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Central Limit Theorems for Mixing Arrays

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Abstract. Two central limit theorems are derived for the triangular arrays of real random variables satisfying a so called ℓ'-mixing condition which weakens and simplifies ℓ-mixing one of Withers [11]. Hence α-mixing, β-mixing, φ-mixing, ρ-mixing, and ℓ-mixing arrays satisfy the results here. Two examples are given for the arrays which are not mixing in any known sense but still satisfying a C.L.T. here.

1. Introduction

Central limit theorems (C.L.T.s for shot) have been derived for α-mixing (strong mixing), β-mixing (absolute regularity), φ-mixing (uniform strong mixing), ρ-mixing and ψ-mixing random variables (r.v.s for short) (see for examples [3, 6, 8, 11, 12]). In [11] Withers proved C.L.T.s for r.v.s satisfying ℓ-mixing condition defined below.

Let $(X_{n,k}),\ n = 1, 2, \dots,\ k = 1, 2, \dots, k_n$, be a triangular array (briefly an array) of real r.v.s $X_{n,k}$ and let

$$\ell(c,t) := \sup_{\substack{n,a,b \\ a+b+c\le k_n}} \sup_{\substack{d_j \\ d_j\in\{0,1\}}} \left|\operatorname{cov}\{\exp(it\sum_{j=1}^{a} d_j X_{n,j}),\ \exp(-it\sum_{j=a+c}^{a+b+c} d_j X_{n,j})\}\right|.$$

An array $(X_{n,k})$ is called ℓ-mixing if $\ell(c,t) \to 0$ as $c \to \infty$ for all real t. It is called strong ℓ-mixing if there exist $\ell(c)$ and $K(t)$ such that $\ell(c,t) \le \ell(c)K(t)$ for all t, c and $\ell(c) \to 0$ as $c \to \infty$ [11]. It is showed in [11] that for any array $4\alpha(c) \le \rho(c) \le 2\sqrt{\varphi(c)},\ \beta(c) \le \varphi(c),\ \ell(c,t) \le 16\alpha(c)$, and $\ell(c,t) \le 4\beta(c)$ for all real t. So any array mixing in other sense is strong ℓ-mixing.

In this paper we try to weaken ℓ-mixing notion. We call an array $(X_{n,k})$, $k \le k_n$, *ℓ'-mixing* if for all real t

$$\ell'_n(c,t) := \sup_{\substack{a,b \\ a+b+c\le k_n}} \Big|\operatorname{cov}\{\exp(it\sum_{j=1}^{a} X_{n,j}),\ \exp(-it\sum_{j=a+c}^{a+c+b} X_{n,j})\}\Big| \longrightarrow 0 \quad (1.1)$$

as c and $n \to \infty$ such that $c < k_n$. In other words $(X_{n,k})$ is ℓ'-mixing if for all real t and any $\epsilon > 0$ there exists an $A = A(t,\epsilon)$ such that $\ell'_n(c,t) < \epsilon$ for all $n > A$ and $k_n > c > A$. By the definition $\ell'_n(c,t)$ is simpler than $\ell(c,t)$, without using the d_j factors in its definition. We have $\ell'(c,t) := \sup_n \ell'_n(c,t) \le \ell(c,t)$. So all ℓ-mixing arrays, consequently all α-, β-, φ-, and ρ-mixing ones, are also ℓ'-mixing.

Two C.L.T.s are proved for ℓ'-mixing arrays in this paper. One of them is similar to Theorem 2.2 of Withers [11] but without assuming the stationarity of the array.

Then there are given two examples of ℓ'-mixing arrays which are not ℓ-mixing and hence not mixing in the other sense. These arrays also satisfy all conditions of Theorem 3.1 but not those of Theorem 3.2. Hence somewhat unfamiliar conditions of Theorem 3.1 may not be too general.

This paper is self-contained to make the content easier to read.

2. A Comparison Method

Consider a triangular array of real r.v.s $(X_{n,k})$, $n = 1, 2, \ldots,\ k = 1, 2, \ldots, k_n$, defined on any probability space $(\Omega, \mathcal{F}, P)$. Suppose in what follows that $k_n \to \infty$ as $n \to \infty$ and $E\,X_{n,k} = 0$ for all n and k. Denote the partial sums of r.v.s of $(X_{n,k})$ by

$$S_{n,a,b} := \sum_{k=a+1}^{b} X_{n,k}, \quad S_{n,b} := S_{n,0,b} \quad \text{and} \quad S_n := S_{n,k_n}.$$

Related to $(X_{n,k})$, let us define the array $(X^*_{n,k})$, $k \le k_n$, of independent r.v.s $X^*_{n,k}$ such that the distribution function of each $X^*_{n,k}$ coincides with that of $X_{n,k}$ for all n and $k \le k_n$. Denote similarly $S^*_{n,a,b} = \sum_{a+1}^{b} X^*_{n,k}$.

Now suppose

$$E\,e^{itS_n} - E\,e^{itS^*_n} \underset{n}{\longrightarrow} 0 \tag{2.1}$$

for all real t, that is, the difference of the characteristic functions of the sums of $X_{n,k}$'s and of those of $X^*_{n,k}$'s tends to zero. By the main limit theorem for characteristic functions $S_n \xrightarrow{D} F$ if and only if $S^*_n \xrightarrow{D} F$, for any given distribution function F. Hence under condition (2.1) in order to derive C.L.T.s for $(X_{n,k})$ it suffices to do that for $(X^*_{n,k})$ of independent r.v.s.

Let us write the term in (2.1) in other way. We have for each fixed real t

$$
\begin{aligned}
E\,e^{itS_n} - E\,e^{itS_n^*} &= \sum_{k=2}^{k_n} (E\,e^{itS_{n,k}} - E\,e^{itS_{n,k-1}} E\,e^{itX_{n,k}})\, E\,e^{it\sum_{j=k+1}^{k_n} X^*_{n,j}} \\
&= \sum_{k=2}^{k_n} (E\,e^{itS_{n,k}} - E\,e^{itS_{n,k-1}} E\,e^{itX_{n,k}}) \prod_{j=k+1}^{k_n} E\,e^{itX_{n,j}} \\
&= \sum_{k=2}^{k_n} \Pi_k(X,t) \mathrm{cov}\,\{ e^{itS_{n,k-1}}, e^{-itX_{n,k}} \},
\end{aligned}
$$

where

$$
\Pi_k(X,t) := \prod_{j=k+1}^{k_n} E\,e^{itX_{n,j}}
$$

and $\sum_{j=k_n+1}^{k_n} X^*_{n,j} = 0$ by convention. So, for each fixed real t, condition (2.1) is equivalent to

$$
\sum_{k=2}^{k_n} \Pi_k(X,t) \mathrm{cov}\,\{ e^{itS_{n,k-1}}, e^{-itX_{n,k}} \} \underset{n}{\longrightarrow} 0. \tag{2.2}
$$

Note that in this form condition (2.1) describes the weak dependence between $X_{n,k}$'s.

Since $|\Pi_k(X,t)| \le \prod_{j=k+1}^{k_n} E\,|e^{itX_{n,j}}| = 1$, condition (2.2) is implied by

$$
\sum_{k=2}^{k_n} |\mathrm{cov}\,\{\exp(\,it\,S_{n,k-1}),\ \exp(-itX_{n,k})\}| \underset{n}{\longrightarrow} 0, \tag{2.3}
$$

for each real t.

Proposition 2.1. *Let $(X_{n,k})$, $k \le k_n \to \infty$, be any array of zero mean-valued r.v.s such that*

$$
\sum_{k=1}^{k_n} E\,X^2_{n,k} \underset{n}{\longrightarrow} 1 \tag{2.4a}
$$

$$
\sum_{k=1}^{k_n} E\,X^2_{n,k} I\,(|X_{n,k}| > \varepsilon) \underset{n}{\longrightarrow} 0 \text{ for every } \varepsilon > 0. \tag{2.4b}
$$

Then for any given real t $E\,e^{itS_n} \to e^{-t^2/2}$ if and only if (2.2) *holds for that t, consequently $S_n \xrightarrow{D} N(0,1)$ if and only if $(X_{n,k})$ satisfies* (2.2) *for all real t.*

Conversely, if $(X_{n,k})$ satisfies (2.2) *for all real t, $S_n \xrightarrow{D} N(0,1)$, (2.4a) holds and*

$$
\max_k\ E\,X^2_{n,k} \underset{n}{\longrightarrow} 0,
$$

then the Lindeberg condition (2.4b) follows.

Proof. Conditions (2.4a) and (2.4b) also hold for the related array $(X^*_{n,k})$. By Lindeberg-Feller theorem $S^*_n \xrightarrow{D} N(0,1)$,that is, $E\, e^{itS^*_n} - e^{-t^2/2} \to 0$ for all real t. Noting that (2.2) is the same as (2.1) for each t, we obtain the first conclusions. For the second part of the proposition we note that all the assumptions written for $X_{n,k}$ also hold for $X^*_{n,k}$. So by Lindeberg-Feller theorem again (2.4b) holds for $(X^*_{n,k})$, hence for $(X_{n,k})$ too. ∎

As a direct consequence we have the following

Corollary 1. *For any array of zero mean-valued r.v.s* $(X_{n,k}),\ k \le k_n \to \infty$, *if* (2.4*a*) *and* (2.4*b*) *hold and*

$$a_n(t) := \sum_{k=2}^{k_n} \Pi_k(X,t)\, \mathrm{cov}\,\{\, \exp\,(\, it\, S_{n,k-1}), X_{n,k}\} \underset{n}{\longrightarrow} 0 \tag{2.5}$$

$$b_n(t) := \sum_{k=2}^{k_n} \Pi_k(X,t)\, \mathrm{cov}\,\{\, \exp\,(\, it\, S_{n,k-1}), X^2_{n,k}\} \underset{n}{\longrightarrow} 0 \tag{2.6}$$

for all real t, *then* $S_n \xrightarrow{D} N(0,1)$.

Conversely if (2.4*a*), (2.4*b*) *hold and* $S_n \xrightarrow{D} N(0,1)$, *then*

$$a_n(t) - \frac{t^2}{2} b_n(t) \underset{n}{\longrightarrow} 0$$

for all real t.

Proof. By Taylor expansion

$$\exp\,(\, it\, X_{n,k}) \quad = \quad 1 + itX_{n,k} - \frac{t^2}{2} X^2_{n,k} + R(tX_{n,k}),$$

where for $R(x)$ $|R(x)| \le x^2$ and $|R(x)| \le |x|^3$ for all real x [10]. So

$$|\{\text{ the term in (2.2) }\} - ita_n(t) + \frac{t^2}{2} b_n(t)| \le 2 \sum_{k=2}^{k_n} E\,|R(tX_{n,k})|.$$

Since for any $\varepsilon > 0$ by the property of $R(x)$

$$|R(tX_{n,k})| \le t^2 X^2_{n,k} I(|X_{n,k}| > \varepsilon) + |tX_{n,k}|^3 I\,(|X_{n,k}| < \varepsilon),$$

and since

$$|tX_{n,k}|^3 I\,(|X_{n,k}| < \varepsilon) \le |t|^3\, \varepsilon X^2_{n,k},$$

the last sum tends to zero by (2.4a) and (2.4b). So (2.2) holds if and only if $ita_n(t) - \frac{t^2}{2} b_n(t) \to 0$ for all real t. ∎

We can apply Prop. 2.1 not only to $(X_{n,k})$ directly but to the arrays of its partial sums, as well. By this way we shall have more chance to decide whether $S_n \xrightarrow{D} N(0,1)$.

An array $(Y_{n,k})$, $k \le k_n$, such that $k_n \to \infty$, is said to be *the array of partial sums* of any array $(X_{n,i})$, $i \le i_n$ if for each n there is a partition of $[0, i_n]$: $0 = m_{n,0} < m_{n,1} < \ldots < m_{n,k_n} = i_n$ such that $Y_{n,k} = S_{n,m_{n,k-1},m_{n,k}}$, $k = 1, 2, \ldots, k_n$. Note that $S_n(Y) := \sum_{k=1}^{k_n} Y_{n,k} = S_n$.

Proposition 2.2. *For any array of zero mean-valued r.v.s $(X_{n,i})$, $i \le i_n \to \infty$, and any fixed real t suppose there exists an its partial sums array $(Y_{n,k}(t)), k \le k_n = k_n(t) \to \infty$, such that (2.4a) and (2.4b) hold for $(Y_{n,k}(t))$. Suppose further that there are non-negative integers $d_{n,k} = d_{n,k}(t)$ such that*

$$\sum_{k=2}^{k_n} E\,|S_{n,m_{n,k}-d_{n,k},m_{n,k}}(t)| \underset{n}{\longrightarrow} 0. \tag{2.8a}$$

Then $E\,e^{itS_n} \to e^{-t^2/2}$ if and only if

$$\sum_{k=2}^{k_n} \Pi_k(Y,t) \mathrm{cov}\{\exp(it\,S_{n,m_{n,k-1}-d_{n,k-1}}(t)), \exp(-itY_{n,k}(t))\} \underset{n}{\longrightarrow} 0, \tag{2.8b}$$

consequently if

$$\sum_{k=2}^{k_n} |\mathrm{cov}\{\exp(it\,S_{n,m_{n,k-1}-d_{n,k-1}}(t)), \exp(-itY_{n,k}(t))\}| \underset{n}{\longrightarrow} 0. \tag{2.8c}$$

Here, and in the sequel, we use the convention $\sum_a^b(\cdot)_k = \sum_1^b(\cdot)_k$ if $a \le 0$, and $\sum_a^b(\cdot)_k = 0$ if $b < a$ (to treat the cases when $m_{n,k} - d_{n,k}$ is not positive).

Proof. If $d_{n,k} = 0$ for all k, n then Proposition 2.1 applied to $(Y_{n,k})$ yields the conclusions.

We need only show that (2.8b) with $d_{n,k} = 0$ is equivalent to (2.8b) with $d_{n,k}$ satisfying condition (2.8a). But we have

| the term in (2.8b) - the term in (2.8b) with $d_{n,k} = 0$ |

$$\le \sum_{k=2}^{k_n} |E\,[e^{itS_{n,m_{n,k-1}}} - e^{itS_{n,m_{n,k-1}-d_{n,k-1}}}][e^{itY_{n,k}} - Ee^{itY_{n,k}}]|$$

$$\le 2\sum_{k=2}^{k_n} E|e^{itS_{n,m_{n,k-1}-d_{n,k-1},m_{n,k-1}}} - 1|.$$

Since $|\exp(ix) - 1| \le |x|$ for all real x, the right-hand side of the above chain of inequalities tends to 0 by (2.8a). So we obtain the first conclusion which obviously implies the last one. ∎

The following corollary could be useful for martingale-like sequences with which we should deal later.

Corollary 2. *The direct part of the conclusion of Proposition 2.2 remains valid if, beside the other conditions, (2.8b) is replaced by*

$$\sum_{k=2}^{k_n} \Pi_k(Y,t) \mathrm{cov}\,\{\,\exp\,(it\,S_{n,m_{n,k-1}-d_{n,k-1}}), Y^p_{n,k}\} \underset{n}{\longrightarrow} 0 \tag{2.10}$$

and (2.8c) *by*

$$\sum_{k=2}^{k_n} |\mathrm{cov}\,\{\,\exp\,(it\,S_{n,m_{n,k-1}-d_{n,k-1}}), Y^p_{n,k}\}| \underset{n}{\longrightarrow} 0, \tag{2.11}$$

where p*, as the powers of* $Y_{n,k} = Y_{n,k}(t)$*, takes the values* 1 *and* 2.

Proof. Similar to the proof of Proposition 2.2, by the previous corollary we need only to show that

$$\sum_{k=2}^{k_n} |E\,[\,(\exp\,(\,it\,S_{n,m_{n,k-1}}) - \exp\,(\,it\,S_{n,m_{n,k-1}-d_{n,k-1}}))(Y^p_{n,k} - E\,Y^p_{n,k})\,]| \underset{n}{\longrightarrow} 0 \tag{2.12}$$

for $p = 1$ and 2. However since $Y^p_{n,k} = Y^p_{n,k} I(|Y_{n,k}| \leq 1) + Y^p_{n,k} I(|Y_{n,k}| > 1)$ and by (2.4b)

$$\sum_{k=2}^{k_n} E\,|Y_{n,k}| I\,(|Y_{n,k}| > 1) \leq \sum_{k=2}^{k_n} E\,Y^2_{n,k} I\,(|Y_{n,k}| > 1) \underset{n}{\longrightarrow} 0,$$

we need only to show that (2.12) remains valid if $Y^p_{n,k}$ is replaced by $Y^p_{n,k} I\,(|Y_{n,k}| \leq 1)$. This can be done in a similar way as in the proof of Prop. 2.2 utilizing also the condition (2.8a). ∎

3. Central Limit Theorems for ℓ'-Mixing Arrays

For any array $(X_{n,i}),\ i \leq i_n$ and any sequence of positive integers m_n let us define its m_n*-size partial sums array* $(Y_{n,k}), k \leq k_n$ as the array of its partial sums with the partition $\{m_{n,k} := m_n k\}$ for $0 \leq k \leq [i_n/m_n]$ with the last point $k = k_n := [i_n/m_n]$ if i_n/m_n is an integer and $k_n := [i_n/m_n] + 1$ otherwise, where $[(\cdot)]$ denotes the integer part of $(\cdot)$. So by setting $n(\sum X_{n,\cdot})$ = the number of summands $X_{n,\cdot}$ in $\sum X_{n,\cdot}$ we have $n(Y_{n,k}) = m_n$ for $k \leq [i_n/m_n]$ and $0 \leq n(Y_{n,k_n}) \leq m_n$.

Theorem 3.1. *Let* $(X_{n,i}),\ i \leq i_n \to \infty$*, be any* ℓ'*-mixing array of zero mean-valued r.v.s for which there are numbers* $d_n \to \infty$ *such that*

$$\sup_a E\,|S_{n,a,a+d_n}| \underset{n}{\longrightarrow} 0. \tag{3.1a}$$

Suppose further that for every sequence of positive integers (m_n) *such that* $m_n \to \infty$ *and* $i_n/m_n \to \infty$ m_n*-size partial sums array* $(Y_{n,k})$ *of* $(X_{n,i})$ *satisfies the following conditions:*

$$\sum_{k=1}^{k_n} E\,Y_{n,k}^2 \underset{n}{\longrightarrow} 1 \tag{3.1b}$$

$$\sum_{k=1}^{k_n} E\,Y_{n,k}^2 I\left(|Y_{n,k}| > \varepsilon\right) \underset{n}{\longrightarrow} 0 \quad \textit{for every} \quad \varepsilon > 0. \tag{3.1c}$$

Then $S_n \xrightarrow{D} N(0,1)$.

To prove Theorem 3.1 we need the following lemma.

Lemma 3.1. *For a given array of non-negative numbers* $(B_{m,n})$ *and a given sequence of positive integers* (i_n) $(m, n = 1, 2, \dots)$ *suppose* $B_{m,n} \to 0$ *and* $i_n \to \infty$ *as* $n \to \infty$, *for each* m. *Then there exists a sequence* (m_n) *such that:* $m_n \to \infty$, $i_n/m_n \to \infty$ *and* $(i_n/m_n)B_{m_n,n} \to 0$, *as* $n \to \infty$.

Proof. Put

$$m_n := \min\left(m \le i_n\,;\ B_{m,n} \ + \ \frac{1}{m} \ \le \ \frac{m^2}{i_n^2} \right)$$

if the set in min(.) function is not empty, and put $m_n := i_n$ otherwise. Since $B_{m,n} \to 0$ and $i_n \to \infty$ we obtain that $m_n < i_n$ for large enough n. Hence for large n we have

$$\frac{i_n}{m_n^2} \le \Big(B_{m_n,n} + \frac{1}{m_n}\Big)\frac{i_n}{m_n} \le \frac{m_n}{i_n},$$

consequently $m_n^3 \ge i_n^2$ which implies that $m_n \to \infty$ as $i_n \to \infty$. By the definition of m_n and since $m_n \to \infty$

$$\Big(\frac{m_n - 1}{i_n}\Big)^2 < B_{m_n-1,n} + \frac{1}{m_n - 1} \ \longrightarrow \ 0$$

as $n \to \infty$. So all the terms in the above chain of inequalities tend to zero as $n \to \infty$ which imply the other conclusions of the lemma. ∎

Proof of Theorem 3.1. For any positive integer m let $(Y_{n,k})$ be the m-size partial sums array of $(X_{n,i})$ and put

$$L_{m,n}(t) := \max_{2 \le k \le k_n} \left|\operatorname{cov}\left\{ \exp\left(it\,S_{n,m(k-1)-d_n}\right),\ \exp\left(-itY_{n,k}\right)\right\}\right|$$

for $k_n \ge 2$ and $L_{m,n}(t) := 0$ for $k_n < 2$. Since by (1.1) $L_{m,n}(t) \le \ell_n'(d_n + 1, t)$, by (3.1a) and that $(X_{n,i})$ is ℓ'-mixing we have

$$B_{m,n}(t) := L_{m,n}(t) + \sup_a E\left|S_{n,a,a+d_n}\right| \underset{n}{\longrightarrow} 0$$

for all real t. By Lemma 3.1 applying to $(B_{m,n}(t))$ and (i_n) for each real t there exists a sequence $(m_n(t))$ such that $m_n(t) \to \infty$, $i_n/m_n(t) \to \infty$, $(i_n/m_n(t))L_{m_n(t),n}(t) \to 0$ and $(i_n/m_n(t))\sup_a E|S_{n,a,a+d_n}| \to 0$, as $n \to \infty$.

For $m_n(t)$-size partial sums array $(Y_{n,k}(t))$, $k \le k_n(t)$, because $k_n(t) \le (i_n/m_n(t)) + 1$, the third convergence implies (2.8c) with $d_{n,k} = d_n$ and that t. And the last one implies (2.8a).

So Proposition 2.2, applied to $(Y_{n,k}(t))$, implies that $E\exp(itS_n) \to \exp(-t^2/2)$ for the related t. Since this fact holds for all real t we obtain the conclusion. ■

Condition (3.1a) is a mild moment condition as it follows from condition

$$\max_i E\,|X_{n,i}| \le K/\sqrt{i_n}$$

for all n, where K is a constant. And the last one is satisfied for weakly stationary arrays. Since then the term in (3.1a) $\le Kd_n/\sqrt{i_n} \to 0$ by putting $d_n := \sqrt[3]{i_n}$. In the case when $X_{n,i} = X_i/\sqrt{n}$ for $i \le n = i_n$, as usually considered in C.L.T.s for sequences of r.v.s, condition (3.1a) is implied by

$$E\,|X_n| \le K,$$

for all n, which is a quite mild condition.

Conditions (3.1b) and (3.1c) have weaker but more clear forms in the theorem below which needs some notations.

Let us denote as in [11], for any array $(X_{n,i})$, $i \le i_n$

$$C_n(m) = \sum_{a=m}^{i_n-1} \sup_{\substack{|i-j|=a \\ 1\le i,j\le i_n}} |\mathrm{cov}\,\{\,X_{n,i}, X_{n,j}\}|.$$

The numbers $C_n(m)$ can be considered as rough but easy to measure dependence coefficients of the involved array. Also denote $\bar{X} := X/\sqrt{E\,X^2}$ for any r.v. X such that $E\,X = 0$ and $0 < EX^2 < \infty$, and $\bar{X} = 0$ if $EX^2 = 0$. So $E\bar{X} = 0$ and $E\bar{X}^2 = 1$, i.e., $\bar{X}$ is the normalization of X if $X \ne 0$.

Theorem 3.2. *For any ℓ'-mixing array of zero mean-valued r.v.s $(X_{n,i})$, $i \le i_n \to \infty$, suppose*

$$E\,S_n^2 \underset{n}{\longrightarrow} 1 \tag{3.2a}$$

$$i_nC_n(0) < A \quad \text{for all } n \text{ and some constant } A \tag{3.2b}$$

$$i_nC_n(m) \longrightarrow 0 \quad \text{as } m \text{ and } n \to \infty \tag{3.2c}$$

$$\limsup_{\substack{n,a,b \\ a<b\le i_n}} E\,\bar{S}_{n,a,b}^2 I\,(|\bar{S}_{n,a,b}| > C) \longrightarrow 0 \quad \text{as } C \to \infty. \tag{3.2d}$$

Then $S_n \xrightarrow{D} N(0,1)$.

To prove the theorem we make use of the following notations and a lemma. Let $(Y_{n,k})$, $k \le k_n$, be the m_n-size partial sums array of any array of zero mean-valued r.v.s $(X_{n,i})$, $i \le i_n$, for any given sequence of positive integers (m_n) such that $m_n \to \infty$. For (m_n) suppose there are given two sequences of non-negative integers (p_n) and (q_n) such that $m_n = p_n + q_n$. Denote $Y_{n,k}^p$ as the sum of the first p_n summands of $Y_{n,k}$ i.e. $Y^p_{n,k} := S_{n,m_n(k-1),m_n(k-1)+p_n}$ for $1 \le k \le k_n$, otherwise if $n(Y_{n,k_n}) < p_n$ denote $Y^p_{n,k_n} := Y_{n,k_n}$. Also denote $Y^q_{n,k} := Y_{n,k} - Y^p_{n,k}$ for all $k \le k_n$. So $n(Y^p_{n,k}) \le p_n$ and $n(Y^q_{n,k}) \le q_n$ for all $k \le k_n$.

Lemma 3.2. [11] *For each fixed* n

$$\begin{aligned}
&E\, S^2_{n,a,a+b} \le 2bC_n(0),\\
&E\,(\sum_{k=1}^{k_n} Y^q_{n,k})^2 \le 2k_nq_nC_n(0) \quad \text{and}\\
&|E\,(\sum_{k=1}^{k_n} Y^p_{n,k})^2 - \sum_{k=1}^{k_n} E\,Y^{p2}_{n,k}| \le 2k_np_nC_n(q_n+1).
\end{aligned}$$

Proof of Lemma 3.2. We have

$$ES^2_{n,a,a+b} \le 2\sum_{d=0}^{b-1}\sum_{i=a+1}^{a+b-d} EX_{n,i}X_{n,i+d} \le 2\sum_{d=0}^{b-1} b \sup_i |EX_{n,i}X_{n,i+d}| \le 2bC_n(0).$$

For the second inequality, since $n(\sum_{k=1}^{k_n} Y^q_{n,k}) \le k_nq_n$ similarly we have its left-hand side

$$\le 2\sum_{d=0}^{i_n-1}\sum_{i,i+d\in I_q} EX_{n,i}X_{n,i+d} \le 2\sum_{d=0}^{i_n-1} k_nq_n \sup_i |EX_{n,i}X_{n,i+d}| \le 2k_nq_nC_n(0),$$

where I_q is the set of all indexes i such that $X_{n,i}$ appears in $\sum_{k=1}^{k_n} Y^q_{n,k}$.

For the third inequality of the lemma we have the left-hand side of it

$$\begin{aligned}
&= 2|\sum_{j=2}^{k_n}\sum_{i=1}^{j-1} EY^p_{n,i}Y^p_{n,j}| = 2|\sum_{j=2}^{k_n}\sum_{i=1}^{j-1} E(\sum_{u=m_n(i-1)+1}^{m_n(i-1)+p_n} X_{n,u})(\sum_{v=m_n(j-1)+1}^{m_n(j-1)+p_n} X_{n,v})|\\
&\le 2\sum_{j=2}^{k_n}\sum_{v=m_n(j-1)+1}^{m_n(j-1)+p_n}(\sum_{i=1}^{j-1}\sum_{u=m_n(i-1)+1}^{m_n(i-1)+p_n} |EX_{n,u}X_{n,v}|)\\
&\le 2\sum_{j=2}^{k_n}\sum_{v=m_n(j-1)+1}^{m_n(j-1)+p_n}\sum_{s=q_n+1}^{i_n-1}\sup_{|u-v|=s} |EX_{n,u}X_{n,v}| \le 2k_np_nC_n(q_n+1),
\end{aligned}$$

using the convention $X_{n,v} = 0$ if $v > i_n$. ∎

Proof of Theorem 3.2. In order to apply Theorem 3.1 to our array we shall check conditions (3.1a), (3.1b) and (3.1c) for it.

Condition (3.1a) holds for our array because by Lemma 3.2 and (3.2b) for any sequence $d_n \to \infty$ such that $i_n/d_n \to \infty$ we have

$$E\,|S_{n,a,a+d_n}| \le \sqrt{E\,S^2_{n,a,a+d_n}} \le \sqrt{d_n C_n(0)} \underset{n}{\longrightarrow} 0.$$

To check out conditions (3.1b) and (3.1c) let us fix an arbitrary sequence (m_n) such that $m_n \to \infty$ and $i_n/m_n \to \infty$ and let $(Y_{n,k})$, $k \le k_n$, be the m_n-size partial sums array of $(X_{n,i})$.

Then (3.1b) holds if

$$a_n := |\sum_{k=1}^{k_n} E\,(Y^2_{n,k} - Y^{p2}_{n,k})| \underset{n}{\longrightarrow} 0 \quad \text{and} \quad b_n := \sum_{k=1}^{k_n} E\,Y^{p2}_{n,k} \underset{n}{\longrightarrow} 1.$$

Applying Schwarz inequality twice we get

$$\begin{aligned} a_n &= |\sum_{k=1}^{k_n} E\,(Y^{q2}_{n,k} + 2Y^q_{n,k}Y^p_{n,k})| \le \sum_{k=1}^{k_n} \{E\,Y^{q2}_{n,k} + 2(E\,Y^{q2}_{n,k} E\,Y^{p2}_{n,k})^{\frac{1}{2}}\} \\ &\le \sum_{k=1}^{k_n} E\,Y^{q2}_{n,k} + 2\big(\sum_{k=1}^{k_n} E\,Y^{q2}_{n,k}\big)^{\frac{1}{2}}\big(\sum_{k=1}^{k_n} E\,Y^{p2}_{n,k}\big)^{\frac{1}{2}}. \end{aligned}$$

Choose $q_n = \sqrt{m_n}$ and $p_n = m_n - \sqrt{m_n}$. By the first conclusion of Lemma 3.2 and (3.2b)

$$\sum_{k=1}^{k_n} E\,Y^{q2}_{n,k} \quad \le \quad 2(\frac{i_n}{m_n} + 1) q_n C_n(0) \underset{n}{\longrightarrow} 0.$$

So by the above chain of inequalities $a_n \to 0$ if $b_n \to 1$, which we shall show next.

We have, considering $(E\,X^2)^{\frac{1}{2}}$ as $||X||_2$ for any r.v. X,

$$|(E\,S^2_n)^{\frac{1}{2}} - (E\,(\sum_{k=1}^{k_n} Y^p_{n,k})^2)^{\frac{1}{2}}| \le (E\,(\sum_{k=1}^{k_n} Y^q_{n,k})^2)^{\frac{1}{2}}$$

in which $E\,S^2_n \to 1$ by (3.2a) and the term on the right-hand side tends to zero as $n \to \infty$ by the second conclusion of Lemma 3.2 and (3.2b). So the middle term of the last inequality tends to 1 as $n \to \infty$. However by the third conclusion of Lemma 3.2 and (3.2c) we also have

$$\big|E\,(\sum_{k=1}^{k_n} Y^p_{n,k})^2 - E\,(\sum_{k=1}^{k_n} Y^{p2}_{n,k})\big| \le 2(\frac{i_n}{m_n} + 1) p_n C_n(q_n + 1) \underset{n}{\longrightarrow} 0,$$

which implies that $b_n \to 1$.

For checking condition (3.1c) we have the term in (3.1c)

$$\begin{aligned} &= \sum_{k=1}^{k_n} E\,Y^2_{n,k} E\,\bar{Y}^2_{n,k} I\,(|\bar{Y}_{n,k}| > \varepsilon/(E\,Y^2_{n,k})^{\frac{1}{2}}) \\ &\le \left[\sum_{k=1}^{k_n} E\,Y^2_{n,k}\right] \sup_k E\,\bar{Y}^2_{n,k} I\,(|\bar{Y}_{n,k}| > \varepsilon/\sup_k (E\,Y^2_{n,k})^{\frac{1}{2}}), \end{aligned}$$

where the term in the square brackets tends to 1 as we have just showed and by Lemma 3.2 and (3.2b)

$$\sup_k E\,Y_{n,k}^2 \le m_n C_n(0) \underset{n}{\longrightarrow} 0.$$

So by (3.2d) the last term of above chain of inequalities tends to zero as $n \to \infty$ from which (3.1c) follows.

So by Theorem 3.1 we have the conclusion. ∎

This theorem improves Theorem 2.2 of [11] in some aspects. Here we do not assume the stationarity of $X_{n,i}$ but only (3.2b) and (3.2c).

4. Examples of Non-mixing Arrays

We shall give here two examples of the arrays which are not ℓ-mixing but ℓ'-mixing. These examples satisfy all conditions of Theorems 3.1 but not those of Theorem 3.2.

Example 4.1. Let (X_n) and (V_n), $n = 1, 2, \dots$, be two sequences of independent identically distributed r.v.s with zero mean and 1 variance, such that (X_n) is independent from (V_n). The two distribution functions may be different. Suppose further that the distribution functions of $2V_1$ and $V_1 + V_2$ are not the same. Define

$$X_{n,i} := \frac{1}{\sqrt{n}}\{\, X_i + (-1)^i V_n\}, \quad i \le i_n = n.$$

Then by the definition of ℓ-mixing, for $t \neq 0$ and $c > 0$ we have for the array $(X_{n,i})$, $i \le n$,

$$\ell(c,t) \ge |E\,e^{\frac{it}{\sqrt{n}}(A+B)} - E\,e^{\frac{it}{\sqrt{n}}A} E\,e^{\frac{it}{\sqrt{n}}B}|, \tag{4.1}$$

where A and B are the sums of $X_k + (-1)^k V_n$ in which k takes only even numbers belonging to $P = [1, 2\sqrt{n}]$ and $Q = [2\sqrt{n} + c, 4\sqrt{n}]$, respectively.

As k is even the sum of $(-1)^k V_n$ in A and B equal to $N(P)V_n$ and $N(Q)V_n$ where $N(P) = [\sqrt{n}]$ and $N(Q) = [\sqrt{n} - c/2]$, as the numbers of k in the related intervals P and Q. So by the independence of r.v.s the right-hand side of (4.1) which equals to

$$\big|E\,e^{\frac{it}{\sqrt{n}}(\sum\limits_{k \text{ even}}^{P\cup Q} X_k)}\big[E\,e^{\frac{it}{\sqrt{n}}V_n(N(P)+N(Q))} - E\,e^{\frac{it}{\sqrt{n}}V_n N(P)} E\,e^{\frac{it}{\sqrt{n}}V_n N(Q)}\big]\big|, \tag{4.2}$$

tends to

$$\big|E\,e^{it2V_1} - E\,e^{itV_1} E\,e^{itV_1}\big|,$$

as $n \to \infty$, noting that

$$\frac{1}{n}E\,(\sum_{k \text{ even } \in P\cup Q} X_k)^2 = \frac{1}{n}(N(P) + N(Q)) \underset{n}{\longrightarrow} 0.$$

By choosing t such that the above limit is not zero and by (4.1) we see $(X_{n,i})$ is not ℓ-mixing.

To show this $(X_{n,i})$ is ℓ'-mixing note that the $\operatorname{cov}(.,.)$ in (1.1) can also be written in the form of (4.2) in which k takes all the numbers in $P := [1, a]$ and in $Q := [a+c, a+c+b]$. But in this case $N(P)$ and $N(Q)$ can be only ± 1 or zero. Hence $\ell'_n(c,t) \to 0$ as $n \to \infty$ for all c and t.

Let us show $(X_{n,i})$ satisfies all conditions of Theorem 3.1.

Condition (3.1a) holds with $d_n = [\sqrt[3]{n}]$ because by Minkowski's inequality

$$E\,|S_{n,a,a+d_n}| \le \sqrt{E\,S^2_{n,a,a+d_n}} \le \frac{d_n}{\sqrt{n}}\sqrt{EX_1^2} + \frac{1}{\sqrt{n}}\sqrt{EV_1^2}.$$

Let $(Y_{n,k})$, $k \le k_n$, be the m_n-size partial sums array of $(X_{n,i})$ for any given sequence (m_n) such that m_n and $n/m_n \to \infty$ as $n \to \infty$. Then $Y_{n,k}$ can be written in the form $(\sum_j X_j + eV_n)/\sqrt{n}$ where e can be only ± 1 or zero. Since X_k are normalized i.i.d. and since $k_n \le (n/m_n) + 1$ we have $1 \le$ the sum in (3.1b) $\le 1 + (k_n/n)EV_n^2 \le 1 + (1/m_n) + (1/n) \to 1$. So (3.1b) holds.

To verify condition (3.1c) let us denote $S_{nme} = (\sum_1^{m_n} X_k + eV_1)/\sqrt{m_n}$ where $e = \pm 1$ or 0. Then since V_n and X_n are i.i.d. and X_i and V_1 are independent $Y_{n,k} = (\sqrt{m}_n/\sqrt{n})S_{nme}$ in distribution for any value of e, for all $k < k_n$. So for any δ and n such that $\varepsilon\sqrt{n/m_n} > \delta$ the term in (3.1c)

$$\le (k_n - 1)\frac{m_n}{n} \max_e \; E\,S^2_{nme} I\left(|S_{nme}| > \varepsilon\sqrt{\frac{n}{m_n}}\right) + E\,Y^2_{n,k_n}$$
$$\le \max_e (1 + \frac{1}{m_n} - E\,S^2_{nme} I\,(|S_{nme}| < \delta)) + \frac{m_n}{n}EX_1^2 + \frac{1}{n}EV_1^2.$$

Note that $S_{nme} \xrightarrow{D} N(0,1)$ for all values of e because $V_1/\sqrt{m_n} \to 0$ in probability. So by Helly-Bray lemma ([2] p. 251) the right-hand side of above inequality tends to $E\,N^2 I\,(|N| > \delta)$ as $n \to \infty$ where N is any $N(0,1)$-distributed r.v.. Since δ can be chosen arbitrarily large as $n/m_n \to \infty$ we obtain that the term in (3.1c) tends to zero for any $\varepsilon > 0$.

Theorem 3.2 can not apply to this example because conditions (3.2b) and (3.2c) are not satisfied for it. As we have

$$|\operatorname{cov}(X_{n,i}, X_{n,j})| = |E\,V_n^2(-1)^{i+j}/n| = 1/n,$$

for $i \ne j$, so $C_n(m) = (n-m)/n$ if $m < n$ and $C_n(0) = 1 + 1/n$ which do not tend to zero as n and $m \to \infty$.

Example 4.2. Let (X_n) be as in Example 4.1 and (U_n), $n = \dots, -1, 0, 1, \dots$ be a sequence of independent $N(0,1)$-distributed r.v.s., such that (X_n) and (U_n) are independent. Define $V_k := \sum_{j=0}^{\infty} U_{k-j} g_j$ where $g_j = \frac{1}{\sqrt{j}\ln j}$, for $j \ge 2$ and $g_0 = g_1 = 1$. Since $\sum_0^\infty g_j^2$ is finite, the characteristic function of $\sum_{j=0}^n U_{k-j} g_j = e^{-(\sum_0^n g_j^2)t^2/2} \to e^{-(\sum_0^\infty g_j^2)t^2/2}$ as $n \to \infty$. So V_k exists as the a.s. limit of $\sum_{j=0}^n U_{k-j} g_j$ and is normally distributed with zero mean and variance $= \sum_0^\infty g_j^2$ ([2, Theorem 3, p. 266 and Theorem 4 p. 269]).

Consider the array $(X_{n,i})$, $i \le 2n$, defined by

$$X_{n,i} \;\; := \;\; \frac{1}{\sqrt{n}}\{\, X_i + (-1)^i V_{[(i+1)/2]}\,\}.$$

Apply (4.1) to this array with the sums A and B of $X_{n,i}$ which are taken over even k such that $1 \le k \le \sqrt{n \ln n}$ and $\sqrt{n \ln n} + c < k \le 2\sqrt{n \ln n}$, respectively. Denote $a_n := [\sqrt{n \ln n}/2]$, $b_n := [(\sqrt{n \ln n} + c)/2]$ and $c_n := [\sqrt{n \ln n}]$. Then the sums of $(-1)^k V_{[(k+1)/2]}$ in A and B equal

$$\sum_{k=1}^{a_n} V_k = \sum_{i=-a_n}^{\infty} U_{-i}(\sum_{k=1}^{a_n} g_{k+i}) = \sum_{i=-a_n}^{\infty} U_{-i} G_i(a_n)$$

and

$$\sum_{k=b_n+1}^{c_n} V_k = \sum_{i=-c_n}^{\infty} U_{-i}(\sum_{k=b_n+1}^{c_n} g_{k+i}) = \sum_{i=-c_n}^{\infty} U_{-i} G_{i+b_n}(c_n - b_n),$$

respectively, where $G_i(a) := \sum_{k=1}^{a} g_{k+i}$ and using the convention $g_n = 0$ for $n < 0$. So

$$\sum_{k=1}^{a_n} V_k + \sum_{k=b_n+1}^{c_n} V_k = \sum_{i=-c_n}^{\infty} U_{-i}(G_i(a_n) + G_{i+b_n}(c_n - b_n)).$$

As showed, these terms are normally distributed with zero mean and variance $E(\sum_{k=1+a}^{b+a} V_k)^2 = \sum_{i=-b}^{\infty} G_i^2(b)$ if the last term is finite. Hence similar to (4.2) the right-hand side of (4.1) becomes

$$\Big| E e^{\frac{it}{\sqrt{n}} \sum X_k} \Big[e^{-\frac{t^2}{2n} \sum\limits_{i=-c_n}^{\infty} (G_i(a_n) + G_{i+b_n}(c_n - b_n))^2} - e^{-\frac{t^2}{2n} (\sum\limits_{i=-a_n}^{\infty} G_i^2(a_n) + \sum\limits_{i=b_n-c_n}^{\infty} G_i^2(c_n - b_n))} \Big] \Big|. \tag{4.3}$$

To continue we need the following fact.

Lemma 4.1. *For any* $a > 0$ $\sum_{i=-a}^{\infty} G_i^2(a)$ *is finite and*

$$\sum_{i=-a}^{\infty} G_i^2(a) \sim \frac{a^2}{\ln a}$$

as $a \to \infty$.

Proof. We have $G_i(a) \le \int_i^{i+a} \frac{1}{\sqrt{x} \ln x} dx \le G_{i-1}(a)$ for any $i \le 2$ and $a \ge 1$. Let us estimate the integral. Since the function $f(x) = \sqrt{x+1}$ is derivable and concave, for any small $0 < \epsilon < 1/4$ there exists a $0 < \delta_\epsilon < 1$ such that for all $0 \le x \le \delta_\epsilon$ we have

$$(\frac{1}{2} - \epsilon)x \le \sqrt{(1+x)} - 1 \le \frac{x}{2}. \tag{4.4}$$

Since $\ln x \le x$ for all $x > 0$ and $\frac{2\sqrt{x}}{\ln x} = \int \frac{1}{\sqrt{x} \ln x}(1 - \frac{2}{\ln x})dx$, for any small $\epsilon > 0$ by (4.4) there exists a constant K_ϵ such that for all $i \ge \max(K_\epsilon, a/\delta_\epsilon)$ we have

$$\begin{aligned}\frac{a(1-4\epsilon)}{\sqrt{i}\ln i} &\le \frac{2a}{\sqrt{i}\ln i}\Big[\frac{1}{2}-\epsilon-\frac{\sqrt{2}}{\ln(i+a)}\Big]\\ &\le \frac{2\sqrt{i}}{\ln i}\Big[\Big(1+\frac{a}{i}\Big)^{1/2}-1-\Big(1+\frac{a}{i}\Big)^{1/2}\frac{\ln\Big(1+\frac{a}{i}\Big)}{\ln(i+a)}\Big]\\ &= \Big[\frac{2\sqrt{x}}{\ln x}\Big]_i^{i+a} \le \int_i^{i+a}\frac{1}{\sqrt{x}\ln x}dx \le \int_i^{i+a}\frac{1}{\sqrt{x}\ln x}\big(1-\frac{2}{\ln x}\big)(1+\epsilon)dx\\ &= (1+\epsilon)\Big[\frac{2\sqrt{x}}{\ln x}\Big]_i^{i+a} \le (1+\epsilon)\frac{2\sqrt{i}}{\ln i}\Big(\Big(1+\frac{a}{i}\Big)^{1/2}-1\Big) \le \frac{a(1+\epsilon)}{\sqrt{i}\ln i}.\end{aligned}$$

So by putting $B_a := \sum_{i=a/\delta_\epsilon}^{\infty} G_i^2(a)$ for all a such that $a/\delta_\epsilon \ge K_\epsilon$ we have

$$\frac{(1-4\epsilon)^2a^2}{\ln(a/\delta_\epsilon+1)} \le \sum_{i=a/\delta_\epsilon+1}^{\infty}\frac{(1-4\epsilon)^2a^2}{i\ln^2 i} \le B_a \le \sum_{i=a/\delta_\epsilon}^{\infty}\frac{(1+\epsilon)^2a^2}{i\ln^2 i} \le \frac{(1+\epsilon)^2a^2}{\ln(a/\delta_\epsilon-1)}.$$

Hence

$$(1-4\epsilon)^2 \le \liminf_a B_a \ln a/a^2 \le \limsup_a B_a \ln a/a^2 \le (1+\epsilon)^2.$$

Since $G_i(a) \le (1+\epsilon)\big[\frac{2\sqrt{x}}{\ln x}\big]_i^{i+a}$ for all $i \ge K_\epsilon$ as a part of the above chain of inequalities using the convention $G_i(a) = 0$ if $a < 0$, for the remainder term $A_a := \sum_{i=-a}^{\infty} G_i^2(a) - B_a$ we have

$$\begin{aligned}A_a &\le \sum_{i=-a}^{a/\delta_\epsilon-1}(G_i(K_\epsilon-i)+G_{K_\epsilon}(a+i-K_\epsilon))^2\\ &\le 2\sum_{i=-a}^{a/\delta_\epsilon-1}(G_i^2(K_\epsilon-i)+G_{K_\epsilon}^2(a+i-K_\epsilon))\\ &\le 2\Big(\frac{a}{\delta_\epsilon}+a\Big)G_{-1}^2(K_\epsilon)+2+8(1+\epsilon)^2\sum_{i=K_\epsilon-a+1}^{a/\delta_\epsilon-1}\frac{(a+i)}{\ln^2(a+i)}.\end{aligned}$$

We have the last sum

$$\le \int_{K_\epsilon}^{\frac{a}{\delta_\epsilon}+a}\frac{x}{\ln^2 x}dx \le \int_2^{\frac{a}{\delta_\epsilon}+a}\frac{x}{\ln^2 x}\Big(1-\frac{1}{\ln x}\Big)Kdx = K\Big[\frac{x^2}{2\ln^2 x}\Big]_2^{\frac{a}{\delta_\epsilon}+a},$$

where $K = 1/(1-1/\ln 2)$. Consequently $A_a \ln a/a^2 \to 0$ when $a \to \infty$.

Hence we obtain the conclusions, as ϵ can be chosen arbitrarily small. ■

By a similar way we can prove that

$$\sum_{i=-c_n}^{\infty}(G_i(a_n)+G_{i+b_n}(c_n-b_n))^2 \sim \frac{(a_n+c_n-b_n)^2}{\ln(a_n+c_n-b_n)} \sim 2n \qquad (4.5)$$

as $n \to \infty$, noting that in this case we have $G_i(a_n) + G_{i+b_n}(c_n - b_n) \le \int_i^{i+a_n} \frac{1}{\sqrt{x}\ln x}dx + \int_{i+b_n}^{i+c_n} \frac{1}{\sqrt{x}\ln x}dx \le G_{i-1}(a_n) + G_{i+b_n-1}(c_n - b_n)$ and the middle term, i.e., the sum of integrals is bounded between

$$\frac{(a_n + c_n - b_n)(1-4\epsilon)}{\sqrt{i+b_n}\ln(i+b_n)} \le \frac{a_n(1-4\epsilon)}{\sqrt{i}\ln i} + \frac{(c_n - b_n)(1-4\epsilon)}{\sqrt{i+b_n}\ln(i+b_n)}$$

$$\text{and} \quad \frac{(a_n + c_n - b_n)(1+\epsilon)}{\sqrt{i}\ln i},$$

and instead of B_a consider $B_n := \sum_{i=(a_n+c_n-b_n)/\delta_\epsilon}^{\infty} (G_i(a_n) + G_{i+b_n}(c_n - b_n))^2$.

Since $\frac{a_n^2}{\ln a_n} \sim \frac{n}{2} \sim \frac{(c_n-b_n)^2}{\ln(c_n-b_n)}$, by (4.3), Lemma 4.1 and (4.5), going similarly to the Example 4.1 we can obtain that the right-hand side of (4.1) tends to $|e^{-t^2} - e^{-t^2/2}|$ as $n \to \infty$. So $(X_{n,i})$ is not ℓ-mixing.

Since any sum of $X_{n,i}$ where i is taken from a to b equals to the related sum of X_i plus zero or $V_{a/2}$ or $-V_{(b+1)/2}$ or $V_{a/2} - V_{(b+1)/2}$, divided by $\sqrt{n}$, and since V_k are normally distributed with variance $\sum_0^\infty g_j^2$, in a similar way as in Example 4.1 we can see that $(X_{n,i})$ is an ℓ'-mixing array satisfying all assumptions of Theorem 3.1.

To show condition (3.2b) is not satisfied for this array, hence Theorem 3.2 can not apply to it, let us compute

$$\operatorname{cov}(X_{nj}, X_{ni}) = \frac{1}{n}(-1)^{i+j} EV_{\left[\frac{i+1}{2}\right]} V_{\left[\frac{j+1}{2}\right]}$$

for any $i \ne j$. Put $V_i^m := \sum_{k=0}^m U_{i-k} g_k$. Then we have $EV_iV_j = EV_i^m V_j^m + EV_i^m(V_j - V_j^m) + EV_j(V_i - V_i^m)$, where by Schwarz inequality

$$|EV_i^m(V_j - V_j^m)| + |EV_j(V_i - V_i^m)| \le 2(\sum_{k=0}^{\infty} g_k^2)^{1/2}(\sum_{k=m}^{\infty} g_k^2)^{1/2} \to 0$$

as $m \to \infty$. So $EV_iV_j = \lim_m EV_i^m V_j^m = \lim_m \sum_{k=0}^m \sum_{l=0}^m E(U_{j-k}U_{i-l}) g_k g_l = \lim_m \sum_{k=0}^{m-(i-j)} g_k g_{k+(i-j)} = \sum_{k=0}^{\infty} g_k g_{k+(i-j)}$. Hence for the term in condition (3.2b) we have

$$\begin{aligned} C_n(0)i_n &\ge 2 \sum_{\substack{a=0 \\ a \text{ even}}}^{2n-1} \sup_{i-j=a} \sum_{k=0}^{\infty} g_k g_{k+\left[\frac{i+1}{2}\right]-\left[\frac{j+1}{2}\right]} \\ &= 2 \sum_{a=0}^{n-1} \sum_{k=0}^{\infty} g_k g_{k+a} \\ &\ge 2 \sum_{a=0}^{n-1} \sum_{k=a}^{\infty} g_k^2 \ge 2n \sum_{k=n}^{\infty} g_k^2 \ge 2n \Big[\frac{-1}{\ln x}\Big]_n^\infty \underset{n}{\to} \infty. \end{aligned}$$

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