi, s, t') - G(\xi, s, t)\|_{L(X,X')} ds \to 0, \quad as \quad |t - t'| \to 0.$$

in I, uniformly with respect to ξ in M.

(G.3) *For each $\xi \in M, t \in I, x' \in X'$ the function $G(\cdot, s, t)x' : M \to Y$ is strongly continuous at ξ for all s in a subset of full measure $I^t(\xi)$ of I^t.*

Moreover, assume

(F.1) *For each $\xi \in M, F(\xi, s, x)$ is closed, convex and contained in $\alpha(s)[1+\|x\|]K$ for all s in a subset of full measure $I(\xi)$ of I and for all $x \in X$, where $\alpha(\cdot) \geq 0$ is a locally integrable function on I and K is a relatively weakly compact set in Y.*

(F.2) *For almost $s \in I, F(\cdot, s, \cdot)$ is a upper semicontinuous (u.s.c) set-valued mapping from $M \times X_\sigma$ to Y_σ*

(F.3) *For each $(\xi, x) \in M \times X, F(\xi, \cdot, x)$ admists at leat one ds-measurable selection.*

Then for each $\xi \in M, S(\xi)$ is a nonempty compact set in $C_{X_\sigma}(I)$ and the set-valued mapping S is u.s.c. when $C_{X_\sigma}(I)$ is endowed with the topology of compact convergence.

Proof. Set

$$c = 1 + \sup_{(\xi,t) \in M \times I} \|a(\xi, t)\|,$$

$$c_1 = 1 + \sup_{(\xi,t) \in M \times I} \text{ess.} \sup_{s \in [T_0, t]} \|G(\xi, s, t)\|_{L(Y,X)}, k = \sup\{\|y\|_Y : y \in K\}.$$

Define $z(\cdot)$ and Σ literally as in the proof of Theorem 2.1. Here also, we can suppose that K is convex weakly compact, containing the origin in Y. For each $\xi \in M$ and $f \in L^1_Y(I)$, the integral $\int_{T_0}^t G(\xi, s, t) f(s) ds$ exists in X for all $t \in I$. Let $\Gamma(\xi, f)$ denote the set of all ds-measurable functions $g : I \to Y$ such that $g(s) \in F(\xi, s, x_{\xi,f}(s))$ a.e. on I, where $x_{\xi,f}(t) = a(\xi, t) + \int_{T_0}^t G(\xi, s, t) f(s) ds$. Exactly as in the proof of Theorem 2.1 for each $\xi \in M$ we have $\emptyset \neq S(\xi) = \{x_{\xi,f}(\cdot) : f \in S_\Sigma \cap \Gamma(\xi, f)\} \subset \mathcal{X} \subset C_X(I)$, where $\mathcal{X}$ is the set of all functions $x_{\xi,f}(\cdot)$ with $(\xi, f) \in M \times S_\Sigma$.

We are going to show that $\mathcal{X}$ is a relatively compact metrizable subset of $C_{X_\sigma}(I)$. First, by what noted in the proof of Theorem 2.1, $\mathcal{X}$ is an equicontinuous set in $C_{X_\sigma}(I)$. In addition, as in [3] it can be shown that for each $t \in I$ the function $(\xi, f) \to x_{\xi,f}(t)$ from $M \times S_{\Sigma|I_t}$ into X (where $I_t = [T_0, t]$) is weakly continuous when $S_{\Sigma|I_t}$ is equipped with the topology $\sigma(L^1_Y(I_t), L^\infty_{Y'}(I_t))$. Since $S_{\Sigma|I_t}$ is compact, by [3, Cor. V. 4], the set $\mathcal{X}(t) = \{x_{\xi,f}(t) : (\xi, f) \in M \times S_{\Sigma|}\} = \{x_{\xi,f}(t) : (\xi, f) \in M \times S_{\Sigma|I_t}\}$ is weakly compact. Hence, by the Ascoli's theorem (see e.g. [4, Th. 0.4.11]) $\mathcal{X}$ is relatively compact in $C_{X_\sigma}(I)$. Next, it follows from the equicontinuity of X that the topology (of compact convergence) in this set coincides with the topology of pointwise convergence at the rational points in I. In the other words, $\mathcal{X}$ can be regarded as a subspace of the product $\prod_{i=1}^\infty \mathcal{X}(r_i)$ of the spaces $\mathcal{X}(r_i)$ endowed with the $\sigma(X, X')$-topology. Since each $\mathcal{X}(r_i)$ is metrizable, $\prod_{i=1}^\infty \mathcal{X}(r_i)$ is also metrizable, hence so is $\mathcal{X}$.

Since $S(\xi) \subset \mathcal{X}$ for each $\xi \in M$, the proof of Theorem 2.2 is therefore reduced to showing that Graph S is a sequentially complete subset of $M \times \mathcal{X}$. To this end, let $\{x_n(\cdot)\}_{n=1}^{\infty}$ be a Cauchy sequence in $\mathcal{X}$ and $\xi_n \to \xi$ in M such that $x_n(\cdot) \in S(\xi_n)$. One has $x_n(\cdot) = x_{\xi_n,f_n}(\cdot)$ with $f_n(s)$ satisfying:

$$f_n(s) \in F(\xi_n, s, x_n(s)) \quad \text{a.e. on } I \quad (\forall n \in \mathbb{N}). \tag{3}$$

By the same argument as in the end of the proof of Theorem 2.1, it can be supposed that $\{f_n\}_{n=1}^{\infty}$ converges weakly in each space $L^1_Y([T_0, t])$ $(\forall t \in I)$ to a function $f \in S_\Sigma$. Then, clearly, $x_n(t) = x_{\xi_n,f_n}(t) \to x_{\xi,f}(t)$ in X_σ as $n \to +\infty$, for all $t \in I$. Since the pointwise convergence in $\mathcal{X}$ is equivalent to the compact convergence, this means that $x_n(\cdot) \to x_{\xi,f}(\cdot)$ in $C_{X_\sigma}(I)$ as $n \to +\infty$. On the other hand, by [3, Th. VI. 4], it follows from (3) that $f(s) \in F(\xi, s, x_{\xi,f}(s))$ a.e. on I, i.e., $x_{\xi,f}(\cdot) \in S(\xi)$. This shows that Graph S is complete, as to be shown.

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