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On Chain Decompositions of Ordered Semigroups

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Abstract. Characterizations of an ordered semigroup which is a natural ordered chain of $\mathcal{N}$-simple and $\mathcal{J}_n$-simple subsemigroups, respectively, are given. Some new characterizations of intra-regular ordered semigroups are considered. As corollaries, characterizations of an ordered semigroup which is a natural ordered chain of simple and archimedean subsemigroups, respectively, are obtained.

1. Introduction and Preliminaries

An *ordered semigroup* (:*po-semigroup*) $(S, \cdot, \leq)$ is a partially ordered set $(S, \leq)$ at the same time a semigroup $(S, \cdot)$ such that: $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$ for every $a, b, x \in S$. (For $H \subseteq S$, $(H] := \{x \in S | (\exists h \in H) x \leq h\}$). In this paper, S stands for an arbitrary ordered semigroup.

Let I be a nonempty subset of S. The *radical* $\sqrt{I}$ of I is defined by: $\sqrt{I} := \{x \in S | (\exists m \in Z^+) x^m \in (I]\}$. I is called an *ideal* of S if: (i) $SI \subseteq I$, $IS \subseteq I$; and (ii) $a \in I$, $b \in S$, $b \leq a$ implies $b \in I$ (see [1]). I is said to be *prime* (resp. *semiprime*) if: for any $a, b \in S$, $ab \in I$ implies $a \in I$ or $b \in I$ (resp. for any $a \in S$, $a^2 \in I$ implies $a \in I$) (see [2]). As in [1], $J(x)$ will denote the least ideal of S containing x $(x \in S)$ and $\mathcal{J}$ will denote the well-known Green's relation on S defined by $\mathcal{J} := \{(x, y) \in S \times S | J(x) = J(y)\}$. Let F be a subsemigroup of S. F is called a *filter* of S if: (i) $a, b \in S$, $ab \in F$ implies $a \in F$ and $b \in F$; and (ii) $a \in F$, $b \in S$, $a \leq b$ implies $b \in F$ (see [3]). We denote by $N(x)$ the least filter of S containing x $(x \in S)$ and, as in [3], by $\mathcal{N}$ the equivalence relation on

S defined by $\mathcal{N} := \{(x, y) \in S \times S | N(x) = N(y)\}$.

Lemma 1.1. *Let F be a nonempty subset of an ordered semigroup S. Then F is a filter of S if and only if F=S or $S \backslash F$ is a prime ideal of S.*

Let S be a semilattice (chain) Y of subsemigroups S_α $(\alpha \in Y)$. As in [8], the semilattice (chain) congruence determined by the partion $\{S_\alpha;\ \alpha \in Y\}$ is said to be *natural ordered* if: $(\forall \alpha, \beta \in Y)(\forall a \in S_\alpha)(\forall b \in S_\beta)\ a \leq b \Longrightarrow \alpha \leq \beta$, where the natural ordering "$\leq$" on Y is defined by: for any $\alpha, \beta \in Y$, $\alpha \leq \beta$ if and only if $\alpha\beta = \alpha$; and in this case, S is called the *natural ordered semilattice* (*chain*) *Y of ordered semigroups S_α* $(\alpha \in Y)$ and Y is called a *natural ordered semilattice* (*chain*) *homomorphic image* of S.

Lemma 1.2. (cf. [4]) *For any ordered semigroup S, $\mathcal{N}$ is the least natural ordered* (: *complete*, see [6]) *semilattice congruence of S.*

Let Z^+ be the set of all positive integers and let $n \in Z^+$. In [7, Definition 1.3] and [9, Definition 3.1], some relations on S are defined as follows: for any $a, b \in S$,

$$a\tau b \Longleftrightarrow b \in J(a), \quad a\sigma b \Longleftrightarrow a\tau b^2, \quad a\eta b \Longleftrightarrow (\exists m \in Z^+)a\tau b^m,$$
$$\rho = \cup_{k\in Z^+}\sigma^k = \cup_{k\in Z^+}\eta^k, \quad \xi = \rho \cap \rho^{-1};$$
$$J_n(a) = \{x \in S | a\eta^n x\}, \quad \mathcal{J}_n = \{(x, y) \in S \times S | J_n(x) = J_n(y)\}.$$

Lemma 1.3. [9, Lemma 3.1] *Let S be an ordered semigroup and let $n \in Z^+$.*
(i) *$\mathcal{J}_n$ is an equivalence relation on S satisfying $\mathcal{J}_n \subseteq \eta^n \cap (\eta^n)^{-1}$.*
(ii) *$\mathcal{J} = \tau \cap \tau^{-1} \subseteq \mathcal{J}_1 \subseteq \cdots \subseteq \mathcal{J}_n \subseteq \mathcal{J}_{n+1} \subseteq \cdots \subseteq \xi = \mathcal{N}$.*

Lemma 1.4. [9, Theorem 2.4] *Let an ordered semigroup S be a natural ordered semilattice Y of ordered semigroups S_α $(\alpha \in Y)$ and let $n \in Z^+$. Let $a \in S_\alpha$, $b \in S_\beta$ with $\alpha, \beta \in Y$.* (i) *If $a\eta^n b$ in S, then $\beta \leq \alpha$.* (ii) *If $\alpha = \beta$ and $a\eta^n b$ in S, then $a\eta^n b$ in S_α.*

Lemma 1.5. [9, Theorems 2.5 and 2.6] *For any ordered semigroup S, we have*
(i) *$M(a) := \{x \in S | a\rho x\}$ is the least semiprime ideal of S containing a, for all $a \in S$;*
(ii) *$M(ab) = M(a) \cap M(b)$ for all $a, b \in S$* (also see [8, Lemma 7]);
(iii) *$N(a) = \{x \in S | x\rho a\}$;*
(iv) *$\mathcal{N} = \{(x, y) \in S \times S | M(x) = M(y)\}$.*

Let ω be one of the equivalence relations in Lemma 1.3 (ii) above, as in [9, Definition 3.2], S is said to be *ω-simple* if $\omega = S \times S$.

Lemma 1.6. [9, Theorem 2.7] *Each $\mathcal{N}$-class of an ordered semigroup S is an $\mathcal{N}$-simple subsemigroup of S.*

It is clear that S is $\mathcal{J}$-simple if and only if S is simple (cf. [1]) and S is $\mathcal{J}_1$-simple if and only if S is archimedean (i.e., for every $a, b \in S$, $b^m \in (SaS]$

for some $m \in Z^+$) (cf. [7]). Ordered semigroups which are chains of simple (i.e., $\mathcal{J}$-simple) ordered semigroups are considered by Kehayopulu [5]. Semilattices of archimedean (i.e., $\mathcal{J}_1$-simple) ordered semigroups are described by Kehayopulu [6] and Cao [7, Theorem 2.7], etc. In Sec. 2 of this paper, we give characterizations of ordered semigroups which are chains of $\mathcal{N}$-simple ordered semigroups. Then we consider characterizations of intra-regular ordered semigroups and obtain some new characterizations of ordered semigroups which are chains of simple ordered semigroups. In Sec. 3, we give characterizations of ordered semigroups which are natural ordered chains of $\mathcal{J}_n$-simple ordered semigroups. As a corollary, we obtain characterizations of ordered semigroups which are natural ordered chains of archimedean ordered semigroups.

2. Chains of $\mathcal{N}$-Simple Ordered Semigroups

Lemma 2.1. *Let I be a semiprime ideal of an ordered semigroup S and let $a \in S$, $a \notin I$. Then there exists a prime ideal P of S such that $I \subseteq P$ and $a \notin P$.*

Proof. Suppose that $N(a) = S$. Then by Lemma 1.5 (iii) we have $a \in M(x)$ for all $x \in S$, especially for $z \in I$ we have $a \in M(z) \subseteq I$, this is a contradiction. Thus $N(a) \neq S$. Let $P = S \backslash N(a)$. Then P is a prime ideal of S with $a \notin P$, by Lemma 1.1. Suppose that $I \not\subseteq P$. Then $I \cap N(a) \neq \emptyset$. We select $x \in I \cap N(a)$, by Lemma 1.5 (iii) it follows that $a \in M(x) \subseteq I$, which is a contradiction. Thus $I \subseteq P$. ∎

Corollary 2.2. *Every semiprime ideal of an ordered semigroup S is an intersection of some prime ideals of S.*

Theorem 2.3. *Let S be an ordered semigroup. Then the following conditions are equivalent:*

(i) *S is a natural ordered chain of $\mathcal{N}$-simple ordered semigroups*;
(ii) *S is a chain of $\mathcal{N}$-simple subsemigroups*;
(iii) $(\forall a, b \in S)\ N(ab) = N(a) \cup N(b)$;
(iv) *For every nonempty family $\{I_\lambda;\ \lambda \in \Lambda\}$ of prime ideals of S, $\cap_{\lambda \in \Lambda} I_\lambda$ is a prime ideal of S*;
(v) *Every semiprime ideal of S is a prime ideal of S*;
(vi) *For every $a \in S$, $M(a)$ is a prime ideal of S*;
(vii) *$\mathcal{N}$ is the unique natural ordered chain congruence on S such that each of its congruence classes is $\mathcal{N}$-simple*;
(viii) *The set of all prime ideals of S, with $\subseteq$ as partial ordering, is a chain*;
(ix) $(\forall a, b \in S)\ M(a) \subseteq M(b)$ or $M(b) \subseteq M(a)$;
(x) *$M_S := \{M(a) | a \in S\}$, with $\cap$ as multiplication, is the unique natural ordered chain homomorphic image of S.*

Proof.
(i) ⇒ (ii), (v) ⇒ (vi) and (vii) ⇒ (i). The implications are obvious.
(ii) ⇒(iii). Let S be a chain C of $\mathcal{N}$-simple subsemigroups S_i ($i \in C$). Let $a \in S_i$, $b \in S_j$ with $i, j \in C$. It is obvious that $N(a) \cup N(b) \subseteq N(ab)$. Since

C is a chain, we have $i \leq j$ or $j \leq i$, i.e., $ij = i$ or $ij = j$. If $ij = i$, then $ab, a \in S_i$, since S_i is an $\mathcal{N}$-simple subsemigroup of S, we have $ab\mathcal{N}a$ in S_i, whence $ab \in N^{(i)}(a) \subseteq N(a)$ by Lemma 1.5 (iii), where $N^{(i)}(a)$ is the least filter of S_i containing a. Thus $N(ab) \subseteq N(a) \subseteq N(a) \cup N(b)$, so that $N(ab) = N(a) \cup N(b)$. Similarly we can prove that $N(ab) = N(a) \cup N(b)$ if $ij = j$.

(iii) $\Rightarrow$(iv). It is obvious that $J = \cap_{\lambda\in\Lambda} I_\lambda$ is also an ideal of S. Let $a, b \in S$ be such that $ab \in J$. Suppose that $a \notin J$ and $b \notin J$. Then $a, b \in S\backslash J = \cup_{\lambda\in\Lambda}(S\backslash I_\lambda)$, whence $a \in S\backslash I_1$ and $b \in S\backslash I_2$ for some $1, 2 \in \Lambda$. By Lemma 1.1, $S\backslash I_1$ and $S\backslash I_2$ are filters of S, from these it follows that $N(a) \subseteq S\backslash I_1$ and $N(b) \subseteq S\backslash I_2$, whence $ab \in N(ab) = N(a) \cup N(b) \subseteq (S\backslash I_1) \cup (S\backslash I_2) = S\backslash(I_1 \cap I_2)$ by (iii), and so $ab \notin I_1 \cap I_2 \supseteq J$, we get a contradiction. Thus we have $a \in J$ or $b \in J$, so that J is prime.

(iv) $\Rightarrow$ (v). This follows immediately from Corollary 2.2.

(vi) $\Rightarrow$ (vii). In view of Lemmas 1.2 and 1.6, $\mathcal{N}$ is the least natural ordered semilattice congruence on S such that each of its congruence classes is $\mathcal{N}$-simple. Let $a, b \in S$. By $ab \in M(ab)$ and (vi) we have $a \in M(ab)$ or $b \in M(ab)$, whence $ab \in N(a)$ or $ab \in N(b)$ by Lemma 1.5 (iii), from this it follows that $N(ab) = N(a)$ or $N(ab) = N(b)$ respectively. Thus $(a\mathcal{N})(b\mathcal{N}) = (ab)\mathcal{N} = a\mathcal{N}$ or $(a\mathcal{N})(b\mathcal{N}) = b\mathcal{N}$, i.e., $a\mathcal{N} \leq b\mathcal{N}$ or $b\mathcal{N} \leq a\mathcal{N}$ in $S/\mathcal{N}$. Hence $S/\mathcal{N}$ is a chain.

Let ω be also a natural ordered chain congruence on S such that each of its congruence classes is $\mathcal{N}$-simple. Then $\mathcal{N} \subseteq \omega$. Let $(a, b) \in \omega$. Then $a, b \in A$ for some ω-class A of S. Since A is $\mathcal{N}$-simple, we have $(a, b) \in \xi = \rho \cap \rho^{-1}$ in A, whence $(a, b) \in \rho$ and $(b, a) \in \rho$ in A. If $(a, b) \in \rho$ in A, then $a\eta^n b$ in A for some $n \in Z^+$, so $(a, b) \in \eta^n \subseteq \rho$ in S. Similarly, we have $(b, a) \in \rho$ in S by the fact that $(b, a) \in \rho$ in A. Thus $(a, b) \in \rho \cap \rho^{-1} = \mathcal{N}$ in S. Therefore, $\omega = \mathcal{N}$.

(iv) $\Rightarrow$ (viii). Let I, J be prime ideals of S. Assume that $I \not\subseteq J$. Then $I \cap J \neq I$. For any $x \in J$, we select $z \in I\backslash(I \cap J)$, then $zx \in IJ \subseteq I \cap J$. Since $I \cap J$ is also a prime ideal of S by (iv), we have $z \in I \cap J$ or $x \in I \cap J$,whence $x \in I \cap J$ since $z \notin I \cap J$, so that $J \subseteq I \cap J$. Thus $I \cap J = J$, i.e., $J \subseteq I$.

(viii) $\Rightarrow$ (ix). Let $a, b \in S$. If $N(a) = S$, then $b \in N(a)$, whence $a \in M(b)$ by Lemma 1.5 (iii), so that $M(a) \subseteq M(b)$. Similarly we have $M(b) \subseteq M(a)$ if $N(a) = S$. Assume that $N(a) \neq S$ and $N(b) \neq S$. Then $I = S\backslash N(a)$ and $J = S\backslash N(b)$ are prime ideals of S by Lemma 1.1, from these it follows by (viii) that $I \subseteq J$ or $J \subseteq I$, whence $N(b) \subseteq N(a)$ or $N(a) \subseteq N(b)$, so that $M(a) \subseteq M(b)$ or $M(b) \subseteq M(a)$ by Lemma 1.5 (iii).

(ix) $\Leftrightarrow$ (x). This follows from Lemma 1.2 and Lemma 1.5 (iv).

(ix) $\Rightarrow$ (iii). Let $x \in N(ab)$. Then $ab \in M(x)$, by Lemma 1.5 (ii) and (iii) we have $M(a) \cap M(b) = M(ab) \subseteq M(x)$. Since $M(a) \cap M(b) = M(a)$ or $M(a) \cap M(b) = M(b)$ by (ix), we have $a \in M(x)$ or $b \in M(x)$, whence $x \in N(a)$ or $x \in N(b)$, so that $x \in N(a) \cup N(b)$. Thus $N(ab) \subseteq N(a) \cup N(b)$ and so $N(ab) = N(a) \cup N(b)$. ■

By the following lemma, we give some new characterizations of intra-regular ordered semigroups, which are complements and deepening of [5].

Lemma 2.4. *The following conditions are equivalent on an ordered semigroup* S:

(i) S *is intra-regular* (i.e., $a \in (Sa^2S]$, $\forall a \in S$, see [5]);
(ii) $\rho = \tau$;
(iii) $(\forall a \in S)\ M(a) = J(a)$;
(iv) $(\forall a \in S)\ N(a) = \{x \in S | a \in J(x)\}$;
(v) $\mathcal{N} = \mathcal{J}$;
(vi) $(\forall a, b \in S)\ J(ab) = J(a) \cap J(b)$;
(vii) *Every ideal of* S *is semiprime.*

Proof.
(i) $\Rightarrow$ (ii). Let S be intra-regular. We first show that $\eta = \tau$. Obviously $\tau \subseteq \eta$. Let $a, b \in S$ be such that $(a, b) \in \eta$. Then there exists $m \in Z^+$ such that $b^m \leq xay$ for some $x, y \in S^1$. We select $k \in Z^+$ such that $m \leq 2^k$, by $b \in (Sb^2S] \subseteq (S(S(b^2)^2S]S] \subseteq (Sb^4S] \subseteq (Sb^8S] \subseteq \cdots$, we have $b \in (Sb^{2^k}S] = ((Sb^{2^k - m})b^mS] \subseteq (SxayS] \subseteq (SaS]$, whence $(a, b) \in \tau$. Thus $\eta \subseteq \tau$, and so $\eta = \tau$. Since τ is transitive, we have $\eta^n = \tau^n = \tau$ for every $n \in Z^+$, whence $\rho = \tau$.
(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). These follow immediately from Lemma 1.5 (i) and (iii).
(iv) $\Rightarrow$ (v). By Lemma 1.3 (ii) we need only to show that $\mathcal{N} \subseteq \mathcal{J}$. In fact: for every $(a, b) \in \mathcal{N}$, i.e., $N(a) = N(b)$, we have $a \in N(b)$ and $b \in N(a)$, whence $b \in J(a)$ and $a \in J(b)$, so $(a, b) \in \mathcal{J}$.
(v) $\Rightarrow$ (vi). For every $a \in S$, let $x \in S$ be such that $x^2 \in J(a)$. Then $J(x^2) \subseteq J(a)$. By Lemma 1.2 we have $x\mathcal{N}x^2$, whence $x\mathcal{J}x^2$ by (v), so that $x \in J(x) = J(x^2) \subseteq J(a)$. Thus $J(a)$ is semiprime, and hence $J(a) = M(a)$. Then the assertion follows immediately from Lemma 1.5 (ii).
(vi) $\Rightarrow$ (vii). Let I be any ideal of S and let $x \in S$ be such that $x^2 \in I$. By (vi) we have $x \in J(x) = J(x) \cap J(x) = J(x^2) \subseteq I$. Thus I is semiprime.
(vii) $\Rightarrow$ (i). This is [5, Remark 3]. ■

Corollary 2.5. *The following conditions are equivalent on an ordered semigroup* S:

(i) S *is a natural ordered chain of simple ordered semigroups*;
(ii) S *is a chain of simple subsemigroups*;
(iii) $(\forall a, b \in S)\ N(a) = \{x \in S | a \in J(x)\}$, and $N(ab) = N(a) \cup N(b)$;
(iv) *For every* $a \in S$, $J(a)$ *is a prime ideal of* S;
(v) *Every ideal of* S *is prime*;
(vi) $\mathcal{J}$ *is the unique natural ordered chain congruence on* S *such that each of its congruence classes is a simple subsemigroup*;
(vii) S *is intra-regular, and the set of all ideals of* S*, with* $\subseteq$ *as partial ordering, is a chain*;
(viii) $(\forall a, b \in S)\ J(ab) = J(a) \cap J(b)$, *and* $J(a) \subseteq J(b)$ *or* $J(b) \subseteq J(a)$;
(ix) $J_S := \{J(a) | a \in S\}$, *with* $\cap$ *as multiplication, is the unique natural ordered chain homomorphic image of* S;
(x) $(\forall a, b \in S)\ a \in (SabS]$ or $b \in (SabS]$.

Proof. (vii) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (x). These are [5, Theorem 2 and Lemma 3].

The other assertions follow immediately from Theorem 2.3 and Lemma 2.4. ■

3. Natural Ordered Chains of $\mathcal{J}_n$-Simple Ordered Semigroups

Theorem 3.1. *Let $n \in Z^+$. Then the following conditions are equivalent on an ordered semigroup S:*

(i) *S is a natural ordered chain of $\mathcal{J}_n$-simple ordered semigroups*;

(ii) $(\forall a, b \in S)$ $a\eta^n b \Longrightarrow a^2\eta^n b$, *and* $a\eta^n b$ *or* $b\eta^n a$;

(iii) *For every $a \in S$, $J_n(a)$ is a prime ideal of S*;

(iv) *$J_n(a)$ is a prime subset of S and $J_n(ab) = J_n(a) \cap J_n(b)$ for every $a, b \in S$*;

(v) $(\forall a, b, c \in S)$ $ab\eta^n a$ *or* $ab\eta^n b$, *and* $a\eta^n c$ & $b\eta^n c \Longrightarrow ab\eta^n c$;

(vi) *Every $\mathcal{J}_n$-class of S is a prime subsemigroup of S*;

(vii) $(\forall a, b \in S)$ $ab\mathcal{J}_n a$ *or* $ab\mathcal{J}_n b$;

(viii) $(\forall a, b \in S)$ $N(a) = \{x \in S | x\eta^n a\}$ *and* $N(ab) = N(a) \cup N(b)$;

(ix) *$\mathcal{J}_n = \eta^n \cap (\eta^n)^{-1}$ that is the unique natural ordered chain congruence on S such that each of its congruence classes is $\mathcal{J}_n$-simple*;

(x) *$J_S^{(n)} := \{J_n(x) | x \in S\}$, with $\cap$ as multiplication, is the unique natural ordered chain homomorphic image of S.*

Proof.
(i) $\Rightarrow$ (ii). Let S be a natural ordered chain C of $\mathcal{J}_n$-simple ordered semigroups S_i $(i \in C)$. Let $a \in S_i$, $b \in S_j$ with $i, j \in C$, be such that $a\eta^n b$. By Lemma 1.4 (i) it follows that $j \le i$, whence $ij = j$, so that $a^2 b \in S_j$. Since S_j is $\mathcal{J}_n$-simple, we have $b \in J_n(b) = J_n(a^2 b)$ in S_j, whence $a^2 b\eta^n b$ in S_j, so that $a^2\eta^n b$ in S.

For every $a \in S_i$ and $b \in S_j$, since C is a chain, we have $ij = i$ or $ij = j$. If $ij = i$, then $ab, a \in S_i$, from this a simple argument shows that $b\eta^n a$ in S since S_i is $\mathcal{J}_n$-simple. Similarly we have $a\eta^n b$ in S if $ij = j$.
(ii) $\Rightarrow$ (iii). Let $a, b \in S$ be such that $a\eta^{n+1} b$. Then $a\eta c\eta^n b$ for some $c \in S$. By (ii) it follows that $c^k\eta^n b$ for every $k \in Z^+$. On the other hand, there exists $k \in Z^+$ such that $c^k \in (SaS]$. Let $d \in S$ be such that $c^k\eta d\eta^{n-1} b$ if $n \ge 2$, and $d = b$ if $n = 1$. Then there exists $m \in Z^+$ such that $d^m \in (Sc^k S] \subseteq (SaS]$, whence $a\eta d$. Therefore, $\eta^{n+1} = \eta^n$, from this it follows that $\eta^n = \rho$.

For any $a \in S$, by Lemma 1.5 (i) it follows that $J_n(a) = \{x \in S | a\rho x\} = M(a)$ is an ideal of S. For every $x, y \in S$, by (ii) we have $x\eta^n y$ or $y\eta^n x$, i.e., $x\rho y$ or $y\rho x$, whence $y \in M(x)$ or $x \in M(y)$, so that $M(y) \subseteq M(x)$ or $M(x) \subseteq M(y)$. From this it follows by Theorem 2.3 that $J_n(a)$ is a prime ideal of S $(\forall a \in S)$.
(iii) $\Rightarrow$ (iv). For every $a \in S$, by (iii) it follows that $J_n(a)$ is a prime ideal of S containing a, whence $J_n(a)$ is semiprime, so that $M(a) \subseteq J_n(a)$ by Lemma 1.5 (i). But $J_n(a) \subseteq \{x \in S | a\rho x\} = M(a)$ and hence $J_n(a) = M(a)$. For every $a, b \in S$, by Lemma 1.5 (ii) we have $J_n(ab) = J_n(a) \cap J_n(b)$.
(iv) $\Rightarrow$ (v). Let $a, b \in S$. Since $J_n(ab)$ is prime and $ab \in J_n(ab)$, we have $a \in J_n(ab)$ or $b \in J_n(ab)$, i.e., $ab\eta^n a$ or $ab\eta^n b$. Let $c \in S$ be such that $a\eta^n c$ and $b\eta^n c$. Then $c \in J_n(a) \cap J_n(b) = J_n(ab)$ by (iv), so that $ab\eta^n c$.
(v) $\Rightarrow$ (vi). For every $a \in S$, let A be the $\mathcal{J}_n$- class of S containing a. Let $u, v \in A$. Then $J_n(u) = J_n(v) = J_n(a)$. If $x \in J_n(a)$, then $u\eta^n x$ and $v\eta^n x$, whence $uv\eta^n x$ by (v), so that $x \in J_n(uv)$. Thus $J_n(a) \subseteq J_n(uv)$. If $y \in J_n(uv)$,

then $uv\eta^n y$, from this it follows that $u\eta^n y$, whence $y \in J_n(u) = J_n(a)$, so that $J_n(uv) \subseteq J_n(a)$. Hence $J_n(uv) = J_n(a)$, i.e., $uv\mathcal{J}_n a$, and so $uv \in A$. Therefore, A is a subsemigroup of S.

Let $u, v \in S$. It is obvious that $J_n(uv) \subseteq J_n(u)$ and $J_n(uv) \subseteq J_n(v)$. If $uv \in A$, then $J_n(a) = J_n(uv) \subseteq J_n(u) \cap J_n(v)$. By (v) it follows that $uv\eta^n u$ or $uv\eta^n v$, whence $u \in J_n(uv) = J_n(a)$ or $v \in J_n(uv) = J_n(a)$, and so $a\eta^n u$ or $a\eta^n v$. Let $x \in S$ be such that $a\eta^n x$. Then by $a\eta^n x$, $a\eta^n x$ and (v) it follows that $a^2\eta^n x$, from this by the proof of "(ii)⇒(iii)" it follows that $a\eta^{n+k}x$ implies $a\eta^n x$ for any $k \in Z^+$. If $a\eta^n u$, then for every $x \in J_n(u)$ we have $u\eta^n x$, whence $a\eta^{n+n}x$ which implies $a\eta^n x$, i.e., $x \in J_n(a)$, so $J_n(u) \subseteq J_n(a)$, and hence $J_n(u) = J_n(a)$, i.e., $u \in A$. Similarly, we have $v \in A$ if $a\eta^n v$. Thus A is prime.

(vi) ⇒ (vii). For every $a, b \in S$, since $J_{ab}^{(n)}$, that is the $\mathcal{J}_n$-class of S containing ab, is a prime subsemigroups of S, we have $a \in J_{ab}^{(n)}$ or $b \in J_{ab}^{(n)}$, and hence $ab\mathcal{J}_n a$ or $ab\mathcal{J}_n b$.

(vii) ⇒ (ii). Let $a, b \in S$. By $ab\mathcal{J}_n a$ or $ab\mathcal{J}_n b$, we have $a \in J_n(a) = J_n(ab)$ or $b \in J_n(b) = J_n(ab)$, whence $ab\eta^n a$ or $ab\eta^n b$, from these it follows that $b\eta^n a$ or $a\eta^n b$ respectively. If $a\eta^n b$, then $b \in J_n(a) = J_n(a^2)$ by $a^2 = aa\mathcal{J}_n a$, whence $a^2\eta^n b$.

(ii) ⇒ (viii). By the proof of "(ii)⇒(iii)" we have that $\eta^n = \rho$. For every $a \in S$, by Lemma 1.5 (iii) we have $N(a) = \{x \in S | x\rho a\} = \{x \in S | x\eta^n\}$. For every $a, b \in S$, by $a\eta^n b$ or $b\eta^n a$ we have $a\rho b$ or $b\rho a$, whence $M(b) \subseteq M(a)$ or $M(a) \subseteq M(b)$ by Lemma 1.5 (i), so that $N(ab) = N(a) \cup N(b)$ by Theorem 2.3.

(viii) ⇒ (ix). By Lemma 1.3 (ii), (iii) it follows that $\mathcal{J}_n \subseteq \eta^n \cap (\eta^n)^{-1} \subseteq \rho \cap \rho^{-1} = \mathcal{N}$. On the other hand, let $(a, b) \in \mathcal{N}$. Then $N(a) = N(b)$. If $x \in J_n(a)$, then $a\eta^n x$, from this by (viii) we have $a \in N(x)$, whence $b \in N(b) = N(a) \subseteq N(x)$, so $b\eta^n x$, i.e., $x \in J_n(b)$. Thus $J_n(a) \subseteq J_n(b)$. Symmetrically, we have $J_n(b) \subseteq J_n(a)$, whence $J_n(a) = J_n(b)$, so $(a, b) \in \mathcal{J}_n$. Therefore, $\mathcal{N} \subseteq \mathcal{J}_n$, and so $\mathcal{J}_n = \eta^n \cap (\eta^n)^{-1} = \mathcal{N}$.

Let A be any $\mathcal{J}_n$-class of S and let $a, b \in A$. Then $a\mathcal{J}_n b$ in S, whence $a\eta^n b$ and $b\eta^n a$ in S, so $a\eta^n b$ and $b\eta^n a$ are in A by Lemma 1.4 (ii), from these a simple argumemt shows that $a\mathcal{J}_n b$ in A. Thus A is $\mathcal{J}_n$-simple.

By (viii) and Theorem 2.3 we see that $\mathcal{J}_n$ is the unique natural ordered chain congruence on S such that each of its congruence classes is a $\mathcal{J}_n$-simple subsemigroup.

As stated above, we conclude that (i)–(ix) are equivalent.

(ix) ⇒ (x). By (ix) ⇒ (iv) it is clear that the mapping $\theta : x \mapsto J_n(x)$ $(\forall x \in S)$ is an ordered semigroup homomorphism of $(S, \cdot, \leq)$ onto $(J_S^{(n)}, \cap, \subseteq)$, with kernel $Ker(\theta) = \{(a, b) \in S \times S | J_n(a) = J_n(b)\} = \mathcal{J}_n$, from this by (ix) we conclude that (x) holds.

(x) ⇒ (ix). Obvious. ■

Corollary 3.2. *Let S be an ordered semigroup. Then the following conditions are equivalent:*

(i) *S is a natural ordered chain of archimedean ordered semigroups*;

(ii) $(\forall a, b \in S)$ $a\eta b$ or $b\eta a$, and $a\eta b$ implies $a^2\eta b$;

(iii) *For every* $a \in S$, $J_1(a) := \{x \in S | a\eta x\} = \{x \in S | (\exists m \in Z^+) x^m \in (SaS]\}$ *is a prime ideal of* S;
(iv) *For every* $a \in S$, $J_1(a)$ *is a prime ideal of* S;
(v) *The radical of every ideal of* S *is a prime ideal of* S;
(vi) $(\forall a, b \in S)$ $ab\eta a$ or $ab\eta b$:
(vii) *Every* $\mathcal{J}_1$*-class of* S *is a prime subset of* S;
(viii) $(\forall a, b \in S)$ $ab\mathcal{J}_1 a$ or $ab\mathcal{J}_1 b$;
(ix) $(\forall a, b \in S)$ $N(a) = \{x \in S | x\eta a\}$ *and* $N(ab) = N(a) \cup N(b)$;
(x) $\mathcal{J}_1 = \eta \cap \eta^{-1}$ *that is the unique natural ordered chain congruence on* S *such that its each congruence class is an archimedean subsemigroup*;
(xi) $J_S^{(1)} := \{J_1(x) | x \in S\}$, *with* $\cap$ *as multiplication, is the unique natural ordered chain homomorphic image of S.*

Proof. By Theorem 3.1 we have (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Rightarrow$ (iv), (v) $\Rightarrow$ (vi), (vii) $\Rightarrow$ (viii) $\Leftrightarrow$ (ix) $\Leftrightarrow$ (x) $\Leftrightarrow$ (xi) $\Leftrightarrow$ (i) and (ii) $\Rightarrow$ (vii). To show that (vi) $\Rightarrow$ (vii), in view of Theorem 3.1, we need only to show that (vi) $\Rightarrow$ (ii). In fact:

Let $a, b \in S$ with $a\eta b$. Then there exists $m \in Z^+$ such that $b^m \leq xay$ for some $x, y \in S^1$. By (vi) we have that $(yxa)(ayx)\eta ayx$ or $(yxa)(ayx)\eta yxa$. If $yxa^2yx\eta ayx$, then $a^2\eta ayx$, whence $(ayx)^k \leq ua^2v$ for some $k \in Z^+$ and $u, v \in S^1$, so that $b^{m(k+1)} \leq (xay)^{k+1} \leq x(ayx)^k ay \leq xua^2vay$, and hence $a^2\eta b$. Similarly, we have $a^2\eta b$ if $yxa^2yx\eta yxa$.

For every $a, b \in S$, by (vi) we have $ab\eta a$ or $ab\eta b$, from these it follows that $b\eta a$ or $a\eta b$.

(iv) $\Rightarrow$ (v). For every ideal I of S, we have $I = \cup_{a\in I} J(a)$, from this it follows that $\sqrt{I} = \{x \in S | (\exists m \in Z^+) x^m \in I\} = \bigcup_{a\in I} \{x \in S | (\exists m \in Z^+) x^m \in J(a)\} = \bigcup_{a \in I} \{x \in S | a\eta x\} = \bigcup_{a\in I} J_1(a)$, so that $\sqrt{I}$ is a prime ideal of S by (iv). ■

Example. Let $S = Z^+ \times Z^+$. For any $a, b, c, d \in Z^+$, define

$$(a, b) \circ (c, d) = (\min\{a, c\}, b + d),$$
$$(a, b) \leq (c, d) \Longleftrightarrow a \leq c \text{ and } b \leq d \text{ in } Z^+.$$

Then $(S, \circ, \leq)$ is an ordered semigroup. In the notations of Corollary 3.2, some straightforward calculations show that

$$(a, b)\eta(c, d) \Longleftrightarrow c \leq a \text{ in } Z^+,$$
$$J_1((a, b)) = \{(x, y) \in S \mid x \leq a \text{ in } Z^+\},$$
$$J_1((a, b)) \cap J_1((c, d)) = \{(x, y) \in S \mid x \leq \min\{a, c\} \text{ in } Z^+\} = J_1((a, b) \circ (c, d)),$$
$$\mathcal{J}_1 = \eta \cap \eta^{-1} = \{((a, b), (c, d)) \in S \times S \mid a = c\},$$
$$J^{(1)}_{(a,b)} := \{(x, y) \in S \mid (a, b)\mathcal{J}_1(x, y)\} = \{(a, y) \mid y \in Z^+\}.$$

Since $(a, b) \circ (c, d)\mathcal{J}_1(a, b)$ (when $a \leq c$ in Z^+) or $(a, b) \circ (c, d)\mathcal{J}_1(c, d)$ (when $c \leq a$ in Z^+), by Corollary 3.2 we conclude that S is a natural ordered chain of archimedean ordered semigroups. Moreover, in view of Corollary 3.2 we see that $J_1((a,b))$ and $\sqrt{J_1((a,b))}$ are prime ideals of S, $N((a,b)) = \{(x,y) \in S \mid a \leq x \text{ in } Z^+\}$, $\mathcal{N} = \mathcal{J}_1$ and that $\theta : S \to J_S^{(1)}$ defined by $(a,b)\theta = J_1((a,b))$ $(\forall (a,b) \in S)$ is an ordered semigroup homomorphism of $(S, \circ, \leq)$ onto the natural ordered chain $(J_S^{(1)}, \cap, \subseteq)$ such that $J_1((a,b))\theta^{-1} = J_{(a,b)}^{(1)}$, where $(J_{(a,b)}^{(1)}, \circ, \leq) \cong (Z^+, +, \leq)$, is an archimedean ordered semigroup for every $(a,b) \in S$.

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